New results on Cartan null and pseudo null Bertrand curves in Minkowski 3-space

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Abstract

In this paper, we give new results for Cartan null and pseudo null Bertrand curves in Minkowski 3-space by using a new approach which generalize the notion of Bertrand curves in Euclidean 3-space. According to the this approach, the necessary and sufficient conditions have been obtained for Cartan null and pseudo null curves to be Bertrand curves in Minkowski 3-space. In addition, some examples are given.

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1 Introduction

As a result of the relations between the Frenet vectors of space curve pairs, important curve pairs emerge in terms of differential geometry. In searching for pairs of curves such that the tangents, principal normals or binormals of one may be the tangents, principal normals or binormals of the other, there are six cases to be considered. Undoubtedly, Bertrand curves or in other words Bertrand curve pairs are among these curve pairs and have been the subject of many studies [20].

In 1845, Saint-Venant proposed [16], the question whether upon the ruled surface generated by the principal normals of a curve in the 3-dimensional Euclidean space $\mathbb{E}^3$, a second curve can exist having the same principal normals. This question was answered by Bertrand in the paper published in 1850 [2]. After the paper of Bertrand, the pairs of curves like this have been called Conjugate Bertrand Curves, or more commonly Bertrand Curves [18]. Bertrand Curves have been extensively studied and continue to be studied in different dimensional spaces, especially the 3-dimensional Euclidean space. Bertrand offset surfaces obtained with the help of these curves form another working area (see [12, 17]).

The results obtained in the theory of curves are carried to spaces with different metrics. At the beginning of these spaces are 3-dimensional Minkowski space and Minkowski space-time. The differential geometric properties of curves in the Minkowski space (or more generally in semi-Riemannian manifold) have been extensively studied by both mathematicians and theoretical physicists. In Minkowski 3-space, there exist three families of curves, that is, spacelike, timelike, and null (or lightlike) curves, according to their causal characters of its velocity vector. It is well-known that the study of timelike curves has many analogies and similarities with that of spacelike curves [21]. However, the fact that the induced metric on a null curve is degenerate leads to a much more complicated study and also different from the non-degenerate case. It is known that the Minkowski space can be represented by the 2D- complex plane, where the Lorentz transformation represents a rotation of four-vector about the origin of the Minkowski space [8]. In the Minkowski complex plane, the vertical y-axis is the imaginary axis. That is, imaginary number i represents a

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90° counterclockwise rotation from horizontal x-axis of the complex plane. In the 2D-Minkowski complex plane, the three-vector spatial component magnitude of the complex four-vector is usually projected onto the horizontal x-axis, and the single temporal component magnitude of the complex four-vector is projected onto the vertical y-axis.

A pseudo null curve is a spacelike curve with the constraint that its acceleration vector is null ([3], [6]). They also are of strong interest in mathematics and encompass a rich source of geometrical problems ([3, 9]). Null curves and pseudo null curves are intensively studied in different fields with the help of developments in the construction of Frenet equations of null curves in different dimension and different indexes ([6, 10, 11, 14, 15]). Null Bertrand curves in Minkowski 3-space were first studied by Balgetir et al in [1] . In this study, it has been proven that the necessary and sufficient condition for a null curve to be a Bertrand curve is that the second curvature of the curve is constant. In this study, both the main curve and the Bertrand mate curve are treated as Cartan null curves. Also Null Bertrand curves with a timelike Bertrand mates and a spacelike Bertrand mates with spacelike princinpal normal are studied in [11]. The curve related by transformation of Combescure are also the examples of Bertrand curves [4] .

Ç. Camcı, et al. gave important results and examples for Bertrand curves by giving a new approach to Bertrand curves in 3-dimensional Euclidean space ([5]).

In this study, Bertrand curvature characterizations of Cartan null and pseudo null curves were examined according to this approach and necessary theorems were obtained. The results are supported with examples and the graphics of these examples are given.

2 Preliminaries

Minkowski space $\mathbb{E}_1^3$ is a three-dimensional affine space endowed with an indefinite flat metric $\langle\ ,\ \rangle$ with signature $(-,+,+)$. This means that metric bilinear form can be written as

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3,$$

for ant two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{E}_1^3$. Recall that a vector $v \in \mathbb{E}_1^3\backslash\{0\}$ can be *spacelike* if $g(v, v) > 0$, *timelike* if $g(v, v) < 0$ and *null (lightlike)* if $g(v, v) = 0$ and $v \neq 0$. In particular, the vector $v = 0$ is a spacelike. The norm of a vector v is given by $||v|| = \sqrt{|g(v, v)|}$, and two vectors v and w are said to be orthogonal, if $g(v, w) = 0$. An arbitrary curve $\varphi(s)$ in $\mathbb{E}_1^3$, can locally be *spacelike*, *timelike* or *null* (*lightlike*), if all its velocity vectors $\varphi'(s)$ are respectively spacelike, timelike or null. A null curve φ is parameterized by pseudo-arc s if $g(\varphi''(s), \varphi''(s)) = 1$. A spacelike or a timelike curve $\varphi(s)$ has unit speed, if $g(\varphi'(s), \varphi'(s)) = \pm 1$ [9, 22].

Let $\{T, N, B\}$ be the moving Frenet frame along a curve φ in $\mathbb{E}_1^3$, consisting of the tangent, the principal normal and the binormal vector fields respectively. Depending on the causal character of φ, the Frenet equations have the following forms. For Cartan and Frenet frames of curves in Minkowski 3-space we can refer to works [3, 6, 9, 22]

Case I. If φ is a non-null curve, the Frenet equations are given by:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2 k_1 & 0 \\ -\varepsilon_1 k_1 & 0 & \varepsilon_3 k_2 \\ 0 & -\varepsilon_2 k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.1}$$

where k_1 and k_2 are the first and the second curvature of the curve respectively. Moreover, the following conditions hold:

$$g(T,T)=\varepsilon_1=\pm 1, g(N,N)=\varepsilon_2 \ =\pm 1, g(B,B)=\varepsilon_3=\pm 1$$

and

$$g(T,N)=g(T,B)=g(N,B)=0.$$

Case II. If φ is a null curve, the Frenet equations are given by

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_2 & 0 & -k_1 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.2}$$

where the first curvature $k_1 = 0$ if φ is straight line, or $k_1 = 1$ in all other cases. In particular, the following conditions hold:

$$g(T,T)=g(B,B)=g(T,N)=g(N,B)=0, g(N,N)=g(T,B)=1.$$

Case III. If φ is a pseudo null curve, the Frenet equations are given by

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ 0 & k_2 & 0 \\ -k_1 & 0 & -k_2 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.3}$$

where the first curvature $k_1 = 0$ if φ is straight line, or $k_1 = 1$ in all other cases. In particular, the following conditions hold:

$$g(N,N)=g(B,B)=g(T,N)=g(T,B)=0, g(T,T)=g(N,B)=1.$$

3 New results on Cartan null and pseudo null Bertrand curves in Minkowski 3-space

In this section, we will reassess the Bertrand curves in Minkowski 3-space $\mathbb{E}_1^3$. In the following definition, we give the well-known definition of Bertrand curve in $\mathbb{E}_1^3$.

Definition 3.1. A curve $\varphi : I \to \mathbb{E}_1^3$ with non-zero curvatures is a Bertrand curve if there is a curve $\varphi^\star : I^\star \to \mathbb{E}_1^3$ and and a diffeomorphism $f : I \to I^\star$ such that the principal normal vectors of $\varphi(s)$ and $\varphi^*(s)$ at $s \in I$, $f(s) = s^* \in I^*$ coincide. In this situation, $\varphi^\star(s^\star)$ is called the Bertrand mate of $\varphi(s)$.

Let φ be a Cartan null Bertrand curve with the Cartan frame $\{T(s), N(s), B(s)\}$ and non-zero curvatures $k_1(s), k_2(s)$, and $\varphi^\star$ be a Bertrand mate curve of φ. Then $\varphi^\star(s)$ can be written as

$$\varphi^\star\left(s^\star\right)=\varphi^\star\left(f\left(s\right)\right)=\varphi\left(s\right)+\mu_1\left(s\right)T\left(s\right)+\mu_2\left(s\right)N\left(s\right)+\mu_3\left(s\right)B\left(s\right)$$

where $\mu_1\left(s\right), \mu_2\left(s\right)$ and $\mu_3\left(s\right)$ are differentiable function on I.

Since the principal normal vector of the Cartan null curve $\varphi(s)$ is spacelike, the Bertrand mate curve $\varphi^\star(s^\star)$ can be a timelike curve, a spacelike curve with spacelike principal normal or a Cartan null curve. We will assess all situations severally in the following theorem.

Theorem 3.2. Let $\varphi : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a unit speed Cartan null curve with spacelike principal normal with the non-zero curvatures $k_1(s) = 1, k_2(s)$. Then the curve $\varphi(s)$ is a Bertrand curve with Bertrand mate curve φ^* if and only if one of the following conditions holds:
(i) there exist differentiable functions μ_1, μ_2, μ_3 and real number λ such that the next system of equations hold:

$$\begin{aligned} \mu_2^{'} + \mu_1 &= \mu_3 k_2 \\ \mu_3^{'} - \mu_2 &= \lambda f^{'} \\ -2(1 + \mu_1^{'} + \mu_2 k_2)(\mu_3^{'} - \mu_2) &= (f^{'})^2 \end{aligned} \tag{3.1}$$

In this situation, the Bertrand mate curve $\varphi^\star$ is a timelike curve.
(ii) there exist differentiable functions μ_1, μ_2, μ_3 and real number λ such that the next system of equations hold:

$$\begin{aligned} \mu_2^{'} + \mu_1 &= \mu_3 k_2 \\ \mu_3^{'} - \mu_2 &= \lambda f^{'} \\ 2(1 + \mu_1^{'} + \mu_2 k_2)(\mu_3^{'} - \mu_2) &= (f^{'})^2 \end{aligned} \tag{3.2}$$

In this situation, the Bertrand mate curve $\varphi^\star$ is a spacelike curve with spacelike principal normal.
(iii) there exist differentiable functions μ_1, μ_2, μ_3 such that the next system of equations hold:

$$\begin{aligned} \mu_2^{'} + \mu_1 &= \mu_3 k_2 \\ \mu_3^{'} - \mu_2 &= 0 \\ 1 + \mu_1^{'} + \mu_2 k_2 &\neq 0 \end{aligned} \tag{3.3}$$

or there exist differentiable functions μ_1, μ_2, μ_3 such that the next system of equations hold:

$$\begin{aligned} \mu_2^{'} + \mu_1 &= \mu_3 k_2 \\ \mu_3^{'} - \mu_2 &\neq 0 \\ 1 + \mu_1^{'} + \mu_2 k_2 &= 0 \end{aligned} \tag{3.4}$$

In this situation, the Bertrand mate curve $\varphi^\star$ is a Cartan null curve.

Proof. Assume that φ is a Cartan null Bertrand curve parametrized by arc-length s with non-zero curvatures $k_1 = 1, k_2$ and the curve $\varphi^\star$ is the Bertrand mate curve of φ parametrized by with arc-length or pseudo arc $s^\star$. Then, we can write the curve $\varphi^\star$ as

$$\varphi^\star(s^\star) = \varphi^\star(f(s)) = \varphi(s) + \mu_1(s) T(s) + \mu_2(s) N(s) + \mu_3(s) B(s) \tag{3.5}$$

for all $s \in I$ where $\mu_1(s), \mu_2(s)$ and $\mu_3(s)$ are differentiable functions on I.

(i) Let $\varphi^\star$ be a timelike curve. Then differentiating (3.5) with respect to s and using the (2.1) and (2.2) Frenet equations, we obtain

$$f^{'} T^\star = \left(1 + \mu_1^{'} + \mu_2 k_2\right) T + \left(\mu_2^{'} + \mu_1 - \mu_3 k_2\right) N + \left(\mu_3^{'} - \mu_2\right) B, \tag{3.6}$$

by taking the scalar product of (3.6) with N, we find

$$\mu_3 k_2 = \mu_1 + \mu_2^{'}. \tag{3.7}$$

If we write instead of (3.7) in (3.6), we obtain

$$f^{'}T^{\star}=\left(1+\mu_{1}^{'}+\mu_{2}k_{2}\right)T+\left(\mu_{3}^{'}-\mu_{2}\right)B. \tag{3.8}$$

By taking the scalar product of (3.8) with itself, we obtain

$$\left(f^{'}\right)^{2}=-2\left(1+\mu_{1}^{'}+\mu_{2}k_{2}\right)\left(\mu_{3}^{'}-\mu_{2}\right), \tag{3.9}$$

if we write instead

$$1+\mu_{1}^{'}+\mu_{2}k_{2}=-\frac{\left(f^{'}\right)^{2}}{2\left(\mu_{3}^{'}-\mu_{2}\right)}\quad\text{and}\quad\lambda=\frac{\mu_{3}^{'}-\mu_{2}}{f^{'}}, \tag{3.10}$$

we get

$$T^{\star}=-\frac{1}{2\lambda}T+\lambda B. \tag{3.11}$$

Differentiating (3.11) with respect to s and using the (2.1) and (2.2) Frenet equations , we find

$$f^{'}k_{1}^{\star}N^{\star}=\left(-\frac{1}{2\lambda}-\lambda k_{2}\right)N, \tag{3.12}$$

by taking the scalar product of (3.12) with itself, we have

$$\left(f^{'}\right)^{2}\left(k_{1}^{\star}\right)^{2}=\left(-\frac{1}{2\lambda}-\lambda k_{2}\right)^{2}. \tag{3.13}$$

Then we have from (3.13),

$$k_{1}^{\star}=m_{1}\frac{-1-2\lambda^{2}k_{2}}{2\lambda f^{'}}, \tag{3.14}$$

where $m_{1}=sgn(-1-2\lambda^{2}k_{2})$.

If we write instead of (3.14) in (3.12), we obtain

$$N^{\star}=m_{1}N. \tag{3.15}$$

Differentiating (3.15) with respect to s and using (2.1) and (2.2) Frenet equations, we find

$$f'k_{1}^{\star}T^{\star}+f'k_{2}^{\star}B^{\star}=m_{1}\left(k_{2}T-B\right), \tag{3.16}$$

by taking the scalar product of (3.16) with itself and using (3.14), we get

$$k_{2}^{\star}=m_{2}\frac{1-2\lambda^{2}k_{2}}{2\lambda f^{'}}, \tag{3.17}$$

where $m_{2}=sgn(1-2\lambda^{2}k_{2})$.

Conversely, let φ be a Cartan null curve parametrized by arc-length s with non-zero curvatures $k_{1}=1,k_{2}$. Firstly assume that φ satisfies the conditions of (3.1) for differentiable functions μ_{1},μ_{2} and μ_{3}. Then we can define a curve $\varphi^{\star}$ as

$$\varphi^{\star}\left(s^{\star}\right)=\varphi^{\star}\left(f\left(s\right)\right)=\varphi\left(s\right)+\mu_{1}\left(s\right)T\left(s\right)+\mu_{2}\left(s\right)N\left(s\right)+\mu_{3}\left(s\right)B\left(s\right). \tag{3.18}$$

Differentiating (3.18) with respect to s, we find

$$\frac{d\varphi^\star}{ds^\star} = \left(1 + \mu_1^{'} + \mu_2 k_2\right) T + \left(\mu_3^{'} - \mu_2\right) B. \tag{3.19}$$

From (3.19), we have

$$f^{'} = \left\| \frac{d^2\varphi^\star}{ds^{\star 2}} \right\| = \sqrt{n_1 \left(1 + \mu_1^{'} + 2\mu_2 k_2 - \mu_3^{'} k_2\right)}, \tag{3.20}$$

where $n_1 = sgn\left(1 + \mu_1^{'} + 2\mu_2 k_2 - \mu_3^{'} k_2\right)$. Rewriting (3.19) , we obtain

$$T^\star = \frac{n_2}{2\lambda}\left(-T + 2\lambda^2 B\right) \quad , g\left(T^\star, T^\star\right) = -1, \tag{3.21}$$

where $n_2 = sgn(\lambda)$.

Differentiating (3.21) with respect to s, we get

$$\frac{dT^\star}{ds^\star} = \frac{n_2\left(-1 - 2\lambda^2 k_2\right)}{2\lambda f^{'}} N, \tag{3.22}$$

which causes that

$$k_1^\star = \left\| \frac{dT^\star}{ds^\star} \right\| = \frac{n_3\left(-1 - 2\lambda^2 k_2\right)}{2\lambda f^{'}}, \tag{3.23}$$

where $n_3 = sgn\left(-1 - 2\lambda^2 k_2\right)$. Now, we can find $N^\star$ as

$$N^\star = n_2 n_3 N, \quad g\left(N^\star, N^\star\right) = 1. \tag{3.24}$$

Differentiating (3.24) with respect to s, using (3.21) and (3.22), we get

$$\frac{dN^\star}{ds^\star} = \frac{n_2 n_3}{f^{'}}\left(k_2 T - B\right). \tag{3.25}$$

Lastly, we define $B^\star$ as

$$B^\star = \frac{n_3}{2\lambda}\left(T + 2\lambda^2 B\right) \qquad , g\left(B^\star, B^\star\right) = 1, \tag{3.26}$$

which bring about that

$$k_2^\star = -g(\frac{dN^\star}{ds^\star}, B^\star) = \frac{n_2\left(1 - 2\lambda^2 k_2\right)}{2\lambda f^{'}},$$

where $n_2 = sgn\left(1 - 2\lambda^2 k_2\right)$. Then $\varphi^\star$ is a timelike curve and a Bertrand mate curve of φ. Thus φ is a Bertrand curve.

(ii) Let $\varphi^\star$ be a spacelike curve . In this case, we omit the proof since it is similar to the case when $\varphi^\star$ is timelike.

(iii) Let $\varphi^\star$ be a Cartan null curve. Then differentiating (3.5) with respect to s and using the (2.2) Frenet equations, we get

$$f^{'} T^\star = \left(1 + \mu_1^{'} + \mu_2 k_2\right) T + \left(\mu_2^{'} + \mu_1 - \mu_3 k_2\right) N + \left(\mu_3^{'} - \mu_2\right) B. \tag{3.27}$$

By taking the scalar product of (3.27) with N, we have

$$\mu_3 k_2 = \mu_2^{'} + \mu_1. \tag{3.28}$$

Substituting (3.28) in (3.27), we find

$$f^{'} T^{\star} = \left(1 + \mu_1^{'} + \mu_2 k_2\right) T + \left(\mu_3^{'} - \mu_2\right) B. \tag{3.29}$$

By taking the scalar product of (3.29) with itself, we obtain

$$2\left(1 + \mu_1^{'} + \mu_2 k_2\right)\left(\mu_3^{'} - \mu_2\right) = 0, \tag{3.30}$$

so

$$1 + \mu_1^{'} + \mu_2 k_2 = 0 \quad \text{or} \quad \mu_3^{'} - \mu_2 \ = 0. \tag{3.31}$$

(i) Let $\mu_3^{'} - \mu_2 = 0$. If we write instead in (3.29), we get

$$f^{'} T^{\star} = \left(1 + \mu_1^{'} + \mu_2 k_2\right) T \quad \text{and} \quad 1 + \mu_1^{'} + \mu_2 k_2 \neq 0 \ . \tag{3.32}$$

Differentiating (3.32) with respect to s and using the (2.2) Frenet equations , we find

$$N^{\star} = \frac{1 + \mu_1^{'} + \mu_2 k_2}{\left(f^{'}\right)^2} N. \tag{3.33}$$

By taking the scalar product of (3.33) with itself, we have

$$\left(f^{'}\right)^2 = a_1\left(1 + \mu_1^{'} + \mu_2 k_2\right) \quad \text{and} \quad g(\frac{dT^{\star}}{ds^{\star}}, N^{\star}) = k_1^{\star} = 1, \tag{3.34}$$

where $a_1 = sgn\left(1 + \mu_1^{'} + \mu_2 k_2\right)$.
Substituting (3.34) in (3.33), we find

$$N^{\star} = a_1 a_2 N \quad , \quad g\left(N^{\star}, N^{\star}\right) = 1 \quad \text{and} \quad a_2 = sgn\left(a_1\left(1 + \mu_1^{'} + \mu_2 k_2\right)\right). \tag{3.35}$$

Differentiating (3.35) with respect to s and using the (2.2) Frenet equations , we find

$$k_2^{\star} T^{\star} - B^{\star} = \frac{a_1 a_2}{f^{'}}\left(k_2 T - B\right). \tag{3.36}$$

By taking the scalar product of (3.36) with itself, we have

$$k_2^{\star} = \frac{k_2}{\left(f^{'}\right)^2}. \tag{3.37}$$

(ii) Let $1 + \mu_1^{'} + \mu_2 k_2 = 0$. If we write instead in (3.29), we get

$$f^{'} T^{\star} = \left(\mu_3^{'} - \mu_2\right) B \quad \text{and} \quad \mu_3^{'} - \mu_2 \neq 0 \ . \tag{3.38}$$

Differentiating (3.38) with respect to s and using the (2.2) Frenet equations , we find

$$N^{\star} = -\frac{\left(\mu_3^{'} - \mu_2\right) k_2}{\left(f^{'}\right)^2} N. \tag{3.39}$$

By taking the scalar product of (3.39) with itself, we have

$$\left(f^{'}\right)^2 = b_1 \left(\mu_3^{'} - \mu_2\right) \quad \text{and} \quad g(\frac{dT^{\star}}{ds^{\star}}, N^{\star}) = k_1^{\star} = 1, \tag{3.40}$$

where $b_1 = sgn\left(\mu_3^{'} - \mu_2\right)$.
Substituting (3.40) in (3.39), we find

$$N^{\star} = b_1 b_2 N \ \ , \ \ g\left(N^{\star}, N^{\star}\right) = 1, \quad \text{and} \quad b_2 = sgn\left(b_1\left(\mu_3^{'} - \mu_2\right)\right) \tag{3.41}$$

Differentiating (3.41) with respect to s and using the (2.2) Frenet equations , we find

$$k_2^{\star} T^{\star} - B^{\star} = \frac{b_1 b_2}{f^{'}} \left(k_2 T - B\right). \tag{3.42}$$

By taking the scalar product of (3.42) with itself, we have

$$k_2^{\star} = \frac{k_2}{\left(f^{'}\right)^2}. \tag{3.43}$$

Conversely, let φ be a Cartan null curve parametrized by arc-length s with non-zero curvatures $k_1 = 1, k_2$. Assume that φ provides the conditions of (3.3) or (3.4) for differentiable functions μ_1, μ_2, μ_3. Then, we can define a curve $\varphi^{\star}$ as

$$\varphi^{\star}\left(s^{\star}\right) = \varphi^{\star}\left(f\left(s\right)\right) = \varphi\left(s\right) + \mu_1\left(s\right) T\left(s\right) + \mu_2\left(s\right) N\left(s\right) + \mu_3\left(s\right) B\left(s\right). \tag{3.44}$$

Differentiating (3.44) with respect to s, we find

$$\frac{d\varphi^{\star}}{ds^{\star}} = \left(1 + \mu_1^{'} + \mu_2 k_2\right) T + \left(\mu_3^{'} - \mu_2\right) B, \tag{3.45}$$

and using (3.3)

$$\frac{d^2\varphi^{*}}{ds^{\star 2}} = \left(1 + \mu_1^{'} + \mu_2 k_2\right)^{'} T + \left(1 + \mu_1^{'} - \mu_3^{'} k_2 + 2\mu_2 k_2\right) N,$$

which leads to that

$$f^{'} = \sqrt{b_3\left(1 + \mu_1^{'} - \mu_3^{'} k_2 + 2\mu_2 k_2\right)}, \tag{3.46}$$

where $b_3 = sgn\left(1 + \mu_1^{'} - \mu_3^{'} k_2 + 2\mu_2 k_2\right)$. Rewriting (3.45) , we obtain

$$T^{\star} = \frac{\left(1 + \mu_1^{'} + \mu_2 k_2\right)}{f^{'}} T, \quad g\left(T^{\star}, T^{\star}\right) = 0,$$

and

$$T^{\star} = \pm T \quad , \quad k_1^{\star} = 1. \tag{3.47}$$

Differentiating (3.47) with respect to s, we get

$$N^{\star} = \pm N, \quad g\left(N^{\star}, N^{\star}\right) = 1. \tag{3.48}$$

Differentiating (3.48) with respect to s, we get

$$f^{'} k_2^{\star} T^{\star} - f^{'} B^{\star} = \pm \left(k_2 T - B\right), \tag{3.49}$$

then,

$$B^{\star} = \pm B, \quad g\left(B^{\star}, B^{\star}\right) = 0, \tag{3.50}$$

and

$$k_2^{\star} = \frac{k_2}{f^{'}}, \tag{3.51}$$

or using (3.4)

$$\frac{d^2\varphi^{*}}{ds^{\star 2}} = \left(1 + \mu_1^{'} - \mu_3^{'} k_2 + 2\mu_2 k_2\right) N + \left(\mu_3^{'} - \mu_2\right)^{'} B, \tag{3.52}$$

which leads to that

$$f^{'} = \sqrt{b_3 \left(1 + \mu_1^{'} - \mu_3^{'} k_2 + 2\mu_2 k_2\right)}, \tag{3.53}$$

where $b_3 = sgn\left(1 + \mu_1^{'} - \mu_3^{'} k_2 + 2\mu_2 k_2\right)$. Rewriting (3.45) , we obtain

$$T^{\star} = \frac{\left(\mu_3^{'} - \mu_2\right)}{f^{'}} B, \tag{3.54}$$

so

$$T^{\star} = \pm k_2 B \qquad \text{and} \qquad g\left(T^{\star}, T^{\star}\right) = 0. \tag{3.55}$$

Differentiating (3.55) with respect to s, we get

$$N^{\star} = \pm \frac{\left(k_2\right)^2}{f^{'}} N, \quad g\left(N^{\star}, N^{\star}\right) = 1. \tag{3.56}$$

Differentiating (3.56) with respect to s, we get

$$f^{'} k_2^{\star} T^{\star} - f^{'} B^{\star} = \pm \frac{\left(k_2\right)^2}{f^{'}} \left(k_2 T - B\right), \tag{3.57}$$

then,

$$B^{\star} = \pm \frac{\left(k_2\right)^2}{\left(f^{'}\right)^2} B, \quad g\left(B^{\star}, B^{\star}\right) = 0, \tag{3.58}$$

and

$$k_2^{\star} = \frac{\left(k_2\right)^2}{\left(f^{'}\right)^2}. \tag{3.59}$$

Then $\varphi^{\star}$ is a Cartan null curve and a Bertrand mate curve of φ. Thus φ is a Bertrand curve. Q.E.D.

Theorem 3.3. Let $\varphi : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a unit speed pseudo null curve with null principal normal with the non-zero curvatures $k_1(s) = 1, k_2(s)$. Then the curve $\varphi(s)$ is a Bertrand curve with Bertrand mate curve $\varphi^\star(s^\star)$ if and only if there exist differentiable functions μ_1, μ_2, μ_3 and $m_1 = \pm 1$ such that the next system of equations hold:

$$\begin{array}{rl} \mu_3^{'} - \mu_3 k_2 = & 0 \\ (1 + \mu_1^{'} - \mu_3)^2 = & (f^{'})^2 \\ \left(\frac{\mu_1 + \mu_2^{'} + \mu_2 k_2}{f^{'}}\right)^{'} + \frac{\mu_1 + \mu_2^{'} + \mu_2 k_2}{f^{'}} k_2 = & m_1 \left(\mu_1^{'} - \mu_3\right) \end{array} \tag{3.60}$$

In this situation, the Bertrand mate curve $\varphi^\star$ is a pseudo null curve with null principal normal vector.

We omit the proof of the theorem since it is similar to the Theorem 3. 1.

4 Examples

In this section, examples of Cartan null and pseudo null Bertrand curves according to the new approach will be given.

Example 4.1. Let us consider a Cartan null in $\mathbb{E}_1^3$ with the equation

$$\varphi(s) = (-\sinh s, -\cosh s, s - 2)$$

with the curvatures $k_1(s) = 1$ and $k_2(s) = \frac{1}{2}$ and the Cartan frame as

$$T(s) = (-\cosh s, -\sinh s, 1),$$

$$N(s) = (-\sinh s, -\cosh s, 0),$$

$$B(s) = \frac{1}{2}(\cosh s, \sinh s, 1).$$

It is easily check that φ is Cartan null helix.

(i) If we take $\mu_1(s) = -\frac{s}{2} - 1, \mu_2 = s, \mu_3(s) = -s$ and $\lambda = 1$ in (i) of Theorem 3. 1, then we get the curve $\varphi^\star(s^\star)$ as follows;

$$\varphi^\star(s^\star) = \varphi^\star(f(s)) = (\cosh s - (1+s)\sinh s, \sinh s - (1+s)\cosh s, -3). \tag{4.1}$$

Differentiating (4.1) with respect to s and considering equation (3.9) we find $f^{'}(s) = -1 - s$. Then, using equations (3.11), (3.15) and (3.16) we get

$$T^\star(s^\star) = (\cosh s, \sinh s, 0).$$

$$N^\star(s^\star) = (\sinh s, \cosh s, 0),$$

$$B^\star(s^\star) = (0, 0, -1)$$

and $k_1^\star(s^\star) = \frac{1}{1+s}$, $k_2^\star(s^\star) = 0$. It can be easily seen that the curve $\varphi^\star(s^\star)$ is a timelike Bertrand mate curve of the curve $\varphi(s)$.

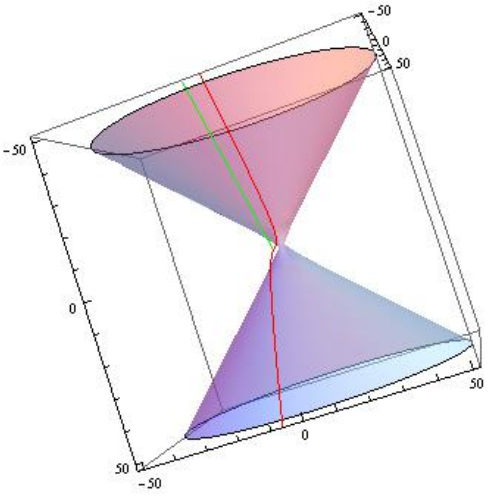


Figure 1: The red graphic is Cartan null Bertrand curve φ and the green graphic is timelike Bertrand mate curve φ^* in (i) of Example 4.1.

(ii) If we take $\mu_1(s) = s - e^s, \mu_2(s) = e^s - 1, \mu_3(s) = 2s$ and $\lambda = 1$ in (ii) of Theorem 3.1, then we get the curve $\varphi^\star(s)$ as follows;

$$\varphi^\star(s^\star) = \varphi^\star(f(s)) = (e^s(\cosh s - \sinh s), e^s(\sinh s - \cosh s), 3s - e^s - 2).$$

In this case, with correct calculations, we get $f'(s) = 3 - e^s$ and the Frenet frame of $\varphi^\star$ as follow,

$$T^\star(s^\star) = (0, 0, 1),$$

$$N^\star(s^\star) = (\sinh s, \cosh s, 0),$$

$$B^\star(s^\star) = (0, 0, 1)$$

and $k_1^\star(s^\star) = \frac{1}{3-e^s}$, $k_2^\star(s^\star) = 0$. It can be easily seen that the curve $\varphi^\star(s^\star)$ is a spacelike Bertrand mate curve of the curve φ.

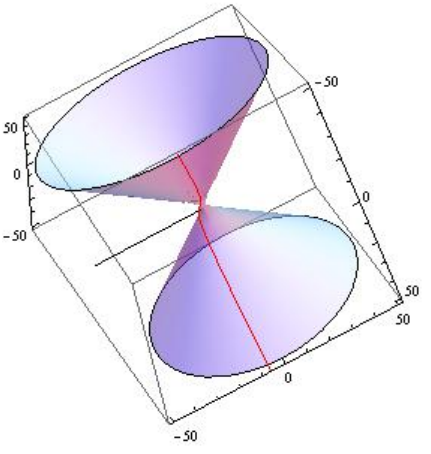


Figure 2: The red graphic is Cartan null Bertrand curve φ and the black graphic is spacelike Bertrand mate curve φ^* in (ii) of Example 4.1.

(iii) If we take $\mu_1(s) = \frac{s}{2}$, $\mu_2(s) = 1, \mu_3(s) = s$ in (iii) of Theorem 3.1, then we get the curve $\varphi^\star$as follows;

$$\varphi^\star(s^\star) = \varphi^\star(f(s)) = (-2\sinh s, -2\cosh s, 2s - 2)$$

With correct calculations, we get $f^{'}(s) = \sqrt{2}$. Then we find the Cartan frame of the curve as follow

$$T^\star(s^\star) = (-\cosh s, -\sinh s, 1),$$

$$N^\star(s^\star) = (-\sinh s, -\cosh s, 0),$$

$$B^\star(s^\star) = (1, -1, 0)$$

and $k_1^\star = 1$, $k_2^\star = 0$. It can be easily seen that the curve $\varphi^\star$ is a Cartan null Bertrand mate curve of the curve φ.

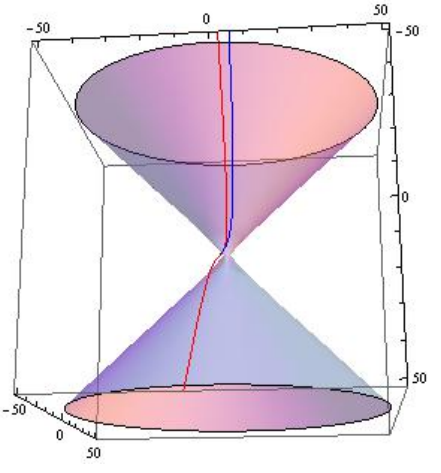


Figure 3: The red graphic is Cartan null Bertrand curve φ and the blue graphic is Cartan null Bertrand mate curve φ^* in (iii) of Example 4. 1.

Example 4.2. Let us consider a pseudo null curve in $\mathbb{E}_1^3$ with the equation

$$\varphi(s) = \left(\frac{s^3}{3}, \frac{s}{2} + \frac{s^3}{2\sqrt{3}}, -\frac{\sqrt{3}s}{2} + \frac{s^3}{6}\right)$$

with the curvatures $k_1(s) = 1$ and $k_2(s) = \frac{1}{s}$ and the Frenet frame with null principle normal vector as

$$\begin{aligned} T(s) &= \left(s^2, \tfrac{1}{2} + \tfrac{\sqrt{3}s^2}{2}, -\tfrac{\sqrt{3}}{2} + \tfrac{s^2}{2}\right) \\ N(s) &= \left(2s, \sqrt{3}s, s\right) \\ B(s) &= \left(-\tfrac{1}{4s} - \tfrac{s^3}{4}, -\tfrac{s}{4} - \tfrac{\sqrt{3}s^3}{8} + \tfrac{\sqrt{3}}{8s}, \tfrac{\sqrt{3}s}{4} + \tfrac{1}{8s} - \tfrac{s^3}{8}\right) \end{aligned}$$

If we take $\mu_1(s) = \frac{s^2}{2}, \mu_2 = -\frac{s^3}{8}, \mu_3(s) = s$ and $m_1 = 1$ in of Theorem 3. 2, then we get the curve $\varphi^\star(s^\star)$ as follows;

$$\varphi^\star(s^\star) = \varphi^\star(f(s)) = \left(-\frac{1}{4} + \frac{s^3}{3}, \frac{1}{24}(3\sqrt{3} + 12s + 4\sqrt{3}s^3), \frac{1}{24}(3 - 12\sqrt{3}s + 4s^3)\right).$$

With correct calculations, we get

$$\begin{aligned} T^\star(s^\star) &= \left(s^2, \tfrac{1}{2}(1 + \sqrt{3}s^2), \tfrac{1}{2}(-\sqrt{3} + s^2)\right), \\ N^\star(s^\star) &= \left(2s, \sqrt{3}s, s\right), \\ B^\star(s^\star) &= \left(-\tfrac{1}{4s} - \tfrac{s^3}{4}, -\tfrac{s}{4} - \tfrac{\sqrt{3}s^3}{8} + \tfrac{\sqrt{3}}{8s}, \tfrac{\sqrt{3}s}{4} + \tfrac{1}{8s} - \tfrac{s^3}{8}\right) \end{aligned}$$

and $k_1^\star(s) = 1$, $k_2^\star(s) = s$. It can be easily seen that the curve $\varphi^\star$ is a pseudo null Bertrand mate curve of the curve $\varphi(s)$.

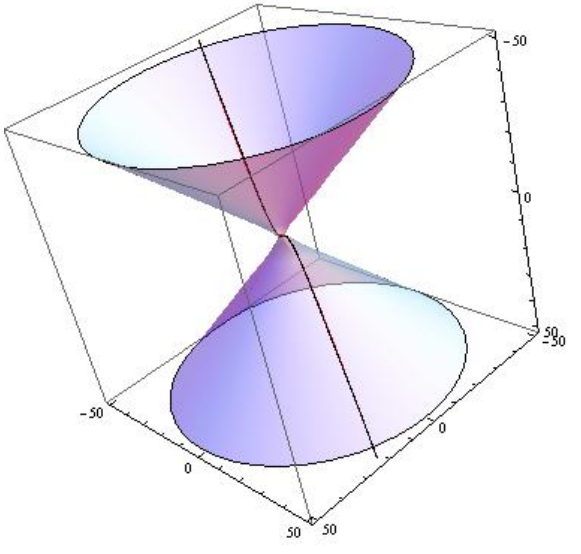


Figure 4: The red graphic is φ (rectifying slant helix) and the blue graphic is φ^* (general helix) in Example 4.2.

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