Timelike Mannheim curves in Minkowski 3-space revisited

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Abstract

An important field of study of the theory of curves in Euclidean 3-space is the study of curve pairs with the help of relations between Frenet vectors of two space curves. In searching for pairs of curves such that the tangents, principal normals or binormals of one may be the tangents, principal normals or binormals of the other, there are six cases to be considered. Mannheim curves is one of the curves that emerged as a result of this relationship. Mannheim curves have been studied extensively for different dimensions in both Euclidean spaces and Minkowski spaces. In this paper, we give new results for the timelike Mannheim curves in Minkowski 3-space by using a new approach. According to this approach, the necessary and sufficient conditions including the known results have been obtained for a timelike curve to be Mannheim curve in $\mathbb{E}_1^3$.

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1 Introduction

In Euclidean 3-space, it is well known that Frenet vectors of a curve, like curvature functions, play an important role in the study of differential geometric properties of the curve with the help of Frenet vectors and curvature functions, we can characterize curves. Examining the relationships between the Frenet vectors of curve pair also an important research topic in the geometry of curves.

In searching for pairs of curves such that the tangents, principal normals or binormals of one may be the tangents, principal normals or binormals of the other, there are six cases to be considered [15].

In one of these cases, Bertrand curves are obtained. In 1845, Saint-Venant proposed [17], the question whether upon the ruled surface generated by the principal normals of a curve in the 3-dimensional Euclidean space $\mathbb{E}^3$, a second curve can exist having the same principal normals. This question was answered by Bertrand in the paper published in 1850 [3]. After the paper of Bertrand, the pairs of curves like this have been called Conjugate Bertrand Curves, or more commonly Bertrand Curves [12].

Another notable case is evolutes and involutes curves. This situation has been discussed in many differential geometry books ([8], [12]). Another case reveals Mannheim curves which have a property that their principal normal lines coincide with the binormal lines of the Mannheim mate (partner) curve at the corresponding points of the curves. It is known that the curvature functions of Mannheim curve in $\mathbb{E}^3$ satify the equality $\kappa_1 = a(\kappa_1^2 + \kappa_2^2)$ for some positive constant number a and its parametric equation is obtained in [13] (see also [15], [16]). Some characterizations of Mannheim curves in $\mathbb{E}^3$ can be found in [12] and [13].

Let φ be a Mannheim curve and $\varphi^\star$ be the Mannheim partner curve of φ in Euclidean 3-space $\mathbb{E}^3$. In the literature, $\varphi^\star$ is given by

$$\varphi^\star(s^\star) = \varphi^\star(f(s)) = \varphi(s) + \lambda(s)\, N(s) \tag{1.1}$$

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where $N(s)$ is the principal normal vector field of φ and f is a diffeomorphism such that $f : I \to I^\star$, $s^\star = f(s)$ [12]. With respect to (1.1), the vector $\overrightarrow{\varphi^\star\varphi}$ must be parallel to N. The Mannheim curve example given in parametric form in 3-dimensional Euclidean space is unknown, except for the example given in Eisenhart's book [8]. One of these examples is circular helix curves.

In [5](see also [4]), the authors gave a new approach to Mannheim curves, taking into account that the $\overrightarrow{\varphi^\star\varphi}$ vector need not be parallel to the N vector.

According to this new approach, the Mannheim partner curve of $\varphi^\star$ is given by

$$\varphi^\star(s^\star) = \varphi^\star(f(s)) = \varphi(s) + \mu_1(s)T(s) + \mu_2(s)N(s) + \mu_3(s)B(s) \tag{1.2}$$

where $\mu_1(s), \mu_2(s)$ and $\mu_3(s)$ are differentiable function on $I \subset \mathbb{R}$.

With the help of this new approach, it has been shown that curves such as slant helix, Salkowski curves and anti-Salkowski curves, which are well known in the theory of curves, can be Mannheim curves and the Mannheim conjugates of these curves are obtianed. This new approach given by the authors in [5] seems to bring important results for Mannheim curves in Euclidean 3-space.

In this study, timelike Mannheim curves in Minkowski 3-space are discussed with the new approach given above. Spacelike and timelike Mannheim conjugate curves of a timelike curve are obtained and examples of these are constructed. It has been shown that a general helix curve cannot be a Mannheim curve in Minkowski 3-space, and the Mannheim conjugate of a slant helix curve is a general helix. According to this approach, the necessary and sufficient conditions including the known results have been obtained for a timelike curve to be Mannheim curve in $\mathbb{E}_1^3$.

2 Preliminaries

The Minkowski space $\mathbb{E}_1^3$ is the Euclidean 3-space $\mathbb{E}^3$ equipped with indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}_1^3$. Recall that a vector $v \in \mathbb{E}_1^3 \backslash \{0\}$ can be *spacelike* if $g(v,v) > 0$, *timelike* if $g(v,v) < 0$ and *null (lightlike)* if $g(v,v) = 0$ and $v \neq 0$. In particular, the vector $v = 0$ is a spacelike. The norm of a vector v is given by $||v|| = \sqrt{|g(v,v)|}$, and two vectors v and w are said to be orthogonal, if $g(v,w) = 0$. An arbitrary curve $\varphi(s)$ in $\mathbb{E}_1^3$, can locally be *spacelike*, *timelike* or *null* (*lightlike*), if all its velocity vectors $\varphi'(s)$ are respectively spacelike, timelike or null [12]. A null curve φ is parameterized by pseudo-arc s if $g(\varphi''(s), \varphi''(s)) = 1$. A spacelike or a timelike curve $\varphi(s)$ has unit speed, if $g(\varphi'(s), \varphi'(s)) = \pm 1$ ([7], [9], [14]).

Let $\{T, N, B\}$ be the moving Frenet frame along a curve φ in $\mathbb{E}_1^3$, consisting of the tangent, the principal normal and the binormal vector fields respectively. Depending on the causal character of φ, the Frenet equations have the following forms.

Case I. If φ is a non-null curve, the Frenet equations are given by [12]:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2\kappa_1 & 0 \\ -\varepsilon_1\kappa_1 & 0 & \varepsilon_3\kappa_2 \\ 0 & -\varepsilon_2\kappa_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.1}$$

where κ_1 and κ_2 are the first and the second curvature of the curve respectively. Moreover, the following conditions hold:

$$g(T,T)=\varepsilon_1=\pm 1, g(N,N)=\varepsilon_2 \ =\pm 1, g(B,B)=\varepsilon_3=\pm 1$$

and

$$g(T,N)=g(T,B)=g(N,B)=0.$$

Case II. If φ is a null curve, the Frenet equations are given by [7]:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 \\ -\kappa_2 & 0 & -\kappa_1 \\ 0 & \kappa_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.2}$$

where the first curvature $\kappa_1 = 0$ if φ is straight line, or $\kappa_1 = 1$ in all other cases. In particular, the following conditions hold:

$$g(T,T)=g(B,B)=g(T,N)=g(N,B)=0, g(N,N)=g(T,B)=1.$$

We will need the definitions and results given below in the next section.

Definition 2.1. A curve is called a slant helix if its principal normal vector field makes a constant angle with a fixed direction in $\mathbb{E}_1^3$ [10].

Lemma 2.2. Let φ be a unit speed timelike curve in $\mathbb{E}_1^3$. Then φ is a slant helix if and only if the geodesic curvature of the spherical image of principal normal indicatrix (n) of φ,

$$\sigma = \frac{\kappa^2}{(\tau^2-\kappa^2)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)',$$

or

$$\sigma = \frac{\kappa^2}{(\kappa^2-\tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)'$$

is constant and $\tau^2-\kappa^2 \neq 0$ [1].

Definition 2.3. A curve is called a rectifying curve when its position vector always lies in its rectifying plane [6].

Lemma 2.4. Let φ be a unit speed non-lightlike curve with timelike or spacelike principal normal vector field in $\mathbb{E}_1^3$, then φ is congruent to a rectifying curve if and only if

$$\frac{\kappa(s)}{\tau(s)} = c_1 s + c_2$$

for some constants $c_1 \neq 0$ and c_2 [11].

3 Timelike Mannheim curves in Minkowski 3-space revisited

In this section, we will reassess the Mannheim curves in Minkowski 3-space $\mathbb{E}_1^3$. In the following definition, we give the well-known definition of Mannheim curve in $\mathbb{E}_1^3$.

Definition 3.1. A curve $\varphi : I \to \mathbb{E}_1^3$ with non-zero curvatures is a Mannheim curve if there is a curve $\varphi^\star : I^\star \to \mathbb{E}_1^3$ and a diffeomorphism $f : I \to I^\star$ such that the principal normal vectors of $\varphi(s)$ and the binormal vectors of $\varphi^\star(s^\star)$ at $s \in I$, $f(s) = s^\star \in I^\star$ coincide. In this situation, $\varphi^\star(s^\star)$ is called the Mannheim mate of $\varphi(s)$.

Let $\varphi : I \to \mathbb{E}_1^3$ be a timelike Mannheim curve with the Frenet frame $\{T(s), N(s), B(s)\}$ and non-zero curvatures κ_1, κ_2 and $\varphi^\star : I^\star \to \mathbb{E}_1^3$ be a Mannheim mate curve of φ with the Frenet frame $\{T^\star(s^\star), N^\star(s^\star), B^\star(s^\star)\}$ and curvatures $\kappa_1^\star, \kappa_2^\star$. Then $\varphi^\star$ can be written as

$$\varphi^\star(s^\star) = \varphi^\star(f(s)) = \varphi(s) + \mu_1(s)\,T(s) + \mu_2(s)\,N(s) + \mu_3(s)\,B(s)$$

where $\mu_1(s), \mu_2(s)$ and $\mu_3(s)$ are differentiable function on I.

Since the principal normal vector of the spacelike curve φ is timelike, the Mannheim mate curve $\varphi^\star$ can be a timelike curve or a spacelike curve. We will assess all situations severally in the following theorem.

Theorem 3.2. Let $\varphi : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a unit speed timelike curve with the non-zero curvatures κ_1, κ_2 and $\kappa_1 \neq \kappa_2$. Then the curve φ is a Mannheim curve with Mannheim mate curve $\varphi^\star$ if and only if one of the following conditions holds:

(i) For $|\kappa_2| < |\kappa_1|$, one of the following equations is provided

$$\begin{aligned} \mu_1\kappa_1 + \mu_2^{'} &= \mu_3\kappa_2, \\ \mu_3^{'} + \mu_2\kappa_2 &\neq 0, \\ (1 + \mu_1^{'} + \mu_2\kappa_1)\kappa_1 &= \left(\mu_3^{'} + \mu_2\kappa_2\right)\kappa_2 \end{aligned} \tag{3.1}$$

where $\mu_1(s), \mu_2(s)$ and $\mu_3(s)$ are differentiable functions on I. In this case, the Mannheim mate curve $\varphi^\star$ is a spacelike curve with timelike principal normal.

(ii) For $|\kappa_1| < |\kappa_2|$, one of the following equations is provided

$$\begin{aligned} \mu_1\kappa_1 + \mu_2^{'} &= \mu_3\kappa_2, \\ \mu_3^{'} + \mu_2\kappa_2 &\neq 0, \\ (1 + \mu_1^{'} + \mu_2\kappa_1)\kappa_1 &= \left(\mu_3^{'} + \mu_2\kappa_2\right)\kappa_2 \end{aligned} \tag{3.2}$$

where $\mu_1(s), \mu_2(s)$ and $\mu_3(s)$ are differentiable functions on I. In this case, the Mannheim mate curve $\varphi^\star$ is a timelike curve.

Proof. Assume that φ is a timelike Mannheim curve with parametrized by arc-length s with non-zero curvatures κ_1, κ_2 and $\kappa_1 \neq \kappa_2$ and the curve $\varphi^\star$ is the Mannheim mate curve of φ parametrized by with arc-length or pseudo arc $s^\star$. Then, we can write the curve $\varphi^\star$ as

$$\varphi^\star(s^\star) = \varphi^\star(f(s)) = \varphi(s) + \mu_1(s)\,T(s) + \mu_2(s)\,N(s) + \mu_3(s)\,B(s) \tag{3.3}$$

for all $s \in I$ where $\mu_1(s), \mu_2(s)$ and $\mu_3(s)$ are differentiable functions on I.

(i) Including $|\kappa_2| < |\kappa_1|$, let $\varphi^\star$ be a spacelike curve with timelike principal normal. Then differentiating (3.3) with respect to s and using the (2.1) Frenet equations, we obtain

$$T^\star f' = (1 + \mu_1' + \mu_2\kappa_1) T + (\mu_1\kappa_1 + \mu_2' - \mu_3\kappa_2) N + (\mu_3' + \mu_2\kappa_2) B. \tag{3.4}$$

By taking the scalar product of (3.4) with N, we find

$$\mu_1\kappa_1 + \mu_2' - \mu_3\kappa_2 = 0. \tag{3.5}$$

If we write instead of (3.5) in (3.4), we obtain

$$T^\star f' = \left(1 + \mu_1' + \mu_2\kappa_1\right) T + \left(\mu_3' + \mu_2\kappa_2\right) B \tag{3.6}$$

By taking the scalar product of (3.6) with itself, we obtain

$$\left(f'\right)^2 = -\left(1 + \mu_1' + \mu_2\kappa_1\right)^2 + \left(\mu_3' + \mu_2\kappa_2\right)^2 \tag{3.7}$$

If we write instead

$$\delta = \frac{1 + \mu_1' + \mu_2\kappa_1}{f'} \quad , \; \gamma = \frac{\mu_3' + \mu_2\kappa_2}{f'} \tag{3.8}$$

we get

$$T^\star = \delta T + \gamma B \tag{3.9}$$

Differentiating (3.9) with respect to s and using the (2.1) Frenet equations , we find

$$-f' \kappa_1^\star N^\star = \delta' T + (\delta\kappa_1 - \gamma\kappa_2) N + \gamma' B \tag{3.10}$$

By taking the scalar product of (3.10) with N, we get

$$\delta\kappa_1 - \gamma\kappa_2 = 0 \tag{3.11}$$

Substituting δ and γ, we get

$$(1 + \mu_1' + \mu_2\kappa_1)\kappa_1 = \left(\mu_3' + \mu_2\kappa_2\right)\kappa_2 \tag{3.12}$$

where $\mu_3' + \mu_2\kappa_2 \neq 0$. Otherwise, $T^\star$and T are linearly dependent because of (3.6). If $T^\star$and T are linearly dependent, then the curves are equivalent curves and there is a parallel frame. So $N^\star$ and N, $B^\star$and B are parallel. In this case it conflicts with $N = B^\star$. So the curve cannot be Mannheim.

Conversely, let $|\kappa_2| < |\kappa_1|$ and φ be a timelike curve with parametrized by arc-length s with non-zero curvatures κ_1, κ_2. Firstly assume that φ satisfies the conditions of (3.2) for differentiable functions μ_1, μ_2 and μ_3. Then we can define a curve $\varphi^\star$ as

$$\varphi^\star(s^\star) = \varphi^\star(f(s)) = \varphi(s) + \mu_1(s) T(s) + \mu_2(s) N(s) + \mu_3(s) B(s). \tag{3.13}$$

Differentiating (3.13) with respect to s, we find

$$\frac{d\varphi^{\star}}{ds^{\star}} = \left(1 + \mu_1^{'} + \mu_2\kappa_1\right) T + \left(\mu_3^{'} + \mu_2\kappa_2\right) B. \tag{3.14}$$

From (3.14), we have

$$f^{'} = \left\| \frac{d\varphi^{\star}}{ds^{\star}} \right\| = \sqrt{\left\langle \frac{d\varphi^{\star}}{ds^{\star}}, \frac{d\varphi^{\star}}{ds^{\star}} \right\rangle} = \frac{m_1 \left(\mu_3^{'} + \mu_2\kappa_2\right) \sqrt{\kappa_1^2 - \kappa_2^2}}{\kappa_1} \tag{3.15}$$

where $m_1 = sgn\left(\mu_3^{'} + \mu_2 k_2\right)$. If we rewrite (3.14), we find

$$T^{\star} = \frac{m_1}{\sqrt{\kappa_1^2 - \kappa_2^2}} \left(\kappa_2 T + \kappa_1 B\right) \quad , g\left(T^{\star}, T^{\star}\right) = 1. \tag{3.16}$$

Differentiating (3.16) with respect to s, we get

$$\kappa_1^{\star} = \left\| \frac{dT^{\star}}{ds^{\star}} \right\| = \frac{m_2 \left(\kappa_1\kappa_2' - \kappa_2\kappa_1'\right)}{f' \left(\kappa_1^2 - \kappa_2^2\right)} = \frac{m_2\kappa_1^2 \left(\frac{\kappa_2}{\kappa_1}\right)'}{f' \left(\kappa_1^2 - \kappa_2^2\right)} \tag{3.17}$$

where $m_2 = sgn\left(\frac{\kappa_1\kappa_2' - \kappa_2\kappa_1'}{\kappa_1^2 - \kappa_2^2}\right)$.Now, we can find $N^{\star}$ as

$$N^{\star} = \frac{-m_1 m_2}{\sqrt{\kappa_1^2 - \kappa_2^2}} \left(\kappa_1 T + \kappa_2 B\right) \quad g\left(N^{\star}, N^{\star}\right) = -1. \tag{3.18}$$

From here we find

$$B^{\star} = T^{\star} \times N^{\star} = -m_2 N, \quad g\left(B^{\star}, B^{\star}\right) = 1 \tag{3.19}$$

Lastly, we define $\kappa_2^{\star}$ as

$$\kappa_2^{\star} = -g\left(\frac{dB^{\star}}{ds^{\star}}, N^{\star}\right) = \frac{m_1\sqrt{\kappa_1^2 - \kappa_2^2}}{f'} \neq 0 \tag{3.20}$$

Then $\varphi^{\star}$ is a spacelike curve with timelike principal normal and a Mannheim mate curve of φ. Thus φ is a Mannheim curve.

(ii) Including $|\kappa_1| < |\kappa_2|$, let $\varphi^{\star}$ be a timelike curve with spacelike principal normal. Then differentiating (3.3) with respect to s and using the (2.1) Frenet equations, we obtain

$$T^{\star} f' = \left(1 + \mu_1' + \mu_2\kappa_1\right) T + \left(\mu_1\kappa_1 + \mu_2' - \mu_3\kappa_2\right) N + \left(\mu_3' + \mu_2\kappa_2\right) B. \tag{3.21}$$

By taking the scalar product of (3.21) with N, we find

$$\mu_1\kappa_1 + \mu_2' - \mu_3\kappa_2 = 0. \tag{3.22}$$

If we write instead of (3.22) in (3.21), we obtain

$$T^{\star} f^{'} = \left(1 + \mu_1^{'} + \mu_2\kappa_1\right) T + \left(\mu_3^{'} + \mu_2\kappa_2\right) B \tag{3.23}$$

By taking the scalar product of (3.23) with itself, we obtain

$$-\left(f^{'}\right)^{2}=-\left(1+\mu_{1}^{'}+\mu_{2}\kappa_{1}\right)^{2}+\left(\mu_{3}^{'}+\mu_{2}\kappa_{2}\right)^{2} \tag{3.24}$$

If we write instead

$$\delta=\frac{1+\mu_{1}^{'}+\mu_{2}\kappa_{1}}{f^{'}}\quad,\ \gamma=\frac{\mu_{3}^{'}+\mu_{2}\kappa_{2}}{f^{'}} \tag{3.25}$$

we get

$$T^{\star}=\delta T+\gamma B \tag{3.26}$$

Differentiating (3.26) with respect to s and using the (2.1) Frenet equations , we find

$$f^{'}\kappa_{1}^{\star}N^{\star}=\delta^{'}T+\left(\delta\kappa_{1}-\gamma\kappa_{2}\right)N+\gamma^{'}B \tag{3.27}$$

By taking the scalar product of (3.27) with N, we get

$$\delta\kappa_{1}-\gamma\kappa_{2}=0 \tag{3.28}$$

Substituting δ and γ, we get

$$(1+\mu_{1}^{'}+\mu_{2}\kappa_{1})\kappa_{1}=\left(\mu_{3}^{'}+\mu_{2}\kappa_{2}\right)\kappa_{2} \tag{3.29}$$

where $\mu_{3}^{'}+\mu_{2}\kappa_{2}\neq 0$. Otherwise, $T^{\star}$and T are linearly dependent because of (3.23). If $T^{\star}$and T are linearly dependent, then the curves are equivalent curves and there is a parallel frame. So $N^{\star}$and N, $B^{\star}$and B are parallel. In this case it conflicts with $N=B^{\star}$. So the curve cannot be Mannheim.

Conversely, let $|\kappa_{1}|<|\kappa_{2}|$ and φ be a timelike curve with parametrized by arc-length s with non-zero curvatures κ_{1},κ_{2}. Firstly assume that φ satisfies the conditions of (3.2) for differentiable functions μ_{1},μ_{2} and μ_{3}. Then we can define a curve $\varphi^{\star}$ as

$$\varphi^{\star}\left(s^{\star}\right)=\varphi^{\star}\left(f\left(s\right)\right)=\varphi\left(s\right)+\mu_{1}\left(s\right)T\left(s\right)+\mu_{2}\left(s\right)N\left(s\right)+\mu_{3}\left(s\right)B\left(s\right). \tag{3.30}$$

Differentiating (3.30) with respect to s, we find

$$\frac{d\varphi^{\star}}{ds^{\star}}=\left(1+\mu_{1}^{'}+\mu_{2}\kappa_{1}\right)T+\left(\mu_{3}^{'}+\mu_{2}\kappa_{2}\right)B. \tag{3.31}$$

From (3.31), we have

$$f^{'}=\left\|\frac{d\varphi^{\star}}{ds^{\star}}\right\|=\sqrt{\left\langle\frac{d\varphi^{\star}}{ds^{\star}},\frac{d\varphi^{\star}}{ds^{\star}}\right\rangle}=\frac{m_{1}\left(\mu_{3}^{'}+\mu_{2}\kappa_{2}\right)\sqrt{\kappa_{2}^{2}-\kappa_{1}^{2}}}{\kappa_{1}} \tag{3.32}$$

where $m_{1}=sgn\left(\mu_{3}^{'}+\mu_{2}\kappa_{2}\right)$. If we rewrite (3.31), we find

$$T^{\star}=\frac{m_{1}}{\sqrt{\kappa_{2}^{2}-\kappa_{1}^{2}}}\left(\kappa_{2}T+\kappa_{1}B\right)\quad,g\left(T^{\star},T^{\star}\right)=-1. \tag{3.33}$$

Differentiating (3.33) with respect to s, we get

$$\kappa_1^\star = \left\| \frac{dT^\star}{ds^\star} \right\| = \frac{m_2 \left(\kappa_2 \kappa_1' - \kappa_1 \kappa_2'\right)}{f' \left(\kappa_2^2 - \kappa_1^2\right)} = \frac{-m_2 \kappa_1^2 \left(\frac{\kappa_2}{\kappa_1}\right)'}{f' \left(\kappa_2^2 - \kappa_1^2\right)} \tag{3.34}$$

where $m_2 = sgn\left(\frac{\kappa_2\kappa_1' - \kappa_1\kappa_2'}{\kappa_2^2 - \kappa_1^2}\right)$. Now, we can find $N^\star$ as

$$N^\star = \frac{m_1 m_2}{\sqrt{\kappa_2^2 - \kappa_1^2}} \left(\kappa_1 T + \kappa_2 B\right) \quad g\left(N^\star, N^\star\right) = 1. \tag{3.35}$$

From here we find

$$B^\star = T^\star \times N^\star = -m_2 N, \quad g\left(B^\star, B^\star\right) = 1 \tag{3.36}$$

Lastly, we define $\kappa_2^\star$ as

$$\kappa_2^\star = -g\left(\frac{dB^\star}{ds^\star}, N^\star\right) = \frac{m_1 \sqrt{\kappa_2^2 - \kappa_1^2}}{f'} \neq 0 \tag{3.37}$$

Then $\varphi^\star$ is a timelike curve with spacelike principal normal and a Mannheim mate curve of φ. Thus φ is a Mannheim curve. Also, there exists a homothety map between φ and $\varphi^\star$. Q.E.D.

From the theorem we have proved above, the following results can be obtained directly. Therefore, they are given without proof.

Corollary 3.3. If we take $\mu_1 = \mu_3 = 0$ in Theorem 3.2, we get the conditions of classical Mannheim curves in the literature with the Mannheim partner curve $\varphi^\star$ given by

$$\varphi^\star(s^\star) = \varphi^\star(f(s)) = \varphi(s) + \mu_2(s) N(s).$$

Corollary 3.4. Let $\varphi : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a unit speed timelike curve with the non-zero curvatures κ_1, κ_2 and $\kappa_1 \neq \kappa_2$. Then the curve φ is a Mannheim curve with Mannheim partner $\varphi^\star$ given by

$$\varphi^\star(s^\star) = \varphi^\star(f(s)) = \varphi(s) + \mu_2(s) N(s)$$

if and only if there exist real number μ_2 satisfying $\kappa_1 = \mu_2\left(\kappa_1^2 - \kappa_2^2\right)$.

Corollary 3.5. Let $\varphi : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a timelike general helix with the non-zero curvatures κ_1, κ_2 and $\kappa_1 \neq \kappa_2$. Then φ does not have the Mannheim partner curve liying fully in $\mathbb{E}_1^3$ (or the Mannheim partner curve is a straight line).

Corollary 3.6. Let $\varphi : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a timelike Mannheim curve with the curvatures κ_1, κ_2 and $\kappa_1 \neq \kappa_2$ and the curve $\varphi^\star$ be a Mannheim partner curve of φ with the curvatures $\kappa_1^\star, \kappa_2^\star$. Then $\varphi^\star$ is a general helix if and only if φ is a slant helix.

4 Examples

In this section, we will give examples of Mannheim curves constructed with the help of theorems that we proved in the previous section.

Example 4.1. Let us assess a timelike curve in $\mathbb{E}_1^3$ with the equation

$$\varphi(s) = \left(\frac{s}{\sqrt{1-s^2}}, -\frac{1}{2}\sqrt{3-3s^2}, \frac{1}{2}\sqrt{1-s^2}\cosh(2\arctan h(s)) \right)$$

with the curvatures

$$\begin{aligned} \kappa_1 &= \frac{\sqrt{3}}{(1-s^2)^{\frac{3}{2}}}, \\ \kappa_2 &= \frac{\sqrt{3}s}{(1-s^2)^{\frac{3}{2}}}. \end{aligned}$$

It is easily check that φ is a timelike rectifying slant helix (see [2]) and the Frenet frame as

$$\begin{aligned} T(s) &= \left(\frac{1}{(1-s^2)^{\frac{3}{2}}}, \frac{3s}{2(3-3s^2)^{\frac{1}{2}}}, -\frac{s(-3+s^2)}{2(1-s^2)^{\frac{3}{2}}} \right), \\ N(s) &= \left(\frac{\sqrt{3}s}{1-s^2}, \frac{1}{2}, -\frac{\sqrt{3}(1+s^2)}{2(-1+s^2)} \right), \\ B(s) &= \left(-\frac{s^3}{(1-s^2)^{\frac{3}{2}}}, -\frac{3}{2(3-3s^2)^{\frac{1}{2}}}, \frac{1-3s^2}{2(1-s^2)^{\frac{3}{2}}} \right). \end{aligned}$$

If we take

$$\begin{aligned} \mu_1 &= \frac{-1-4s^2+2s^4-s\sqrt{1-s^2}}{\sqrt{1-s^2}}, \\ \mu_2 &= \sqrt{3}s, \\ \mu_3 &= -\frac{6s-3s^3+\sqrt{1-s^2}}{\sqrt{1-s^2}} \end{aligned}$$

in (i) of Theorem (3.2), then we get the curve $\varphi^\star$ as follows;

$$\varphi^\star(s^\star) = \varphi^\star(f(s)) = \left(-1-3s^2, -\sqrt{3}s\left(-3+s^2\right), -s\left(3+s^2\right) \right).$$

With correct calculations, we get

$$\begin{aligned} T^\star(s^\star) &= \left(-\frac{s}{-1+s^2}, -\frac{\sqrt{3}}{2}, \frac{1+s^2}{2-2s^2} \right), \\ N^\star(s^\star) &= \left(\frac{1+s^2}{1-s^2}, 0, -\frac{2s}{-1+s^2} \right), \\ B^\star(s^\star) &= \left(-\frac{\sqrt{3}s}{1-s^2}, -\frac{1}{2}, \frac{\sqrt{3}(1+s^2)}{2(-1+s^2)} \right). \end{aligned}$$

and

$$\begin{aligned}\kappa_1^\star &= \frac{1}{6\left(-1+s^2\right)^2},\\ \kappa_2^\star &= \frac{1}{2\sqrt{3}\left(-1+s^2\right)^2}.\end{aligned}$$

It can be easily seen that the curve $\varphi^\star$ is a spacelike Mannheim mate curve of the curve φ.

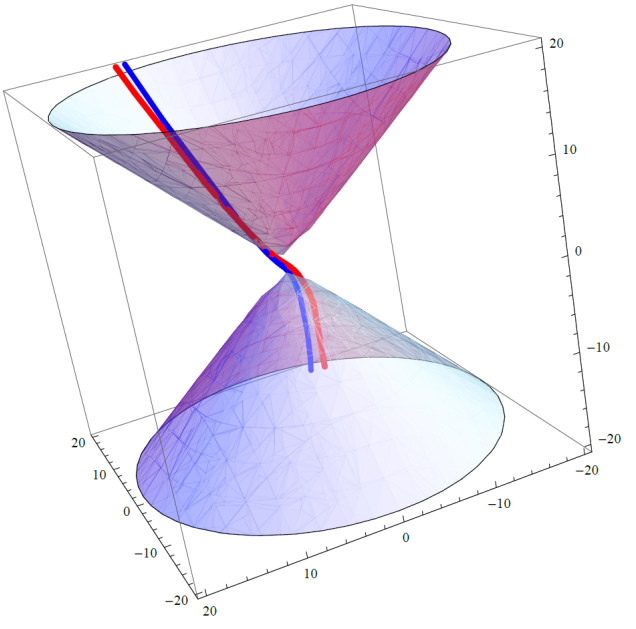


Figure 1: The red graphic is φ Mannheim curve (timelike rectifying slant helix) and the blue graphic is $\varphi^\star$ Mannheim mate curve (general helix) in Example 4.1.

Example 4.2. Let us assess a timelike curve in $\mathbb{E}_1^3$ with the equation

$$\varphi(s) = \left(\frac{9}{400}\sinh(25s) + \frac{25}{144}\sinh(9s), \frac{9}{400}\cosh(25s) - \frac{25}{144}\cosh(9s), \frac{15}{136}\sinh(17s)\right)$$

with the curvatures

$$\begin{aligned}\kappa_1 &= 15\sinh(17s),\\ \kappa_2 &= 15\cosh(17s)\end{aligned}$$

and the Frenet frame as

$$\begin{aligned} T(s) &= \left(\frac{25}{16}\cosh(9s)+\frac{9}{16}\cosh(25s),-\frac{25}{16}\sinh(9s)+\frac{9}{16}\sinh(25s),\frac{15}{8}\cosh(17s)\right), \\ N(s) &= \left(\frac{15}{8}\cosh(8s),\frac{15}{4}\left(\cosh(s)+\cosh(3s)+\cosh(5s)+\cosh(7s)\right)\sinh(s),\frac{17}{8}\right), \\ B(s) &= \left(-\frac{25}{16}\sinh(9s)-\frac{9}{16}\sinh(25s),\frac{25}{16}\cosh(9s)-\frac{9}{16}\cosh(25s),-\frac{15}{8}\sinh(17s)\right). \end{aligned}$$

If we take

$$\begin{aligned} \mu_1 &= \frac{1}{17}\cosh(17s)\left(\sinh(17s)-15\right), \\ \mu_2 &= 1, \\ \mu_3 &= \frac{1}{17}\sinh(17s)\left(\sinh(17s)-15\right) \end{aligned}$$

in (ii) of (3.2), then we get the curve $\varphi^\star$ as follows;

$$\varphi^\star(s^\star)=\varphi^\star(f(s))=\begin{pmatrix} \frac{17(25\sinh(9s)+9\sinh(25s))}{1800}, \\ \frac{17}{225}(\cosh(s))^3(-226+444\cosh(2s)-420\cosh(4s)+380\cosh(6s) \\ -324\cosh(8s)+252\cosh(10s)-189\cosh(12s)+135\cosh(14s) \\ -90\cosh(16s)+54\cosh(18s)-27\cosh(20s)+9\cosh(22s)), \\ \frac{1}{68}(32+15\sinh(17s)) \end{pmatrix}.$$

With correct calculations, we get

$$\begin{aligned} T^\star(s^\star) &= \left(\frac{17}{8}\cosh(8s),\frac{17}{4}\left(-\sinh(s)+\sinh(3s)-\sinh(5s)+\sinh(7s)\right)\cosh(s),\frac{15}{8}\right), \\ N^\star(s^\star) &= \left(\sinh(8s),\cosh(8s),0\right), \\ B^\star(s^\star) &= \left(-\frac{15}{8}\cosh(8s),-\frac{15}{8}\sinh(8s),-\frac{17}{8}\right) \end{aligned}$$

and

$$\begin{aligned} \kappa_1^\star &= \frac{17}{2}\sec h(17s), \\ \kappa_2^\star &= \frac{15}{2}\sec h(17s). \end{aligned}$$

It can be easily seen that the curve $\varphi^\star$ is a timelike Mannheim mate curve of the curve φ.

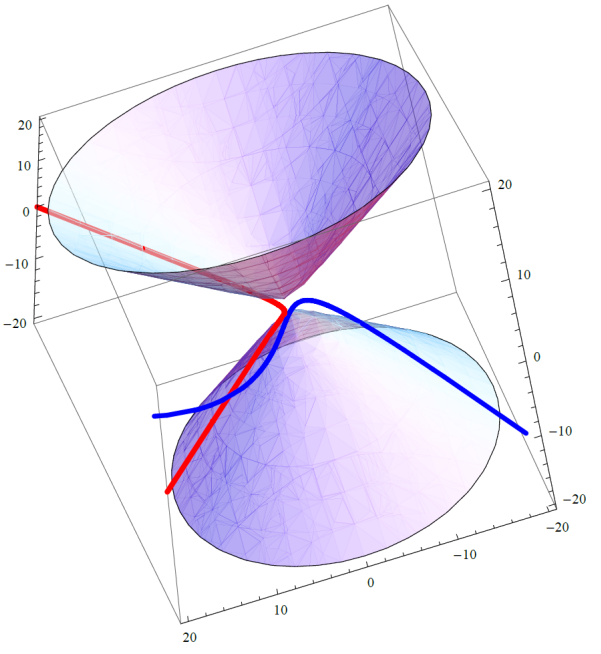


Figure 2: The red graphic is φ Mannheim curve (timelike slant helix) and the blue graphic is $\varphi^{\star}$ Mannheim mate curve (general helix) in Example 4.2.

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