On convergence of positive linear operators related to squared-Durrmeyer operators

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Abstract

The aim of this paper is to study the pointwise behavior of certain sequences of the squared-Durrmeyer operators DB_n^2 and $\widetilde{D}B_n^2$ acting on bounded functions on an interval $[0,1]$, defined by Gavrea and Ivan. Here we estimate the rate of convergence at a point y, which is a Lebesgue point of $g \in L_1([0,1])$ be such that $\psi o\,|g| \in BV([0,1])$, where $\psi o\,|g|$ denotes the composition of the functions ψ and $|g|$. The function $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is continuous and concave with $\psi(0) = 0$, $\psi(u) > 0$ for $u > 0$, which appears from the $(L - \psi)$ Lipschitz conditions.

2010 Mathematics Subject Classification. **41A35**. 41A25, 47G10.
Keywords. Squared-Durrmeyer operators, Bounded variation, Lipschitz condition, Pointwise convergence.

1 Introduction

The examination of convergence properties and calculation of convergence rates for squared-type linear positive operators have been a widely explored topic in approximation theory studies. These operators are generated using well-known Bernstein operators, denoted as B_n, which are defined as follows:

$$(B_n g)(y) = \sum_{\nu=0}^{n} p_{n,\nu}(y) g\left(\frac{\nu}{n}\right) \qquad (y \in [0,1])$$

where g is a bounded function defined on the interval $[0,1]$ and $p_{n,\nu}(y) = \binom{n}{\nu} y^\nu (1-y)^{n-\nu} g\left(\frac{\nu}{n}\right)$, $n = 1, 2, \ldots, \quad y \in [0,1]$. These are known as the Bernstein basis polynomials.

Numerous research works have delved into this subject, aiming to understand the behavior of these operators and quantify their rate of convergence. The focus has been on investigating the convergence characteristics of squared-type operators and providing insights into their convergence rates. This line of study has contributed to advancing the field of approximation theory, shedding light on the efficiency and accuracy of these operators in approximating functions. For example, some studies are historically listed below:

In 2014, Gavrea and Ivan [6] developed a novel expression for the sum of the squared Bernstein polynomials. This expression was then employed to confirm a hypothesis suggesting that the sum represents a convex function. Furthermore, the authors expanded upon this finding and applied it to various other well-known approximation operators, including the Mirakjan-Favard-Szász operators, Meyer-König and Zeller operators, and King-type operators.

In 2015, Abel and Gawronski [1] conducted study on the complete monotonicity of sums of squares of generalized Baskakov basis functions. They are accomplished by establishing the corresponding outcomes for Guassian hypergeometric functions. Additionally, they investigate the

Advanced Studies: Euro-Tbilisi Mathematical Journal 16, supplement issue 3 (2023), pp. 45–58.
DOI: 10.32513/asetmj/1932200823304

Tbilisi Centre for Mathematical Sciences.

Received by the editors: 29 july 2023.
Accepted for publication: 05 October 2023.

distribution of complex zeros in the central Baskakov case, particularly focusing on large parameter values. Finally, they explore the expansion of certain findings to encompass sums of higher powers.

In 2017, Gavrea and Ivan [7], also, presents the authors' calculations of the second central moment of the squared Bernstein polynomials. Additionally, they derived estimates for the rate of approximation by considering the modulus of continuity.

In 2018, 2019, and 2020, Holhoş [8, 9, 10] derived the Voronovskaya formula for a sequence of positive linear operators that were constructed using the squared Bernstein polynomials. They also examined the approximation properties of positive linear operators defined through the utilization of the powered Baskakov basis and also Szász-Mirakyan basis.

In 2019, Abel and Kushnirevych [2] published their work, wherein they successfully derived a comprehensive asymptotic expansion for a specific sequence of positive linear approximation operators. These operators were constructed based on the squared Bernstein basis polynomials. The authors explicitly calculated the first three coefficients and identified a special case known as the Voronovskaja type formula. They also introduced two Durrmeyer-type variants and determined the initial two coefficients for their asymptotic expansions.

In the same year, Abel [3] conducted further investigations on a sequence of positive linear approximation operators, defined using squared Bernstein basis polynomials, Favard-Szász-Mirakjan fundamental functions, and Baskakov fundamental functions. He established a comprehensive asymptotic expansion for these operators and explicitly computed the initial coefficients. Like in the previous case, he derived a Voronovskaja type formula as a special case and also introduced two Durrmeyer-type variants, estimating the initial coefficients of their asymptotic expansions. Additionally, he identified the trivial class for each case and analyzed the asymptotic properties of the operators defined by squared Meyer-König and Zeller fundamental functions.

In 2022, Mohammad, I. J. and Mohammad A. J. [11] proposed a new family of neural networks called multivariate square rational Bernstein operators. These operators were developed by extending the artificial neural networks based on multivariate Bernstein polynomials to incorporate square Bernstein polynomials. The authors extensively investigated the behavior and characteristics of this neural network model.

Very recently, in 2023, Rahman, and Aral [12] introduced a sequence of positive linear operators that utilized the squared Szász-Mirakjan basis functions. They then proceeded to estimate the rate of convergence of these operators by considering the modulus of continuity and the class of Lipschitz functions. Additionally, the authors provided a visual representation through illustrative graphics to showcase the comparison and convergence of these operators.

At the same year, 2023, Altın and Karslı [4] focus on the approximation properties of Squared Chlodovsky operators, they also compute the rate of convergence the operators by means of modulus of continuity and class of Lipschitz functions.

This paper is structured into three distinct sections. The first part serves as the introduction, providing an overview of the research topic. In the subsequent section, we present preliminary findings that are essential for the development of the main theorem. Finally, the last section is dedicated to presenting and proving the main theorem, along with the inclusion of two results that have been derived from it.

1.1 Preliminaries

Recently, Gavrea and Ivan [7] defined the squared-Bernstein operators as

$$B_n(g;y)=\frac{1}{\sum\limits_{\nu=0}^{n}p_{n,\nu}^2(y)}\sum_{\nu=0}^{n}p_{n,\nu}^2(y)\,g\left(\frac{\nu}{n}\right).$$

Let $L_1[0,1]$ be the class of all Lebesgue integrable functions defined on $[0,1]$. For $g\in L_1[0,1]$, we consider the two variants of the Durrmeyer operators of the squared Berstein polynomial defined as, respectively,

$$\begin{aligned}DB_n^2(g;y) &= \frac{2n+1}{\sum\limits_{\nu=0}^{n}p_{n,\nu}^2(y)}\sum_{\nu=0}^{n}p_{n,\nu}^2(y)\frac{\binom{2n}{2\nu}}{\binom{n}{\nu}^2}\int_0^1 g(t)p_{n,\nu}^2(t)dt\\ &= \int_0^1 G_n(y,t)g(t)dt,\end{aligned}$$

where

$$G_n(y,t):=\frac{2n+1}{\sum\limits_{\nu=0}^{n}p_{n,\nu}^2(y)}\frac{\binom{2n}{2\nu}}{\binom{n}{\nu}^2}\sum_{\nu=0}^{n}p_{n,\nu}^2(y)p_{n,\nu}^2(t), \tag{1.1}$$

and

$$\begin{aligned}\widetilde{D}B_n^2(g;y) &= \frac{n+1}{\sum\limits_{\nu=0}^{n}p_{n,\nu}^2(y)}\sum_{\nu=0}^{n}p_{n,\nu}^2(y)\int_0^1 g(t)p_{n,\nu}(t)dt\\ &= \int_0^1 \widetilde{G}_n(y,t)g(t)dt,\end{aligned}$$

where

$$\widetilde{G}_n(y,t):=\frac{n+1}{\sum\limits_{\nu=0}^{n}p_{n,\nu}^2(y)}\sum_{\nu=0}^{n}p_{n,\nu}^2(y)p_{n,\nu}(t). \tag{1.2}$$

Theorem 1.1. For any $g\in C[0,1]$ we have the following estimate

$$\left|\widetilde{D}B_n^2(g;y)-g(y)\right|\le 2\omega\left(g;\sqrt{\left(\widetilde{D}B_n^2(t-y)^2\right)(y)}\right).$$

Proof.

$$\begin{aligned}\left|\widetilde{D}B_n^2(g;y)-g(y)\right| &= \left|\frac{n+1}{\sum\limits_{\nu=0}^{n}p_{n,\nu}^2(y)}\sum_{\nu=0}^{n}p_{n,\nu}^2(y)\int_0^1\left[g(t)-g(y)\right]p_{n,\nu}(t)\,dt\right| \\ &\leq \frac{n+1}{\sum\limits_{\nu=0}^{n}p_{n,\nu}^2(y)}\sum_{\nu=0}^{n}p_{n,\nu}^2(y)\int_0^1\omega(g;\delta)\left[\frac{|t-y|}{\delta}+1\right]p_{n,\nu}(t)\,dt.\end{aligned}$$

In view of Hölder inequality we get

$$\begin{aligned}\left|\widetilde{D}B_n^2(g;y)-g(y)\right| &\leq \omega(g;\delta)+\frac{n+1}{\sum\limits_{\nu=0}^{n}p_{n,\nu}^2(y)}\sum_{\nu=0}^{n}p_{n,\nu}^2(y)\int_0^1\omega(g;\delta)\frac{|t-y|}{\delta}p_{n,\nu}(t)dt \\ &\leq \omega(g;\delta)+\frac{\omega(g;\delta)}{\delta}\left(\frac{n+1}{\sum\limits_{\nu=0}^{n}p_{n,\nu}^2(y)}\sum_{\nu=0}^{n}p_{n,\nu}^2(y)\int_0^1(t-y)^2p_{n,\nu}(t)\,dt\right)^{1/2} \\ &= \omega(g;\delta)+\frac{\omega(g;\delta)}{\delta}\sqrt{\left(\widetilde{D}B_n^2(t-y)^2\right)(y)} \\ &\leq \omega(g;\delta)\left[1+\frac{\sqrt{\left(\widetilde{D}B_n^2(t-y)^2\right)(y)}}{\delta}\right]\end{aligned}$$

so one has

$$\left|\widetilde{D}B_n^2(g;y)-g(y)\right|\leq 2\omega\left(g;\sqrt{\left(\widetilde{D}B_n^2(t-y)^2\right)(y)}\right).$$

Q.E.D.

Similarly one can easily obtain.

Theorem 1.2. For any $g\in C[0,1]$ we have the following estimate

$$\left|DB_n^2(g;y)-g(y)\right|\leq 2\omega\left(g;\sqrt{(DB_n^2(t-y)^2)(y)}\right).$$

Proof.

$$\begin{aligned}
\left|DB_n^2(g;y)-g(y)\right| &= \left|\frac{2n+1}{\sum_{\nu=0}^{n} p_{n,\nu}^2(y)}\sum_{\nu=0}^{n} p_{n,\nu}^2(y)\frac{\binom{2n}{2\nu}}{\binom{n}{\nu}^2}\int_0^1 [g(t)-g(y)]\,p_{n,\nu}^2(t)dt\right| \\
&\leq \frac{2n+1}{\sum_{\nu=0}^{n} p_{n,\nu}^2(y)}\sum_{\nu=0}^{n} p_{n,\nu}^2(y)\frac{\binom{2n}{2\nu}}{\binom{n}{\nu}^2}\int_0^1 \omega(g;\delta)\left[\frac{|t-y|}{\delta}+1\right]p_{n,\nu}^2(t)\,dt \\
&\leq \omega(g;\delta)+\frac{\omega(g;\delta)}{\delta}\sqrt{(DB_n^2(t-y)^2)(y)} \\
&\leq \omega(g;\delta)\left[1+\frac{\sqrt{(DB_n^2(t-y)^2)(y)}}{\delta}\right]
\end{aligned}$$

so one has

$$\left|DB_n^2(g;y)-g(y)\right| \leq 2\omega\left(g;\sqrt{(DB_n^2(t-y)^2)(y)}\right).$$

Q.E.D.

Definition 1.3. For $g\in C[a,b]$ and $\delta>0$, the Peetre-K Functional are defined by

$$K(g,\delta):=\inf_{h\in C^2[a,b]}\left\{\|g-h\|_{C[a,b]}+\delta\|h\|_{C^2[a,b]}\right\}.$$

Theorem 1.4. If $h\in C^2[0,1]$, then

$$\left|DB_n^2(h;y)-h(y)\right| \leq \frac{\left|(DB_n^2(t-y)^2)(y)\right|}{2}\|h\|_{C^2[0,1]}.$$

Proof. Using the Taylor formula with an integral remainder term, we write

$$h(t)=h(y)+(t-y)h'(y)+\int_0^{t-y}(t-y-u)^2h^{''}(y+u)du. \tag{1.3}$$

If we apply the operator to (1.3), we get

$$\begin{aligned}
\left|DB_n^2(h;y)-h(y)\right| &= \left|h'(y)(DB_n^2(t-y))(y)+DB_n^2\left(\int_0^{t-y}(t-y-u)^2h^{''}(y+u)du\right)(y)\right| \\
&\leq \|h'\|_{C[0,1]}\left|(DB_n^2(t-y))(y)\right| \\
&\quad +\left\|h^{''}\right\|_{C[0,1]}\left|DB_n^2\left(\int_0^{t-y}(t-y-u)^2du\right)(y)\right|.
\end{aligned}$$

Since

$$\int_0^{t-y}(t-y-u)^2du=\frac{(t-y)^2}{2},$$

one has

$$\left|DB_n^2(h;y)-h(y)\right|\leq\left\|h^{''}\right\|_{C[0,1]}\frac{\left|(DB_n^2(t-y)^2)(y)\right|}{2}.$$

Here we note that

$$\|h\|_{C^2[a,b]}=\|h\|_{C[a,b]}+\|h'\|_{C[a,b]}+\|h''\|_{C[a,b]}.$$

Finally we have

$$\left|DB_n^2(h;y)-h(y)\right|\leq\frac{\left|(DB_n^2(t-y)^2)(y)\right|}{2}\|h\|_{C^2[0,1]}.$$

Therefore the theorem is proved. Q.E.D.

Now, we are prepared to demonstrate the validity of the following theorem.

Theorem 1.5. If $g\in C[0,1]$, then

$$\left\|DB_n^2g-g\right\|_{C[0,1]}\leq 2K\left(g,\frac{\left|(DB_n^2(t-y)^2)(y)\right|}{2}\right).$$

Proof. By the linearity property of (DB_n^2), we get

$$\begin{aligned}|DB_n^2(g;y)-g(y)| &\leq |DB_n^2(g;y)-DB_n^2(h;y)|+|DB_n^2(h;y)-h(y)|\\ &\quad +|h(y)-g(y)|\\ &\leq \|g-h\|_{C[0,1]}\left|DB_n^2(1;y)\right|+\|g-h\|_{C[0,1]}+\left|DB_n^2(h;y)-h(y)\right|.\end{aligned}$$

From Theorem 1.4, one has

$$\left|DB_n^2(g;y)-g(y)\right|\leq 2\|g-h\|_{C[0,1]}+\frac{\left|(DB_n^2(t-y)^2)(y)\right|}{2}\|h\|_{C^2[0,1]},$$

and hence

$$\left\|DB_n^2g-g\right\|_{C[0,1]}\leq 2\|g-h\|_{C[0,1]}+\frac{\left|(DB_n^2(t-y)^2)(y)\right|}{2}\|h\|_{C^2[0,1]}. \tag{1.4}$$

Taking the infimum the right hand side of (1.4) over all $g\in C^2[0,1]$, then

$$\left\|DB_n^2g-g\right\|_{C[0,1]}\leq 2K\left(g,\frac{\left|(DB_n^2(t-y)^2)(y)\right|}{2}\right).$$

We have now demonstrated the desired theorem. Q.E.D.

When we work on L_1 space, we have the following result

Theorem 1.6. If $g \in L_1[0,1]$, then $DB_n^2 g \in L_1[0,1]$ and

$$\left\|DB_n^2 g\right\|_{L_1[0,1]} \leq \|g\|_{L_1[0,1]}$$

for every $n \in \mathbb{N}$.

Proof. From the definition of the norm on $L_1[0,1]$, and applying the Generalized Minkowsky Inequality, we have

$$\begin{aligned}
\int_0^1 \left|\left(DB_n^2 g\right)(y)\right| dy &= \int_0^1 \left| \int_0^1 G_n(y,t) g(t)\, dt \right| dy \leq \int_0^1 \int_0^1 G_n(y,t) \left|g(t)\right| dt dy \\
&\leq \int_0^1 \left(\int_0^1 G_n(y,t) \left|g(t)\right| dy \right) dt \\
&\leq \int_0^1 \left|g(t)\right| \left(\int_0^1 G_n(y,t) dy \right) dt.
\end{aligned}$$

Since

$$\int_0^1 G_n(y,t) dt = \int_0^1 G_n(y,t) dy,$$

one has

$$\int_0^1 \left|\left(DB_n^2 g\right)(y)\right| dy \leq \int_0^1 \left|g(t)\right| dt = \|g\|_{L_1[0,1]}.$$

As a result, the squared-Durrmeyer operators acting on $L_1[0,1]$ into itself. The proof is now concluded. Q.E.D.

We have the same result for the second variant squared-Durrmeyer operator.

Theorem 1.7. If $g \in L_1[0,1]$, then $\widetilde{D}B_n^2 g \in L_1[0,1]$ and

$$\left\|\widetilde{D}B_n^2 g\right\|_{L_1[0,1]} \leq \|g\|_{L_1[0,1]}$$

for every $n \in \mathbb{N}$.

Proof. As in the proof of the previous theorem, we obtain

$$\begin{aligned}
\int_0^1 \left|\left(\widetilde{D}B_n^2 g\right)(y)\right| dy &= \int_0^1 \left| \int_0^1 \widetilde{G}_n(y,t) g(t)\, dt \right| dy \leq \int_0^1 \int_0^1 \widetilde{G}_n(y,t) \left|g(t)\right| dt dy \\
&\leq \int_0^1 \left(\int_0^1 \widetilde{G}_n(y,t) \left|g(t)\right| dy \right) dt \\
&\leq \int_0^1 \left|g(t)\right| \left(\int_0^1 \widetilde{G}_n(y,t) dy \right) dt.
\end{aligned}$$

Hence we get

$$\left\|\widetilde{D}B_n^2 g\right\|_{L_1[0,1]} = \int_0^1 \left|\left(\widetilde{D}B_n^2 g\right)(y)\right| dy \leq \int_0^1 |g(t)|\, dt = \|g\|_{L_1[0,1]}.$$

As a result, the squared-Durrmeyer operators acting on $L_1[0,1]$ into itself. This completes the proof.

Q.E.D.

We let

$$g_y(t) = \begin{cases} g(t) - g(y+) & , \ y < t \leq 1 \\ 0 & , \ t = y \\ g(t) - g(y-) & , \ 0 \leq t < y \end{cases}, \tag{1.5}$$

and $\bigvee_0^1(|g_y|)$ is the total variation of $(|g_y|)$ on $[0,1]$.

Remark 1.8. For all $y \in (0,1)$, and for each $n \in \mathbb{N}$ with Sikkema's paper [13], let

$$\left(\widetilde{D}B_n^2 g(t-y)^\beta\right)(y) = \int_0^1 \widetilde{G}_n(y,u)\,|u-y|^\beta\, du \leq \frac{B(y)}{n^{\left\lfloor \frac{1+\beta}{2} \right\rfloor}}, \ (\beta \in \mathbb{N}) \tag{1.6}$$

holds, where $\widetilde{G}_n(y,u)$ is as defined in the beginning of this section and $B(y)$ is a bounded function.

The assumptions of the remark lead to the following inequalities.

Lemma 1.9.

$$\lambda_n(y,z) := \int_0^z \widetilde{G}_n(y,u)du \leq \frac{B(y)}{(y-z)^\beta n^{\left\lfloor \frac{1+\beta}{2} \right\rfloor}}, \quad 0 \leq z < y, \tag{1.7}$$

and

$$1 - \lambda_n(y,z) = \int_z^1 \widetilde{G}_n(y,u)du \leq \frac{B(y)}{(z-y)^\beta n^{\left\lfloor \frac{1+\beta}{2} \right\rfloor}}, \quad y < z < 1. \tag{1.8}$$

Proof. We have

$$\begin{aligned} \lambda_n(y,z) &:= \int_0^z \widetilde{G}_n(y,u)du \leq \int_0^z \widetilde{G}_n(y,u)\left(\frac{y-u}{y-z}\right)^\beta du \\ &\leq \frac{1}{(y-z)^\beta}\int_0^1 \widetilde{G}_n(y,u)\,|u-y|^\beta\, du. \end{aligned}$$

According to (1.6), we have

$$\lambda_n(y,z) \leq \frac{B(y)}{(y-z)^\beta n^{\left\lfloor \frac{1+\beta}{2} \right\rfloor}}.$$

Proof of (1.8) is analogous. Q.E.D.

Definition 1.10. A point $y_0 \in \mathbb{R}$ is is referred to as a Lebesgue point of the function g if the following condition

$$\lim_{h\to 0}\frac{1}{h}\int_0^h |g(y_0+t)-g(y_0)|\,dt = 0, \tag{1.9}$$

holds. (Butzer and Nessel [5])

2 Main results

We are at the point where we can now establish the main results of this research.

Theorem 2.1. Let $g \in L_1([0,1])$ be such that $|g| \in BV([0,1])$. Then at each point $y \in (0,1)$ for which (1.9) holds we have for each $\varepsilon > 0$ and for sufficiently large $n \in \mathbb{N}$, there holds

$$\begin{aligned}\left|\left(\widetilde{D}B_n^2 g\right)(y) - g(y)\right| \quad &\leq \quad \varepsilon B^*(y) n^{\beta-1} \\ &\quad + \frac{B^*(y)}{n^\beta}\left[\bigvee_0^1(|g_y|) + \sum_{k=1}^{[n^\beta]}\bigvee_{y-y/k^{1/\beta}}^{y+(1-y)/k^{1/\beta}}(|g_y|)\right],\end{aligned}$$

where $B^*(y) = B(y)\max\{y^{-\beta},(1-y)^{-\beta}\}$,$(\beta > 0)$.

There follows the pointwise convergence result.

Theorem 2.2. Let $g \in L_1([0,1])$ be such that $|g| \in BV([0,1])$. Then at each point $y \in (0,1)$ for which (1.9) holds we have

$$\lim_{n\to\infty}\left|\left(\widetilde{D}B_n^2 g\right)(y) - g(y)\right| = 0.$$

Proof. From Theorem 2.1 we reach the result, by the arbitrariness of $\varepsilon > 0$. Q.E.D.

Corollary 2.3. Let $g \in L_1([0,1])$ be such that $|g| \in BV([0,1])$. Then

$$\lim_{n\to\infty}\left|\left(\widetilde{D}B_n^2 g\right)(y) - g(y)\right| = 0$$

holds almost everywhere in $(0,1)$.
In light of the fact that nearly all y values within the open interval (0, 1) can be regarded as Lebesgue points of the function g, we can conclude the assertion using Theorem 2.2.

Proof the Theorem 2.1. Suppose that

$$y+\delta < 1, \;\; y-\delta > 0, \tag{2.1}$$

for any sufficiently small $0 < \delta$.

Let

$$|I_n(y)| := \left|\left(\widetilde{D}B_n^2 g\right)(y) - g(y)\right|.$$

We can rewrite $|I_n(y)|$ as follows:

$$\begin{aligned} |I_n(y)| &\leq \left| \int_0^1 \widetilde{G}_n(y,t) g(t)\, dt - \int_0^1 \widetilde{G}_n(y,t) g(y)\, dt \right| \\ &\leq \int_0^1 \widetilde{G}_n(y,t) \left| g(t) - g(y) \right| dt. \end{aligned}$$

Let us divide the last integral into three separate terms in the following manner:

$$\begin{aligned} |I_n(y)| &\leq \left(\int_0^{y-y/n} + \int_{y-y/n}^{y+(1-y)/n} + \int_{y+(1-y)/n}^1 \right) |g(t) - g(y)|\, d_t\left(\lambda_n(y,t)\right) \\ &:= I_1(n,y) + I_2(n,y) + I_3(n,y). \end{aligned}$$

We estimate $I_2(n,y)$. We have for $t \in [y - y/n, y + (1-y)/n]$

$$\begin{aligned} |I_2(n,y)| &= \int_{y-y/n}^{y+(1-y)/n} |g(t) - g(y)|\, d_t\left(\lambda_n(y,t)\right) \\ &\leq \int_{y-y/n}^{y} |g(t) - g(y)|\, d_t\left(\lambda_n(y,t)\right) + \int_{y}^{y+(1-y)/n} |g(t) - g(y)|\, d_t\left(\lambda_n(y,t)\right) \\ &= I_{2,1}(n,y) + I_{2,2}(n,y). \end{aligned}$$

Let

$$H(t) := \int_t^y |g(z) - g(y)|\, dz,$$

then, according to (1.9), to each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$H(t) \leq \varepsilon\,(y - t) \tag{2.2}$$

for all $0 < y - t \leq \delta$.

We now fix this δ and estimate $I_{2,1}(n,y)$ and $I_{2,2}(n,y)$ respectively.

Now, by invoking the Lebesgue-Stieltjes integral representation, we can express $I_{2,1}(n,x)$ as follows:

$$I_{2,1}(n,y) = \int_{y-y/n}^{y} d_t\left(\lambda_n(y,t)\right) d\left(H(t)\right). \tag{2.3}$$

By utilizing partial Lebesgue-Stieltjes integration (2.3) and using (2.2) we obtain,

$$\begin{aligned} I_{2,1}(n,y) &= -H(y-y/n)\, d_t\left(\lambda_n(y,y-y/n)\right) + \int_{y-y/n}^{y} H(t)\frac{\partial}{\partial t} d_t\left(\lambda_n(y,t)\right)\, dt \\ &\leq \varepsilon\, y/n\, d_t\left(\lambda_n(y,y-y/n)\right) + \varepsilon \int_{y-y/n}^{y} (y-t)\frac{\partial}{\partial t} d_t\left(\lambda_n(y,t)\right)\, dt. \end{aligned}$$

Integration by parts again gives

$$\begin{aligned} I_{2,1}(n,y) &\leq \varepsilon\, y/n\, d_t\left(\lambda_n(y,y-y/n)\right) \\ &+ \varepsilon \left\{ -y/n d_t\left(\lambda_n(y,y-y/n)\right) + \int_{y-y/n}^{y} d_t\left(\lambda_n(y,t)\right) \right\} \\ &= \varepsilon \int_{y-y/n}^{y} d_t\left(\lambda_n(y,t)\right). \end{aligned}$$

Setting

$$I_{2,1,1}(n,y) := \int_{-y/n}^{0} d_t\left(\lambda_n(y,t+y)\right),$$

according to (1.7) Consequently, we are able to derive the subsequent estimation:

$$\begin{aligned} I_{2,1}(n,y) &= \varepsilon I_{2,1,1}(n,y) = \varepsilon \left(\lambda_n\left(y, y-\frac{y}{n}\right)\right) \int_{-y/n}^{0} dt \\ &\leq \varepsilon B(y) y^{-\beta} n^{\beta-1}. \end{aligned}$$

By using a similar method for $I_{2,2}(n,y)$, and in view of (1.8) we deduce the following inequality,

$$I_{2,2}(n,y) \leq \varepsilon B(y)\left(1-y\right)^{-\beta} n^{\beta-1}.$$

Next, we estimate $I_1(n,y)$. By utilizing partial Lebesgue-Stieltjes integration, we obtain

$$\begin{aligned} |I_1(n,y)| &= \int_{0}^{y-y/n} |g_y(t)|\, d_t\left(\lambda_n(y,t)\right) \\ &= \left|g_y\left(y-\frac{y}{n}\right)\right| \lambda_n\left(y, y-\frac{y}{n}\right) - \int_{0}^{y-y/n} \lambda_n(y,t) d_t\left(|g_y(t)|\right). \end{aligned}$$

Let $z = y - y/n$. By (1.7), it is clear that

$$\lambda_n(y,z) \le B(y) y^{-\beta} n^{\beta-1}. \tag{2.4}$$

Here we note that

$$\left|g_y\left(y-\frac{y}{n}\right)\right| = \left|\left|g_y\left(y-\frac{y}{n}\right)\right| - |g_y(y)|\right| \le \bigvee_{y-y/n}^{y}(|g_y|).$$

Using partial integration and applying (2.4), we obtain

$$\begin{aligned}
|I_1(n,y)| &\le \bigvee_{y-y/n}^{y}(|g_y|)\left|\lambda_n\left(y,y-\frac{y}{n}\right)\right| + \int_0^{y-y/n} \lambda_n(y,t)d_t\left(-\bigvee_t^y(|g_y|)\right) \\
&\le \bigvee_{y-y/n}^{y}(|g_y|)B(y)y^{-\beta}n^{\beta-1} + \frac{B(y)}{n}\int_0^{y-y/n}(y-t)^{-\beta}d_t\left(-\bigvee_t^y(|g_y|)\right) \\
&= \bigvee_{y-y/n}^{y}(|g_y|)B(y)y^{-\beta}n^{\beta-1} + \frac{B(y)}{n}\left[-y^{-\beta}n^{\beta}\bigvee_{y-y/n}^{y}(|g_y|)\right. \\
&\quad \left. + y^{-\beta}\bigvee_0^y(|g_y|) + \int_0^{y-y/n}\bigvee_t^y(|g_y|)\frac{\beta}{(y-t)^{\beta+1}}dt\right] \\
&= \frac{B(y)}{n^{\beta}}\left[y^{-\beta}\bigvee_0^y(|g_y|) + \int_0^{y-y/n}\bigvee_t^y(|g_y|)\frac{\beta}{(y-t)^{\beta+1}}dt\right].
\end{aligned}$$

Changing the variable t by $y - y/u^{1/\beta}$ in the last integral, we have

$$\begin{aligned}
\int_0^{y-y/n}\bigvee_t^y(|g_y|)\frac{\beta}{(y-t)^{\beta+1}}dt &= \frac{1}{y^{\beta}}\int_1^{n^{\beta}}\bigvee_{y-y/u^{1/\beta}}^{y}(|g_y|)\,du \\
&\le \frac{1}{y^{\beta}}\sum_{\nu=1}^{[n^{\beta}]}\bigvee_{y-y/\nu^{1/\beta}}^{y}(|g_y|).
\end{aligned}$$

Consequently, we obtain

$$|I_1(n,y)| \le \frac{B(y)}{n^{\beta}}y^{-\beta}\left[\bigvee_0^y(|g_y|) + \sum_{\nu=1}^{[n^{\beta}]}\bigvee_{y-y/\nu^{1/\beta}}^{y}(|g_y|)\right].$$

By employing a similar approach, we can deduce

$$|I_3(n,y)| \leq \frac{B(y)}{n^{\beta}}(1-y)^{-\beta}\left[\bigvee_{y}^{1}(|g_y|)+\sum_{\nu=1}^{\left[n^{\beta}\right]}\bigvee_{y}^{y+(1-y)/\nu^{1/\beta}}(|g_y|)\right].$$

By combining the aforementioned estimations, we arrive at the desired outcome. Q.E.D.

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