Common fixed point theorems via $\alpha-$series in b-metric spaces

Vildan Oztur[1] and Duran Turkolu[2]

[1] Department Of Mathematics, Polatli Faculty Of Science and Letters, Ankara Haci Bayram Veli University, Ankara, Turkey. https://orcid.org/0000-0001-5825-2030
[2] Department Of Mathematics, Faculty Of Science, Gazi University, Ankara, Turkey. https://orcid.org/0000-0002-8667-1432

E-mail: vildan.ozturk@hbv.edu.tr[1], dturkoglu@gazi.edu.tr[2]

Abstract

In this work, we will define rational type Hardy-Rogers contraction principle by $\alpha-$series using homogeneous functions in $b-$metric spaces and prove common fixed point theorems for sequences of self mappings. This method is based on the convergence series of coefficients. Our results are the generalized version of the results in the literature.

2010 Mathematics Subject Classification. **47H10**. 54H25.
Keywords. common fixed point, $\alpha-$series, b-metric.

1 Introduction

Metric spaces dating about a hundred years ago are an important tool in the transition from classical analysis to modern analysis. Metrics and distances are used many areas of Mathematics and other sciences such as chemistry, biology, networks, computer graphics, astronomy. Metric spaces have many generalizations due to their widespread using. One of them is b-metric space which has Hausdorff topology [6, 7, 8]. Most of fixed point results were gived in $b-$metric spaces [1, 2, 5, 11, 13, 14, 15, 16, 17, 18, 19]

Fixed point and common fixed point theorems in metric spaces have become a popular research topic since 1922, moreover in generalized metric spaces. Researchers proved fixed point theorems with various conditions on contraction, mappings or sets. In [20] authors generalized the contraction principle with $\alpha-$series which are a larger class of convergent series with a new approach. Vats et.al [21] developed tripled fixed point results using $\alpha-$series. Gaba [9, 10] defined λ-sequence of $(0,\infty)$ endowed with max metric specially which is also $\alpha-$sequence. Bonab.et.al. [3] presented the concept of JS-contractive type mappings using an $\alpha-$series.

Definition 1.1. [20] Let $\{a_r\}$ be a sequence of $[0,\infty)$. The series $\sum_{r=1}^{\infty} a_r$ is named $\alpha-$series if there exists $0<\alpha<1$ and $r(\alpha)\in\mathbb{N}$ such that

$$\sum_{r=1}^{\infty} a_r \leq \alpha L$$

for each $L \geq r(\alpha)$.

Example 1.2. [3] Series $\sum_{r=1}^{\infty}\frac{1}{2^r}$ and $\sum_{r=1}^{\infty}\frac{1}{r}$ are $\alpha-$series. Note that all convergent series with non-negative real sentences are $\alpha-$series.

Advanced Studies: Euro-Tbilisi Mathematical Journal 16, supplement issue 3 (2023), pp. 159–167.
DOI: 10.32513/asetmj/1932200823312
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 15 july 2023.
Accepted for publication: 05 October 2023.

In this work, we think a rational type form Hardy and Rogers contraction principle [12] by $\alpha-$series for the sequence of mappings and prove common fixed point theorems in $b-$metric spaces. This method is based on the convergence of an appropriate series of coefficients.

Definition 1.3. [6] Let $B \neq \varnothing$ be and a function $b : B \times B \to [0, \infty)$ satisfy the following properties for all $\xi_0, \xi_1, \xi_2 \in B$ and any constant $K \geq 1$,

(b^1) $b(\xi_0, \xi_1) \geq 0$,

(b^2) $b(\xi_0, \xi_1) = 0 \Leftrightarrow \xi_0 = \xi_1$,

(b^3) $b(\xi_0, \xi_1) = b(\xi_1, \xi_0)$,

(b^4) $b(\xi_2, \xi_1) \leq K(b(\xi_2, \xi_0) + b(\xi_0, \xi_1))$.

Then (B, b) is called a $b-$metric space.
If $K = 1$, then b is a metric.

Let (B, b) be a $b - ms$. For any $\xi_1 \in B$ and $\forall \varepsilon > 0$, $b-$open ball centered at ξ_0 with radius ε is

$$O(\xi_0, \varepsilon) = \{\xi_1 \in B : \quad b(\xi_0, \xi_1) < \varepsilon\}.$$

The topology of $b-$metric is Hausdorff.

Example 1.4. Let $B = [0, 100]$. Define $b : B^2 \to [0, \infty)$, $b(\xi_0, \xi_1) = \frac{1}{\xi_0 + \xi_1}$, then (B, b) is a $b-$metric space. But not a metric space.

Example 1.5. Let $B = \{0\} \cup \{\frac{1}{t} : t \geq 1, t \in \mathbb{N}\}$. Define $b : B^2 \to [0, \infty)$,

$b(\xi_0, \xi_1) = \begin{cases} 1, & \text{if } \xi_0 \neq \xi_1 \\ 0, & \text{if } \xi_0 = \xi_1 \end{cases}$ Then (B, b) is a $b-$metric space.

Example 1.6. Let $B = [0, \infty)$. Define $b : B^2 \to [0, \infty)$, $b(\xi_0, \xi_1) = \min\{1, |\xi_0 - \xi_1|\}$, then (B, b) is a $b-$metric space.

Definition 1.7. [4] Let $\{\xi_n\} \subset B$ be a sequence.

i. $\{\xi_n\}$ is named $b-$convergent $\Leftrightarrow$ there is $\xi \in B$ $\ni \lim_{n\to\infty} b(\xi_n, y) \to 0$.

ii. $\{\xi_n\}$ is named $b-$Cauchy sequence $\Leftrightarrow \lim_{n,m\to\infty} b(\xi_n, \xi_m) = 0$.

iii. A $b-$metric space is named $b-$complete if all b-Cauchy sequence is $b-$convergent.

In a $b-$metric space a $b-$convergent sequence has a unique limit.

2 Common fixed point theorems

We denote the set Γ by
$\Gamma = \{G|G : [0,\infty) \to [0,\infty)$, continuous, nondecreasing, sub-additive, homogeneus and $G^{-1}(\{0\}) = \{0\}\}$.

Theorem 2.1. Let (B,b) be a b-complete metric space and $\{k_n\}$ be a sequence of self mappings on B and $G \in \Gamma$ with degree s. Assume the sequences $\{a_n\}$, $\{c_n\}$, $\{d_n\}$ and $\{e_n\}$ of B such that

$$G\left(b\left(k_i\left(x\right),k_j\left(y\right)\right)\right) \leq G\left(\begin{array}{c}\vartheta_{i,j}b\left(x,y\right)+\beta_{i,j}b(x,k_i\left(x\right))+\gamma_{i,j}b(y,k_j\left(y\right)) \\ +\zeta_{i,j}\frac{b(x,k_i(x))b(y,k_j(y))}{1+b(x,y)}\end{array}\right) \tag{2.1}$$

for all distinct $x,y \in B$, where $\vartheta_{i,j},\beta_{i,j},\gamma_{i,j},\zeta_{i,j} \in [0,1)$, $i,j = 1,2,...$ with $\vartheta_{i,j}+\beta_{i,j}+\gamma_{i,j}+\zeta_{i,j} < 1$ and $\vartheta_{i,j} = b\left(a_i,a_j\right)$, $\beta_{i,j} = b\left(c_i,c_j\right)$, $\gamma_{i,j} = b\left(d_i,d_j\right)$, $\zeta_{i,j} = b\left(e_i,e_j\right)$. If $\sum_{i=1}^{\infty}\frac{(\vartheta_{i,i+1}+\beta_{i,i+1})^s}{1-(\gamma_{i,i+1}+\zeta_{i,i+1})^s}$ is an $\alpha-$series, then $\{k_n\}$ has a unique common fixed point.

Proof. Let $x_0 \in B$ be any arbitrary element. We introduce a sequence $x_n = k_n x_{n-1}$ for all $n \geq 1$. From (2.1) and since G is nondecreasing,

$$\begin{aligned}G\left(b\left(x_1,x_2\right)\right) &= G\left(b\left(k_1\left(x_0\right),k_2\left(x_1\right)\right)\right) \\ &\leq G\left(\begin{array}{c}\vartheta_{1,2}b\left(x_0,x_1\right)+\beta_{1,2}b(x_0,k_1\left(x_0\right))+\gamma_{1,2}b(x_1,k_2\left(x_1\right)) \\ +\zeta_{1,2}\frac{b(x_0,k_1(x_0))b(x_1,k_2(x_1))}{1+b(x_0,x_1)}\end{array}\right) \\ &= G\left(\begin{array}{c}\vartheta_{1,2}b\left(x_0,x_1\right)+\beta_{1,2}b(x_0,x_1)+\gamma_{1,2}b(x_1,x_2) \\ +\zeta_{1,2}\frac{b(x_0,x_1)b(x_1,x_2)}{1+b(x_0,x_1)}\end{array}\right) \\ &\leq G\left(\left(\vartheta_{1,2}+\beta_{1,2}\right)b\left(x_0,x_1\right)+\left(\gamma_{1,2}+\zeta_{1,2}\right)b(x_1,x_2)\right).\end{aligned}$$

By homogeneity and sub-additivity of G, we have

$$G\left(b\left(x_1,x_2\right)\right) \leq \left(\vartheta_{1,2}+\beta_{1,2}\right)^s G\left(b\left(x_0,x_1\right)\right)+\left(\gamma_{1,2}+\zeta_{1,2}\right)^s G(b(x_1,x_2)). \tag{2.2}$$

Hence,

$$G\left(b\left(x_1,x_2\right)\right) \leq \frac{\left(\vartheta_{1,2}+\beta_{1,2}\right)^s}{1-\left(\gamma_{1,2}+\zeta_{1,2}\right)^s}G\left(b\left(x_0,x_1\right)\right).$$

Therefore we get,

$$G\left(b\left(x_2,x_3\right)\right) \leq \frac{\left(\vartheta_{2,3}+\beta_{2,3}\right)^s}{1-\left(\gamma_{2,3}+\zeta_{2,3}\right)^s}\frac{\left(\vartheta_{1,2}+\beta_{1,2}\right)^s}{1-\left(\gamma_{1,2}+\zeta_{1,2}\right)^s}G\left(b\left(x_0,x_1\right)\right).$$

Continuouning this process

$$G\left(b\left(x_n,x_{n+1}\right)\right) \leq \Pi_{i=1}^{n}\frac{\left(\vartheta_{i,i+1}+\beta_{i,i+1}\right)^s}{1-\left(\gamma_{i,i+1}+\zeta_{i,i+1}\right)^s}G\left(b\left(x_0,x_1\right)\right). \tag{2.3}$$

For $p>0$, by triangular inequality of b and definition of G,

$$\begin{aligned} G\left(b\left(x_n,x_{n+p}\right)\right) &\leq K^s\left[G\left(b\left(x_n,x_{n+1}\right)\right)+G\left(b\left(x_{n+1},x_{n+2}\right)\right)+...+G\left(b\left(x_{n+p-1},x_{n+p}\right)\right)\right] \\ &\leq K^s[\Pi_{i=1}^{n}\frac{\left(\vartheta_{i,i+1}+\beta_{i,i+1}\right)^s}{1-\left(\gamma_{i,i+1}+\zeta_{i,i+1}\right)^s}G\left(b\left(x_0,x_1\right)\right) \\ &\quad +\Pi_{i=1}^{n+1}\frac{\left(\vartheta_{i,i+1}+\beta_{i,i+1}\right)^s}{1-\left(\gamma_{i,i+1}+\zeta_{i,i+1}\right)^s}G\left(b\left(x_0,x_1\right)\right) \\ &\quad +... \\ &\quad +\Pi_{i=1}^{n+p-1}\frac{\left(\vartheta_{i,i+1}+\beta_{i,i+1}\right)^s}{1-\left(\gamma_{i,i+1}+\zeta_{i,i+1}\right)^s}G\left(b\left(x_0,x_1\right)\right)] \\ &= K^s[\sum_{k=n}^{n+p-1}\Pi_{i=1}^{n+k}\frac{\left(\vartheta_{i,i+1}+\beta_{i,i+1}\right)^s}{1-\left(\gamma_{i,i+1}+\zeta_{i,i+1}\right)^s}G\left(b\left(x_0,x_1\right)\right)]. \end{aligned}$$

Therefore

$$\begin{aligned} G\left(b\left(x_n,x_{n+p}\right)\right) &\leq K^s[\sum_{k=n}^{n+p-1}\Pi_{i=1}^{n+k}\frac{\left(\vartheta_{i,i+1}+\beta_{i,i+1}\right)^s}{1-\left(\gamma_{i,i+1}+\mu_{i,i+1}\right)^s}G\left(b\left(x_0,x_1\right)\right)] \\ &\leq K^s[\sum_{k=n}^{n+p-1}\lambda^k G\left(b\left(x_0,x_1\right)\right)] \\ &\leq K^s\frac{\lambda^n}{1-\lambda}G\left(b\left(x_0,x_1\right)\right). \end{aligned}$$

Letting limit as $n\to\infty$, using continuity of G and $G(0)=\{0\}$, we get $b\left(x_n,x_{n+p}\right)\to 0$ and $\{x_n\}$ is a b-Cauchy sequence. Since (B,b) is $b-$complete, there exists a $y\in B$ with $x_n\to y$. For any $m\in\mathbb{N}$,

$$\begin{aligned} G\left(b\left(x_n,k_m(y)\right)\right) &\leq G\left(b\left(k_n\left(x_{n-1}\right),k_m\left(y\right)\right)\right) \\ &\leq G\left(\begin{array}{c}\vartheta_{n,m}b\left(x_{n-1},y\right)+\beta_{n,m}b(x_{n-1},k_n\left(x_{n-1}\right)) \\ +\gamma_{n,m}b(y,k_m\left(y\right)) \\ +\zeta_{n,m}\frac{b(x_{n-1},k_n(x_{n-1}))b(y,k_m(y))}{1+b(x_{n-1},y)}\end{array}\right) \end{aligned}$$

As $n\to\infty$,

$$\begin{aligned} G\left(b\left(y,k_m(y)\right)\right) &\leq G\left(\begin{array}{c}\vartheta_{n,m}b\left(y,y\right)+\beta_{n,m}b(y,y) \\ +\gamma_{n,m}b(y,k_m\left(y\right)) \\ +\zeta_{n,m}\frac{b(y,y)b(y,k_m(y))}{1+b(y,y)}\end{array}\right) \\ &= G\left(\gamma_{n,m}b(y,k_m\left(y\right))\right) \\ &= \gamma_{n,m}^s G\left(b(y,k_m\left(y\right))\right). \end{aligned}$$

Thus by $\gamma\in[0,1)$, we have $G\left(b\left(y,k_m(y)\right)\right)=0$. So $y=k_m(y)$ i.e. y is common fixed point of $\{k_m\}$.

Now we show that uniqueness of fixed point. Suppose $\{k_m\}$ has different fixed points y, z. By (2.1),

$$\begin{aligned} G\left(b\left(y,z\right)\right) &= G(b\left(k_m(y),k_m(z)\right)) \\ &\leq G\left(\begin{array}{c}\vartheta_{n,m}b\left(y,z\right)+\beta_{n,m}b(y,k_m(y))+\gamma_{n,m}b(z,k_m(z)) \\ +\zeta_{n,m}\frac{b(y,k_m(y))b(z,,k_m(z)}{1+b(y,z)}\end{array}\right) \\ &= G\left(\begin{array}{c}\vartheta_{n,m}b\left(y,z\right)+\beta_{n,m}b(y,y))+\gamma_{n,m}b(z,z) \\ +\zeta_{n,m}\frac{b(y,y)b(z,z)}{1+b(y,z)}\end{array}\right) \\ &= G(0)=0 \end{aligned}$$

which is a contradiction. Thus $\{k_m\}$ has a unique fixed point. Q.E.D.

Example 2.2. Let $B=[0,1]$. Let define $b:[0,1]\times[0,1]\to\mathbb{R}$, $b\left(x,y\right)=\left(x-y\right)^2$ and $G:[0,\infty)\to[0,\infty)$, $G(x)=x^{1/2}$. Then b is a b-metric on $[0,1]$ and $G\in\Gamma$.
For the sequences $(a_i)=(\frac{1}{2i})$, $(c_i)=(d_i)=(\frac{1}{(1+4^i)})$, $(e_i)=(\frac{1}{1+i^2})$. Thus, we have for $i,j\geq 1$

$$\begin{aligned} \vartheta_{i,j} &= \left(\frac{1}{2i}-\frac{1}{2j}\right)^2,\ \beta_{i,j}=\gamma_{i,j}=\left(\frac{1}{1+4^i}-\frac{1}{1+4^j}\right)^2 \\ \zeta_{i,j} &= \left(\frac{1}{1+i^2}-\frac{1}{1+j^2}\right)^2. \end{aligned}$$

Let $k_i:B\to B$, $k_i(x)=\frac{x}{4i}$. Then, we get

$$\begin{aligned} G\left(b\left(k_i\left(x\right),k_j\left(y\right)\right)\right) &= G\left(\left(\frac{x}{4i}-\frac{y}{4j}\right)^2\right)=\left|\frac{x}{4i}-\frac{y}{4j}\right| \\ &\leq \left|\frac{1}{2i}-\frac{1}{2j}\right||x-y| \\ &= G\left(\vartheta_{i,j}b(x,y)\right) \\ &\leq G\left(\begin{array}{c}\vartheta_{i,j}b\left(x,y\right)+\beta_{i,j}b(x,k_i\left(x\right))+\gamma_{i,j}b(y,k_j\left(y\right)) \\ +\zeta_{i,j}\frac{b(x,k_i(x))b(y,k_j(y))}{1+b(x,y)}\end{array}\right). \end{aligned}$$

All conditions of previous theorem are satisfied. 0 is common fixed point of $\{k_n\}$.

Corollary 2.3. Let (B,b) be a b-complete metric space and $\{k_n\}$ be a sequence of self mappings on B and $G\in\Gamma$ with degree s. Assume the sequences $\{a_n\}\subset B$ such that

$$G\left(b\left(k_i\left(x\right),k_j\left(y\right)\right)\right)\leq G\left(\vartheta_{i,j}[b\left(x,y\right)+b(x,k_i\left(x\right))+b(y,k_j\left(y\right))]\right)$$

for all distinct $x,y\in B$, where $0\leq\vartheta_{i,j}<\frac{1}{3}$, ($i,j=1,2,...$) and $\vartheta_{i,j}=b\left(a_i,a_j\right)$. Then $\{k_n\}$ has a unique common fixed point.

Corollary 2.4. Let (B,b) be a b-complete metric space and $\{k_n\}\subset B$ be a sequence of self mappings and $G\in\Gamma$. Assume the sequence $\{a_n\}\subset B$ such that

$$G\left(b\left(k_i\left(x\right),k_j\left(y\right)\right)\right)\leq G\left(\vartheta_{i,j}[b(x,k_i\left(x\right))+b(y,k_j\left(y\right)]\right)$$

$x, y \in B$ with $x \neq y$ and $0 \leq \vartheta_{i,j} < \frac{1}{2}$, $\vartheta_{i,j} = b(a_i, a_j)$ ($i, j = 1, 2, ...$). Then $\{k_n\}$ has a unique common fixed point.

Corollary 2.5. Let (B, b) be a b-complete metric space and $\{k_n\}$ be a sequence of self mappings on B and $G \in \Gamma$ with degree s. Assume the sequences $\{a_n\}$, $\{c_n\}$, $\{d_n\}$ and $\{e_n\}$ of B such that

$$G(b(k_i(x), k_j(y))) \leq G\left(\vartheta_{i,j}[b(x, k_i(x)) + b(y, k_j(y))] + \zeta_{i,j} \frac{b(x,k_i(x))b(y,k_j(y))}{1+b(x,y)} \right)$$

for all distinct $x, y \in B$, where $\vartheta_{i,j}, \zeta_{i,j} \in [0, 1)$, $i, j = 1, 2, ...$ with $\vartheta_{i,j} + \zeta_{i,j} < 1$ and $\vartheta_{i,j} = b(a_i, a_j)$, $\zeta_{i,j} = b(e_i, e_j)$. Then $\{k_n\}$ has a unique common fixed point.

Corollary 2.6. Let (B, b) be a b-complete metric space and $\{k_n\} \subset B$ be a sequence of self mappings and $G \in \Gamma$. Assume the sequences $\{a_n\}, \{c_n\} \subset B$ such that

$$G(b(k_i(x), k_j(y))) \leq G(\vartheta_{i,j}[b(x, k_i(x)) + b(y, k_j(y)] + \zeta_{i,j} b(x, y))$$

$x, y \in B$ with $x \neq y$ and $\vartheta_{i,j}, \zeta_{i,j} \in [0, 1)$, $\vartheta_{i,j} = b(a_i, a_j)$, $\zeta_{i,j} = b(c_i, c_j)$ ($i, j = 1, 2, ...$). Then $\{k_n\}$ has a unique common fixed point.

Corollary 2.7. Let (B, b) be a b-complete metric space and $\{k_n\}$ be a sequence of self mappings on B. Assume the sequences $\{a_n\}$, $\{c_n\}$, $\{d_n\}$ and $\{e_n\}$ of B such that

$$b(k_i(x), k_j(y)) \leq \vartheta_{i,j} b(x, y) + \beta_{i,j} b(x, k_i(x)) + \gamma_{i,j} b(y, k_j(y)) + \zeta_{i,j} \frac{b(x, k_i(x)) b(y, k_j(y))}{1 + b(x, y)}$$

for all distinct $x, y \in B$, where $\vartheta_{i,j}, \beta_{i,j}, \gamma_{i,j}, \zeta_{i,j} \in [0, 1)$, $i, j = 1, 2, ...$ with $\vartheta_{i,j} + \beta_{i,j} + \gamma_{i,j} + \zeta_{i,j} < 1$ and $\vartheta_{i,j} = b(a_i, a_j)$, $\beta_{i,j} = b(c_i, c_j)$, $\gamma_{i,j} = b(d_i, d_j)$, $\zeta_{i,j} = b(e_i, e_j)$. If $\sum_{i=1}^{\infty} \frac{(\vartheta_{i,i+1} + \beta_{i,i+1})^s}{1 - (\gamma_{i,i+1} + \zeta_{i,i+1})^s}$ is an $\alpha-$series, then $\{k_n\}$ has a unique common fixed point.

3 Application

Let consider the following integral equation

$$x(t) = \int_0^t K_i(t, r, x(r))] dr, \quad t \in [0, a] \tag{3.1}$$

where $a > 0$, $K_i : [0, a] \times [0, a] \times \mathbb{R} \to \mathbb{R}$. Let us consider $B = (C[0, a], \mathbb{R})$ be the set of all continuous functions $x : [0, a] \to \mathbb{R}$ and the sequence of mappings $k_i : B \to B$ be defined by

$$k_i(x)(t) = \int_0^t K_i(t, r, x(r))] dr, \quad t \in [0, a].$$

Let us consider the $b-$metric on B,

$$b(x, y) = \sup_{t \in [0,a]} |x(t) - y(t)|^2$$

for all $x, y \in B$. Then (B, b) is complete $b-$metric space with constant $K = 2$.

Theorem 3.1. Suppose the following hypotheses hold:
(i) the functions K_i are continuous,
(ii) for all $t, r \in [0,a]$ and $x, y \in \mathbb{R}$

$$|K_i(t,r,x) - K_i(t,r,y)| \leq \sqrt{\frac{1}{t}\cdot\frac{1}{2^{i+j+2}}} \left(\begin{array}{c} |x(t)-y(t)|^2 + |x(t)-k_i(x)(t)|^2 \\ +|y(t)-k_j(y)(t)|^2 + \frac{|x(t)-k_i(x)(t)|^2|y(t)-k_j(y)(t)|2}{1+|x(t)-y(t)|^2} \end{array} \right)^{1/2}$$

for $i, j = 1, 2, ...$ Then integral equation (3.1) has a unique solution.

Proof. Endow $B = (C[0,a], \mathbb{R})$ with the b-metric

$$b(x,y) = \sup_{t\in[0,a]} |x(t) - y(t)|^2.$$

Consider the sequence of mappings

$$k_i : B \to B \ , \ \ k_i(x(t)) = \int_0^t K_i(t,r,x(r))\, dr, \quad t \in [0,a]$$

For all $x, y \in B$, using Cauchy Schwarz inequality and (ii), we have

$$\begin{aligned} |k_i(x(t)) - k_j(y(t))|^2 &= \left| \int_0^t K_i(t,r,x(r))\, dr - \int_0^t K_j(t,r,y(r))\, dr \right|^2 \\ &\leq \left(\int_0^t 1 dr \right) \left(\int_0^t |K_i(t,r,x(r)) - K_j(t,r,y(r))|^2\, dr \right) \\ &\leq \frac{1}{t}\frac{1}{2^{i+j+2}} \int_0^t \left(\begin{array}{c} |x(t)-y(t)|^2 + |x(t)-k_i(x)(t)|^2 \\ +|y(t)-k_j(y)(t)|^2 \\ +\frac{|x(t)-k_i(x)(t)|^2|v(t)-k_j(y)(t)|^2}{1+|x(t)-y(t)|^2} \end{array} \right) dr \\ &\leq \frac{1}{t}\frac{1}{2^{i+j+2}} \left(\begin{array}{c} |x(t)-y(t)|^2 + |x(t)-k_i(x)(t)|^2 \\ +|y(t)-k_j(y)(t)|^2 \\ +\frac{|x(t)-k_i(x)(t)|^2|y(t)-k_j(y)(t)|^2}{1+|x(t)-y(t)|^2} \end{array} \right) \int_0^t dr \end{aligned}$$

Thus, we get

$$\begin{aligned} b(k_i(x) - k_j(y)) \leq\ & \vartheta_{i,j} b(x,y) + \beta_{i,j} b(x, k_i(x)) + \gamma_{i,j} b(y, k_j(y)) \\ & + \zeta_{i,j} \frac{b(x,k_i(x))\, b(y,k_j(y))}{1 + b(x,y)}. \end{aligned}$$

for $\vartheta_{i,j}, \beta_{i,j}, \gamma_{i,j}, \zeta_{i,j} = \frac{1}{2^{i+j+2}}$. So, by Corollary 2.7, integral equation (3.1) has a unique solution. Q.E.D.

References

[1] S. Aleksić, H. Huang, Z.D. Mitrović and S. Radenović, *Remarks on some fixed point results in b-metric spaces*, Journal Of Fixed Point Theory Appl. **20(147)** (2018).

[2] A.S. Babu, T. Dosenovic, MD.M. Ali, S. Radenovic and K.P.R. Rao, *Some Presic type results in b-discolated metric spaces*, Constructive Mathematical Analysis **2(1)** (2019) 40-48.

[3] S.H. Bonab, V. Parvaneh, H. Hosseinzadeh, A. Dinmohammadi, B. Mohammadi, *Fixed Point Theory And Fractional Calculus Recent Advanced And Applications*, Springer (2022) 93-104.

[4] M. Boriceanu, M. Bota and A. Petrusel, *Multivalued fractals in b-metric spaces*, Cent. Eur. J. Math. **10** (2010) 367-377.

[5] S. Chandok, V. Ozturk and S. Radenovic, *On fixed points in context of b-metric spaces*, Matematicki Vesnik 71(1-2) (2019), 23-30.

[6] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostrav. **1** (1993) 5-11.

[7] E. Deza and M. Deza, Dictionary of Distances, Elsevier, (2006).

[8] N.V. Dung and V.T. Le Hang, *The generalized Banach contraction conjecture in b-metric spaces*, J. Fixed Point Theory Appl. **20** (2018).

[9] Y.U. Gaba, *Metric type spaces and $\lambda-$sequences*, Quaestiones Mathematicae **40(10)** (2017) 49-55.

[10] Y.U. Gaba, *$\lambda-$sequences and fixed point theorems G-metric type spaces*, Journal Of The Nigerian Mathematical Society **35** (2016) 303-311.

[11] T. Hamaizia and A. Aliouche, *A nonunique common fixed point theorem of Rhoades type in b-metric spaces with applications*, Int. J. Nonlinear Anal. Appl. **12(2)** (2021) 399-413.

[12] G.E. Hardy and T.D. Rogers, *A generalization of a fixed point theorem of Reich*, Canadian Mathematical Bulletin **16** (1973) 201-206.

[13] Z. Kadelburg and S. Radenovic, *Notes on some recent papers concerning F-contractions in b-metric spaces*, Constructive Mathematical Analysis **1(2)** (2018) 108-112.

[14] E. Karapinar, *A short survey on the recent fixed point results on b-metric spaces*, Constructive Mathematical Analysis **1(1)** (2018) 15-44.

[15] G. Mani, L.N. Mishra and V.N. Mishra,*Common fixed point theorems in complex partial b-metric space with an application to integral equations*, Adv. Studies: Euro-Tbilisi Math. J. **15(1)** (2022) 129-149.

[16] M. Nazam, M. Arshad, P. Choonkil, O. Acar, Y. Sungsik and G.A. Anastassiou, *On solution of a system of differential equations via fixed point theorem*, Journal of Computational Analysis and Applications **27(3)** (2019) 417-426.

[17] V. Ozturk and S. Radenovic, *Some remarks on b-E.A-property in b-metric spaces*, Springerplus **5(544)** (2016) 1–10.

[18] V. Ozturk, D. Turkoglu and A.H. Ansari, *Fixed points for generalized (F,h,alpha,mü)-phi contractions in b-metric spaces*, Commun. Fac. Sci. Univ. Ank. Ser. A1: Math. Stat. **67(2)** (2018) 306–316.

[19] V. Ozturk and D. Turkoglu, *Fixed points for generalized alpha-psi contractions in b-metric spaces*, Journal Of Nonlinear And Convex Analysis, **16(10)** (2015) 2059–2066.

[20] V. Sihag, R.K. Vats and C. Vetro, *A fixed point theorems in G-metric spaces via $\alpha-$series*, Quaestiones Mathematicae **37(3)** (2014) 429-434.

[21] R.K. Vats, K. Tas, V. Sihag, and A. Kumar, *Triple fixed point theorems via alpha-series in partially ordered metric spaces*, Journal of Inequalities and Applications **2014(176)** (2014).