Generalized framed helices in Euclidean 3-space

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Abstract

In this study, we introduce framed k-slant helices from the viewpoint of singularity theory in Euclidean 3-space. Framed k-slant helices are regular or singular curves in Euclidean 3-space. Particularly, the family of framed k-slant helices encompasses a comprehensive set of curves, including framed helices, framed slant helices, and framed clad helices in the category of framed curves, as well as helices, slant helices, regular clad helices, and g-clad curves. We construct successive alternative adapted frames to define the framed k-slant helix and express its axis. We obtain a characterization of the framed (k+1)-slant helix in terms of the k-alternative adapted frame. Moreover, we discover interesting relationships between the framed k-slant helix and its generalized spherical images with respect to the adapted frame. Finally, we display appealing visualizations of the framed helix, framed slant helix, and framed clad helix as regular or smooth curves with singular points.

2010 Mathematics Subject Classification. **53A04**. 58K05.
Keywords. framed curve, helix, slant helix, clad helix, k-slant helix, singularity.

1 Introduction

A regular space curve in Euclidean space does not have any singular points, allowing the curvature and torsion functions to be well-defined at every point along the curve. However, not all space curves share this property, as they may possess singular points where the Frenet-Serret frame fails to be defined. Nevertheless, Honda and Takahashi [13] introduced the idea of a framed curve that includes regular and singular space curves with a moving frame in Euclidean space. They defined framed curvature functions that work even at singular points, similar to regular curves. Additionally, Fukunaga and Takahashi [11] studied the conditions for the existence of framed curves. Moreover, Wang et al. [37] proposed an adapted frame as a substitute for a moving frame of a framed curve in Euclidean space. The adapted frame includes the generalized tangent vector, generalized principal normal vector, and generalized binormal vector. Naturally, the theory of framed curves, which includes regular curves as well, has captured the interest of researchers. As a result, concepts that traditionally belonged to the category of regular curves have begun to be transferred to the theory of framed curves.

Recently, researchers have introduced the concepts of framed helix [14], framed slant helix [29], framed clad helix [26], framed rectifying curve [37, 39], framed normal curve [40], framed Bertrand and Mannheim curves [15]. Furthermore, the references [9, 16, 38] are noteworthy other studies that contribute to the theory of framed curves. Furthermore, Legendre curves belong to the category of framed curves as a special case. Therefore, references [10, 21, 22, 24, 32, 35, 42] are other notable studies that contribute to the field of framed curves, specifically in the category of frontal or front curves.

Advanced Studies: Euro-Tbilisi Mathematical Journal 16, supplement issue 3 (2023), pp. 59–76.
DOI: 10.32513/asetmj/1932200823305

Tbilisi Centre for Mathematical Sciences.

Received by the editors: 08 july 2023.
Accepted for publication: 05 October 2023.

In differential geometry, a curve is called a helix curve or helical curve if the vector field associated with its Frenet frame makes a constant angle with a fixed direction along the curve. The helical curve finds applications across various fields of mathematics and other sciences due to its distinctive shape, proving to be a valuable tool for addressing a wide array of problems. For instance, in physics, it helps analyze the motion of particles traveling in circular paths at constant velocities. In engineering, it aids in the design of mechanical components like screws, bolts, and springs. In chemistry, it sheds light on the structures of certain chiral molecules and crystals. Similarly, in biology, it provides insights into the structures of significant molecules such as DNA and proteins. The versatility of the helical curve makes it a fundamental concept, with extensive utility.

In the classical sense, a helix in Euclidean 3-space was first defined by Lancret in 1802 using the tangent vector of the curve. Saint-Venant characterized it in 1845 as a curve whose ratio of its curvatures (τ/κ) is a constant. A helix is also called a general helix or cylindrical helix. Especially, if its curvatures are non-zero constants, then it is called a circular helix. By using the principal normal vector of the curve, Izumiya and Takeuchi introduced the slant helix, which is the first generalization of the helix in 2004. By the same authors, the slant helix is characterized by the constant function σ which corresponds to the geodesic curvature of the spherical image of the unit principal normal of the curve [17]. Scofield introduced a new helical curve whose Darboux vector field (centrode) revolves around a fixed line in space with a constant angle and constant speed. It is called a curve of constant precession [31]. Kula and Yaylı proved that the tangent, principal normal, and binormal images of a slant helix are spherical helices, a circle, and a spherical helix, respectively. Also, they show that a curve of constant precession is a slant helix [18]. By considering a similar idea to the slant helix, Takahashi and Takeuchi in 2014 defined clad helix and g-clad helix if the spherical image of the unit principal normal is a spherical helix and a spherical slant helix, respectively. They give characterizations of the clad helix and g-clad helix by constant functions φ and ψ, respectively [33]. However, as a consecutive slant helix, Ali in 2012 defined a k-slant helix by using a sequentially k-vector field. He examined the axis and characterization for k-slant helices with respect to the Frenet apparatus [1]. We remark that 0-type, 1-type, 2-type, and 3-type slant helix correspond to helix, slant helix, clad helix, and g-clad helix, respectively. Moreover, Uzunoglu et al. in 2016 defined an alternative frame that is generated by the Frenet frame. They defined the C-slant helix by using the vector field C and gave its axis with respect to the alternative frame and its characterization [36]. However, we point out that the C-slant helix corresponds to clad helix (or 2-slant helix) and the vector field C corresponds to the co-Darboux vector field (co-centrode) with respect to the Frenet frame of a regular space curve. Additionally, references [2-6, 8, 12, 19, 23, 25, 27, 30, 33, 34, 41] demonstrate that the research on curves belonging to the helical type within the category of regular curves has maintained its popularity up to the present day. This indicates the enduring significance and interest in studying helical curves.

The references [1, 33, 36] in the category of regular curves, along with references [26, 29] in the category of framed curves, have served as sources of motivation for the creation of this article. Therefore, the aim of this study is to introduce framed k-slant helices from the viewpoint of singularity theory in Euclidean-3 space. Framed k-slant helices are smooth curves which may have singular points in Euclidean-3 space. The family of framed k-slant helices includes the concepts of framed helix [14], framed slant helix [29], and framed clad helix [26] in the category of framed curves as well as helix, slant helix, clad helix, and g-clad helix in the category of regular curves.

The design of the article is as follows: In Section 2, as a motivation, we provide background information on the theory of framed curves in Euclidean 3-space. Especially, in Section 2.1, we give

the construction method of k-alternative adapted frames generated through a consecutive procedure from the adapted frame of a framed curve, which enables us to define framed k-slant helices. Also, we define the unit framed type-(k+1) Darboux vector and the unit framed type-(k+1) co-Darboux vector with respect to the k-alternative adapted frame. In Section 3, we introduce framed (k+1)-slant helix whose the unit framed type-k co-Darboux vector makes a constant angle with a unit fixed direction (i.e. its axis). We give a form of the axis of the framed (k+1)-slant helix with respect to the k-alternative adapted frame. After, we get a characterization for framed (k+1)-slant helix with respect to the k-alternative adapted frame by the constant function $\boldsymbol{\sigma}_{k+1}$. Moreover, we obtain interesting results for framed spherical indicatrices of framed k-slant helix with respect to the adapted frame. Finally, we enrich the concept of generalized framed helices with several illustrative examples that feature visually appealing graphics.

2 Preliminary

Let $\mathbb{R}^3$ denote the Euclidean 3-space, that is, the 3-dimensional real vector space endowed with the standard inner product $\langle x, y\rangle = \sum_{i=1}^{3} x_i y_i$, for all $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$. The norm of a vector $x \in \mathbb{R}^3$ is defined by $\|x\| = \sqrt{\langle x, x\rangle}$. Also, the cross product of vectors x and y is given by $x \wedge y = (x_2y_3 - x_3y_2, -x_1y_3 + x_3y_1, x_1y_2 - x_2y_1)$.

Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ be a space curve. If $\boldsymbol{\gamma}'(t) = \frac{d\boldsymbol{\gamma}}{dt}(t) \neq 0$ for all $t \in I$, then $\boldsymbol{\gamma}$ is called a regular curve. Also, If $\boldsymbol{\gamma}'(t_0) = 0$ at any $t_0 \in I$ then t_0 is called a singular point of $\boldsymbol{\gamma}$. It is easy to see that the Frenet frame of any space curve is not well-defined at any singular point of the curve.

Now, let's give the theory of framed curve, which is a regular or singular space curve in $\mathbb{R}^3$ (see [11,13,14,37] for more detail and background).

Let us take the set $\Delta = \left\{\boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2) \in \mathbb{S}^2 \times \mathbb{S}^2 \,|\langle \boldsymbol{u}_1, \boldsymbol{u}_2\rangle = 0\right\}$ as a 3-dimensional manifold.

Definition 2.1. $(\boldsymbol{\gamma}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) : I \to \mathbb{R}^3 \times \Delta \subset \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{S}^2$ is called a framed curve, if $\langle \boldsymbol{\gamma}'(t), \boldsymbol{\mu}_i(t)\rangle = 0$ for all $t \in I$. $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ is also called a framed curve (or framed base curve) if there exists $\boldsymbol{\mu} = (\boldsymbol{\mu}_1, \boldsymbol{\mu}_2) : I \to \Delta$ such that $(\boldsymbol{\gamma}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2)$ is a framed curve [13].

Now, unlike the Frenet frame, a well-defined moving frame can be constructed along the framed curve $\boldsymbol{\gamma}$, which may have singular points. Let $(\boldsymbol{\gamma}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2)$ be a framed curve, and let $\boldsymbol{\vartheta} : I \to \mathbb{S}^2$ be a regular spherical curve such that $\boldsymbol{\vartheta}(t) = \boldsymbol{\mu}_1(t) \wedge \boldsymbol{\mu}_2(t)$ for all $t \in I$. Hence, $\{\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\vartheta}\}$ is an orthonormal frame, which is a moving frame along the framed curve $\boldsymbol{\gamma}$ in $\mathbb{R}^3$. Then, the Frenet-Serret type formulas are given by:

$$\begin{aligned}
\boldsymbol{\mu}_1'(t) &= \mathfrak{l}(t)\boldsymbol{\mu}_2(t) + \mathfrak{m}(t)\boldsymbol{\vartheta}(t),\\
\boldsymbol{\mu}_2'(t) &= -\mathfrak{l}(t)\boldsymbol{\mu}_1(t) + \mathfrak{n}(t)\boldsymbol{\vartheta}(t),\\
\boldsymbol{\vartheta}'(t) &= -\mathfrak{m}(t)\boldsymbol{\mu}_1(t) - \mathfrak{n}(t)\boldsymbol{\mu}_2(t),
\end{aligned}$$

and there exists a smooth function $\mathfrak{a} : I \to \mathbb{R}$ such that

$$\boldsymbol{\gamma}'(t) = \mathfrak{a}(t)\boldsymbol{\vartheta}(t). \tag{2.1}$$

Here, the quadruple smooth functions $(\mathfrak{l}, \mathfrak{m}, \mathfrak{n}, \mathfrak{a}) = (\langle \boldsymbol{\mu}_1', \boldsymbol{\mu}_2\rangle, \langle \boldsymbol{\mu}_1', \boldsymbol{\vartheta}\rangle, \langle \boldsymbol{\mu}_2', \boldsymbol{\vartheta}\rangle, \langle \boldsymbol{\gamma}', \boldsymbol{\vartheta}\rangle)$ are called the curvature of the framed curve $\boldsymbol{\gamma}$.

Remark 2.2. It is clear that if $t_0 \in I$ is a singular point of $\boldsymbol{\gamma}$, then $\mathfrak{a}(t_0) = 0$. Moreover, since we suppose that $\boldsymbol{\vartheta}$ is a regular spherical curve (i.e. that $\boldsymbol{\vartheta}'(t) \neq 0$) then $(\mathfrak{m}(t), \mathfrak{n}(t)) \neq (0,0)$ for all $t \in I$.

Similar to Bishop frame [7] of regular curves, Wang et al. [37] give the following adapted frame which is an alternative to the moving frame of the framed curve:

Let $(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in \Delta$ and $\theta : I \to \mathbb{R}$ be a smooth function such that

$$\begin{pmatrix} \boldsymbol{\eta}_1(t) \\ \boldsymbol{\eta}_2(t) \end{pmatrix} = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \begin{pmatrix} \boldsymbol{\mu}_1(t) \\ \boldsymbol{\mu}_2(t) \end{pmatrix}.$$

It is easy to see that $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) : I \to \mathbb{R}^3 \times \Delta$ is also a framed curve and $\boldsymbol{\vartheta} = \boldsymbol{\mu}_1 \wedge \boldsymbol{\mu}_2 = \boldsymbol{\eta}_1 \wedge \boldsymbol{\eta}_2$. Now, we assume that $\mathfrak{m}(t) = -\mathfrak{p}(t)\cos\theta(t)$ and $\mathfrak{n}(t) = \mathfrak{p}(t)\sin\theta(t)$ such that $\mathfrak{m}(t)\sin\theta(t) + \mathfrak{n}(t)\cos\theta(t) = 0$, then we have an adapted frame $\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2\}$ along the framed curve $\boldsymbol{\gamma}$ and the following Frenet-Serret type formulas:

$$\boldsymbol{\vartheta}'(t) = \mathfrak{p}(t)\boldsymbol{\eta}_1(t), \quad \boldsymbol{\eta}_1'(t) = -\mathfrak{p}(t)\boldsymbol{\vartheta}(t) + \mathfrak{q}(t)\boldsymbol{\eta}_2(t), \quad \boldsymbol{\eta}_2'(t) = -\mathfrak{q}(t)\boldsymbol{\eta}_1(t), \tag{2.2}$$

where

$$\begin{cases} \mathfrak{p} = \langle \boldsymbol{\vartheta}', \boldsymbol{\eta}_1 \rangle = \|\boldsymbol{\vartheta}'\| = \sqrt{\mathfrak{m}^2 + \mathfrak{n}^2} > 0 \\ \mathfrak{q} = \langle \boldsymbol{\eta}_1', \boldsymbol{\eta}_2 \rangle = (\mathfrak{l} - \theta') = \mathfrak{l} + \left(\dfrac{\mathfrak{m}^2}{\mathfrak{m}^2 + \mathfrak{n}^2}\right)\left(\dfrac{\mathfrak{n}}{\mathfrak{m}}\right)' \\ \mathfrak{a} = \langle \boldsymbol{\gamma}', \boldsymbol{\vartheta} \rangle \end{cases} \tag{2.3}$$

The triple smooth functions $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a})$ are called framed curvature with respect to the adapted frame $\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2\}$ along the framed curve $\boldsymbol{\gamma}$. Moreover, the vectors $\boldsymbol{\vartheta}(t), \boldsymbol{\eta}_1(t), \boldsymbol{\eta}_2(t)$ are called the generalized tangent vector, the generalized principle normal vector, and the generalized binormal vector of the framed curve, respectively.

Proposition 2.3. Let $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with framed curvature $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a})$. If the framed curve $\boldsymbol{\gamma}$ is a regular curve with curvature κ and torsion τ, then we have $\kappa = \frac{\mathfrak{p}}{|\mathfrak{a}|}$ and $\tau = \frac{\mathfrak{q}}{\mathfrak{a}}$ [37].

Now, we give the following framed versions (i.e. that generalized versions) of well-known definitions and characterizations for regular space curves.

Definition 2.4. Let $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta$. Then, $\boldsymbol{\gamma}$ is called a framed spherical curve if it lies on a sphere with a radius r in $\mathbb{R}^3$ [37].

Theorem 2.5. Let $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta$ with framed curvature $(\mathfrak{p}, \mathfrak{q} = 0, \mathfrak{a})$. Then, $\boldsymbol{\gamma}$ is a framed spherical curve which is a circle in $\mathbb{R}^3$ if and only if $\mathfrak{q} = 0$ and $\frac{\mathfrak{p}}{|\mathfrak{a}|}$ is a constant such that $\mathfrak{a} \neq 0$ [26].

Theorem 2.6. Let $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta$ with framed curvature $(\mathfrak{p}, \mathfrak{q} \neq 0, \mathfrak{a})$. Then, $\boldsymbol{\gamma}$ is a framed spherical curve in $\mathbb{S}^2(r)$ if and only if

$$\left(\frac{1}{\mathfrak{q}}\left(\frac{\mathfrak{a}}{\mathfrak{p}}\right)'\right)^2 + \left(\frac{\mathfrak{a}}{\mathfrak{p}}\right)^2 = r^2$$

or equivalently,

$$\left(\frac{1}{\mathfrak{q}}\left(\frac{\mathfrak{a}}{\mathfrak{p}}\right)'\right)' + \frac{\mathfrak{a}\,\mathfrak{q}}{\mathfrak{p}} = 0.$$

[40].

Definition 2.7. Let $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta$ with framed curvature $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a})$. Then, the framed harmonic curvature of $\boldsymbol{\gamma}$ is given by $\mathfrak{h} = \frac{\mathfrak{q}}{\mathfrak{p}}$ [29].

Definition 2.8. Let $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta$ with adapted frame $\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2\}$. Then, $\boldsymbol{\gamma}$ is called a framed helix if its generalized tangent vector $\boldsymbol{\nu}$ makes a constant angle with a fixed unit vector $\boldsymbol{\zeta}$ [14, 37].

Theorem 2.9. Let $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta$ with framed curvature $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a})$. Then, $\boldsymbol{\gamma}$ is a framed helix if and only if $\sigma_0 = \mathfrak{h} = \cot\varphi$ such that φ is a constant angle [37].

Definition 2.10. Let $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta$ with adapted frame $\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2\}$. Then, $\boldsymbol{\gamma}$ is called a framed slant helix if its generalized principal normal vector $\boldsymbol{\eta}_1$ makes a constant angle with a fixed unit vector $\boldsymbol{\zeta}$. That is, $\langle \boldsymbol{\eta}_1, \boldsymbol{\zeta} \rangle = \cos\varphi$ where $\varphi \neq \pi/2$ is a constant angle [29].

Theorem 2.11. Let $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta$ with framed curvature $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a})$. Then, $\boldsymbol{\gamma}$ is a framed slant helix if and only if

$$\boldsymbol{\sigma}_1 = \frac{\mathfrak{p}^2\left(\frac{\mathfrak{q}}{\mathfrak{p}}\right)'}{\left(\mathfrak{p}^2 + \mathfrak{q}^2\right)^{3/2}} = \frac{\mathfrak{h}'}{\mathfrak{p}(1+\mathfrak{h}^2)^{3/2}} \tag{2.4}$$

is a constant function [29].

Definition 2.12. Let $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta$ with adapted frame apparatus $\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, (\mathfrak{p}, \mathfrak{q}, \mathfrak{a})\}$. Then, the framed Darboux (or type-1 Darboux) vector of the framed curve $\boldsymbol{\gamma}$ is defined by $\boldsymbol{\Omega}(t) = \mathfrak{q}(t)\boldsymbol{\vartheta}(t) + \mathfrak{p}(t)\boldsymbol{\eta}_2(t)$ which satisfies the following equations:

$$\boldsymbol{\vartheta}'(t) = \boldsymbol{\Omega}(t) \wedge \boldsymbol{\vartheta}(t), \quad \boldsymbol{\eta}_1'(t) = \boldsymbol{\Omega}(t) \wedge \boldsymbol{\eta}_1(t), \quad \boldsymbol{\eta}_2'(t) = \boldsymbol{\Omega}(t) \wedge \boldsymbol{\eta}_2(t).$$

Moreover, we call that

$$\boldsymbol{\Omega}_1(t) = \frac{\mathfrak{q}(t)\boldsymbol{\vartheta}(t) + \mathfrak{p}(t)\boldsymbol{\eta}_2(t)}{\sqrt{\mathfrak{p}^2(t) + \mathfrak{q}^2(t)}} \tag{2.5}$$

is the unit framed Darboux (or type-1 Darboux) vector of $\boldsymbol{\gamma}$ [26].

Definition 2.13. Let $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta$ with adapted frame apparatus $\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, (\mathfrak{p}, \mathfrak{q}, \mathfrak{a})\}$. Then, the framed co-Darboux (or type-1 co-Darboux) vector of the framed curve $\boldsymbol{\gamma}$ is defined by $\widehat{\boldsymbol{\Omega}}(t) = -\mathfrak{p}(t)\boldsymbol{\vartheta}(t) + \mathfrak{q}(t)\boldsymbol{\eta}_2(t)$. Moreover, we call that

$$\widehat{\boldsymbol{\Omega}}_1(t) = \frac{-\mathfrak{p}(t)\boldsymbol{\vartheta}(t) + \mathfrak{q}(t)\boldsymbol{\eta}_2(t)}{\sqrt{\mathfrak{p}^2(t) + \mathfrak{q}^2(t)}} \tag{2.6}$$

is the unit framed co-Darboux (or type-1 Darboux) vector of $\boldsymbol{\gamma}$. Also, it is easy to see that $\widehat{\boldsymbol{\Omega}}_1 = \frac{\boldsymbol{\eta}_1'}{\|\boldsymbol{\eta}_1'\|}$ [26].

Definition 2.14. Let $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta$. Then $\boldsymbol{\gamma}$ is called a framed clad helix (or co-Darboux helix) with the axis $\boldsymbol{u}$ if its unit framed co-Darboux vector $\widehat{\boldsymbol{\Omega}}_1$ makes a constant angle $\varphi \neq \frac{\pi}{2}$ with a fixed unit vector $\boldsymbol{u}$. That is,

$$\left\langle \widehat{\boldsymbol{\Omega}}_1, \boldsymbol{u} \right\rangle = \cos\varphi, \quad \varphi = \text{constant} \neq \frac{\pi}{2}$$

along the framed curve $\boldsymbol{\gamma}$ [26].

Theorem 2.15. Let $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta$ with alternative adapted frame apparatus $\left\{\boldsymbol{\eta}_1, \widehat{\boldsymbol{\Omega}}_1, \boldsymbol{\Omega}_1, (\mathfrak{f}_1, \mathfrak{g}_1, \mathfrak{a})\right\}$. Then, $\boldsymbol{\gamma}$ is a framed clad helix if and only if

$$\boldsymbol{\sigma}_2 = \frac{\boldsymbol{\sigma}_1'}{\mathfrak{f}_1(1+\boldsymbol{\sigma}_1{}^2)^{\frac{3}{2}}} \tag{2.7}$$

is a constant function where $\boldsymbol{\sigma}_1$ and $\mathfrak{f}_1$ are given by (2.4) and (2.9), respectively [26].

2.1 Alternative adapted frames of framed curves

In [26], Mak introduced the concept of unit type-(k+1) Darboux and co-Darboux vectors with respect to the k-alternative adapted frame of a framed curve, where $k \in \mathbb{N}$. Now, we define the following alternative adapted frame, which is generated by the adapted frame $\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2\}$ along a framed curve $\boldsymbol{\gamma}$.

Definition 2.16. Let $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta$ with adapted (or 0-alternative adapted) frame apparatus $\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, (\mathfrak{p}, \mathfrak{q}, \mathfrak{a})\}$. Then, $\left\{\boldsymbol{\eta}_1, \widehat{\boldsymbol{\Omega}}_1 = \frac{\boldsymbol{\eta}_1'}{\|\boldsymbol{\eta}_1'\|}, \boldsymbol{\Omega}_1 = \boldsymbol{\eta}_1 \wedge \widehat{\boldsymbol{\Omega}}_1\right\}$ is called alternative (or 1-alternative) adapted frame along the framed curve $\boldsymbol{\gamma}$ with the following equations:

$$\boldsymbol{\eta}_1' = \mathfrak{f}_1 \widehat{\boldsymbol{\Omega}}_1, \quad \widehat{\boldsymbol{\Omega}}_1' = -\mathfrak{f}_1 \boldsymbol{\eta}_1 + \mathfrak{g}_1 \boldsymbol{\Omega}_1, \quad \boldsymbol{\Omega}_1' = -\mathfrak{g}_1 \widehat{\boldsymbol{\Omega}}_1, \tag{2.8}$$

where

$$\begin{cases} \mathfrak{f}_1 = \left\langle \boldsymbol{\eta}_1', \widehat{\boldsymbol{\Omega}}_1 \right\rangle = \sqrt{\mathfrak{p}^2 + \mathfrak{q}^2} = \mathfrak{p}\sqrt{1+\mathfrak{h}^2} > 0 \\ \mathfrak{g}_1 = \left\langle \widehat{\boldsymbol{\Omega}}_1', \boldsymbol{\Omega}_1 \right\rangle = \dfrac{\mathfrak{p}^2\left(\frac{\mathfrak{q}}{\mathfrak{p}}\right)'}{\mathfrak{p}^2+\mathfrak{q}^2} = \dfrac{\mathfrak{h}'}{1+\mathfrak{h}^2} = \boldsymbol{\sigma}_1 \mathfrak{f}_1 \\ \mathfrak{a} = \langle \boldsymbol{\gamma}', \boldsymbol{\vartheta} \rangle \end{cases} \tag{2.9}$$

and $\boldsymbol{\sigma}_1$ is given by (2.4). Also, the triple smooth functions $(\mathfrak{f}_1, \mathfrak{g}_1, \mathfrak{a})$ given by (2.8) are called framed curvature with respect to the alternative adapted frame $\left\{\boldsymbol{\eta}_1, \widehat{\boldsymbol{\Omega}}_1, \boldsymbol{\Omega}_1\right\}$ along the framed curve $\boldsymbol{\gamma}$ [26].

Let's give the definition of a generalized version of the unit framed Darboux vector and co-Darboux vector of the framed curve $\boldsymbol{\gamma}$.

Definition 2.17. Let $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta$ with k-alternative adapted frame apparatus $\left\{\widehat{\boldsymbol{\Omega}}_{k-1}, \widehat{\boldsymbol{\Omega}}_k = \frac{\widehat{\boldsymbol{\Omega}}'_{k-1}}{\|\widehat{\boldsymbol{\Omega}}'_{k-1}\|}, \boldsymbol{\Omega}_k = \widehat{\boldsymbol{\Omega}}_{k-1} \wedge \widehat{\boldsymbol{\Omega}}_k, (\mathfrak{f}_k, \mathfrak{g}_k, \mathfrak{a})\right\}$ for $k \in \mathbb{N}$. Then, the unit framed type-(k+1) Darboux vector and the unit framed type-(k+1) co-Darboux vector of the framed curve $\boldsymbol{\gamma}$ are defined by:

$$\boldsymbol{\Omega}_{k+1} = \frac{\mathfrak{g}_k \widehat{\boldsymbol{\Omega}}_{k-1} + \mathfrak{f}_k \boldsymbol{\Omega}_k}{\sqrt{\mathfrak{f}_k^2 + \mathfrak{g}_k^2}},$$
$$\widehat{\boldsymbol{\Omega}}_{k+1} = \frac{\widehat{\boldsymbol{\Omega}}'_k}{\left\|\widehat{\boldsymbol{\Omega}}'_k\right\|} = \frac{-\mathfrak{f}_k \widehat{\boldsymbol{\Omega}}_{k-1} + \mathfrak{g}_k \boldsymbol{\Omega}_k}{\sqrt{\mathfrak{f}_k^2 + \mathfrak{g}_k^2}},$$

with respect to the k-alternative adapted frame, respectively [26].

Let's give the definition of (k+1)-alternative adapted frame, which is generated by the k-alternative adapted frame of the framed curve $\boldsymbol{\gamma}$.

Definition 2.18. Let $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta$ with k-alternative adapted frame apparatus $\left\{\widehat{\boldsymbol{\Omega}}_{k-1}, \widehat{\boldsymbol{\Omega}}_k, \boldsymbol{\Omega}_k, (\mathfrak{f}_k, \mathfrak{g}_k, \mathfrak{a})\right\}$ for $k \in \mathbb{N}$. Then, $\left\{\widehat{\boldsymbol{\Omega}}_k, \widehat{\boldsymbol{\Omega}}_{k+1} = \frac{\widehat{\boldsymbol{\Omega}}'_k}{\|\widehat{\boldsymbol{\Omega}}'_k\|}, \boldsymbol{\Omega}_{k+1} = \widehat{\boldsymbol{\Omega}}_k \wedge \widehat{\boldsymbol{\Omega}}_{k+1}\right\}$ is called (k+1)-alternative adapted frame along the framed curve $\boldsymbol{\gamma}$ with the following equations:

$$\widehat{\boldsymbol{\Omega}}'_k = \mathfrak{f}_{k+1} \widehat{\boldsymbol{\Omega}}_{k+1}, \quad \widehat{\boldsymbol{\Omega}}'_{k+1} = -\mathfrak{f}_{k+1} \widehat{\boldsymbol{\Omega}}_k + \mathfrak{g}_{k+1} \boldsymbol{\Omega}_{k+1}, \quad \boldsymbol{\Omega}'_{k+1} = -\mathfrak{g}_{k+1} \widehat{\boldsymbol{\Omega}}_{k+1}, \tag{2.10}$$

where

$$\begin{cases} \mathfrak{f}_{k+1} = \left\langle \widehat{\boldsymbol{\Omega}}'_k, \widehat{\boldsymbol{\Omega}}_{k+1} \right\rangle = \sqrt{\mathfrak{f}_k^2 + \mathfrak{g}_k^2} = \mathfrak{f}_k \sqrt{1 + \boldsymbol{\sigma}_k^2} > 0, \\ \mathfrak{g}_{k+1} = \left\langle \widehat{\boldsymbol{\Omega}}'_{k+1}, \boldsymbol{\Omega}_{k+1} \right\rangle = \dfrac{\mathfrak{f}_k^2 \left(\frac{\mathfrak{g}_k}{\mathfrak{f}_k}\right)'}{\mathfrak{f}_k^2 + \mathfrak{g}_k^2} = \dfrac{\boldsymbol{\sigma}'_k}{1 + \boldsymbol{\sigma}_k^2} = \boldsymbol{\sigma}_{k+1} \mathfrak{f}_{k+1}, \\ \mathfrak{a} = \langle \boldsymbol{\gamma}', \boldsymbol{\vartheta} \rangle, \end{cases} \tag{2.11}$$

such that

$$\boldsymbol{\sigma}_{k+1} = \frac{\mathfrak{f}_k^2 \left(\frac{\mathfrak{g}_k}{\mathfrak{f}_k}\right)'}{\mathfrak{f}_{k+1}^3} = \frac{\boldsymbol{\sigma}'_k}{\mathfrak{f}_k (1 + \boldsymbol{\sigma}_k^2)^{\frac{3}{2}}}. \tag{2.12}$$

Also, the triple smooth functions $(\mathfrak{f}_{k+1}, \mathfrak{g}_{k+1}, \mathfrak{a})$ given by (2.11) are called framed curvature with respect to the (k+1)-alternative adapted frame along the framed curve $\boldsymbol{\gamma}$ [26].

Remark 2.19. In particular, we use the following notations for the 0-alternative adapted frame, which corresponds to the adapted frame of the framed curve $\boldsymbol{\gamma}$:

$$\left\{\widehat{\boldsymbol{\Omega}}_{-1} = \boldsymbol{\vartheta}, \widehat{\boldsymbol{\Omega}}_0 = \boldsymbol{\eta}_1, \boldsymbol{\Omega}_0 = \boldsymbol{\eta}_2\right\}, \quad \mathfrak{f}_0 = \mathfrak{p}, \ \mathfrak{g}_0 = \mathfrak{q}, \ \boldsymbol{\sigma}_0 = \mathfrak{h}. \tag{2.13}$$

3 Framed k-slant helices

In this section, we introduce framed k-slant helix in $\mathbb{R}^3$. Notably, framed 0-slant helix, framed 1-slant helix, and framed 2-slant helix correspond to framed helix, framed slant helix, and framed clad helix, respectively. So that, the family of framed k-slant helices include framed helices, framed slant helices, framed clad helices, and also regular k-slant helices. Next, we provide a characterization for framed (k+1)-slant helix and explicitly determine its axis with respect to the k-alternative adapted frame. Moreover, we obtain interesting results for framed spherical indicatrices of framed (k+3)-slant helix.

Now, we give the following definition of the framed (k+1)-slant helix and its axis with respect to the k-alternative adapted frame.

Definition 3.1. Let $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta$. Then $\boldsymbol{\gamma}$ is called a framed (k+1)-slant helix with axis $\boldsymbol{u}$ if its unit framed type-k co-Darboux vector $\widehat{\boldsymbol{\Omega}}_k$ makes a constant angle $\varphi \neq \frac{\pi}{2}$ with a fixed unit vector $\boldsymbol{u}$ such that

$$\left\langle \widehat{\boldsymbol{\Omega}}_k, \boldsymbol{u} \right\rangle = \cos\varphi, \quad \varphi = \text{constant} \neq \frac{\pi}{2} \tag{3.1}$$

along the framed curve $\boldsymbol{\gamma}$.

Theorem 3.2. Let $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta$ with the k-alternative adapted frame apparatus $\left\{\widehat{\boldsymbol{\Omega}}_{k-1}, \widehat{\boldsymbol{\Omega}}_k, \boldsymbol{\Omega}_k, (\mathfrak{f}_k, \mathfrak{g}_k, \mathfrak{a})\right\}$ for $k \in \mathbb{N}$. If $\boldsymbol{\gamma}$ is a framed (k+1)-slant helix, then its axis $\boldsymbol{u}$ is given by

$$\boldsymbol{u} = \left(\frac{\mathfrak{f}_k \boldsymbol{\sigma}_k \left(1 + \boldsymbol{\sigma}_k^2\right)}{\boldsymbol{\sigma}_k'} \widehat{\boldsymbol{\Omega}}_{k-1} + \widehat{\boldsymbol{\Omega}}_k + \frac{\mathfrak{f}_k \left(1 + \boldsymbol{\sigma}_k^2\right)}{\boldsymbol{\sigma}_k'} \boldsymbol{\Omega}_k \right) \cos\varphi, \tag{3.2}$$

where $\boldsymbol{\sigma}_k$ is given by (2.12).

Proof. Assume that $\boldsymbol{\gamma}$ is a framed (k+1)-slant helix with axis $\boldsymbol{u}$. Then, we have $\left\langle \widehat{\boldsymbol{\Omega}}_k, \boldsymbol{u} \right\rangle = \cos\varphi$. After successive differentiation of this equation and by using equations (2.10), (3.1), we get the following equations:

$$\begin{cases} -\mathfrak{f}_k \left\langle \widehat{\boldsymbol{\Omega}}_{k-1}, \boldsymbol{u} \right\rangle + \mathfrak{g}_k \left\langle \boldsymbol{\Omega}_k, \boldsymbol{u} \right\rangle = 0, \\ -\mathfrak{f}_k' \left\langle \widehat{\boldsymbol{\Omega}}_{k-1}, \boldsymbol{u} \right\rangle + \mathfrak{g}_k' \left\langle \boldsymbol{\Omega}_k, \boldsymbol{u} \right\rangle = \left(\mathfrak{f}_k^2 + \mathfrak{g}_k^2\right) \cos\varphi, \end{cases} \tag{3.3}$$

respectively. After the solution of system (3.3) and by using equations (2.11), we have

$$\begin{aligned} \left\langle \widehat{\boldsymbol{\Omega}}_{k-1}, \boldsymbol{u} \right\rangle &= \frac{\mathfrak{g}_k \left(\mathfrak{f}_k^2 + \mathfrak{g}_k^2\right)}{\mathfrak{f}_k^2 \left(\frac{\mathfrak{g}_k}{\mathfrak{f}_k}\right)'} \cos\varphi = \frac{\mathfrak{f}_k \boldsymbol{\sigma}_k \left(1 + \boldsymbol{\sigma}_k^2\right)}{\boldsymbol{\sigma}_k'} \cos\varphi \\ \left\langle \boldsymbol{\Omega}_k, \boldsymbol{u} \right\rangle &= \frac{\mathfrak{f}_k^2 + \mathfrak{g}_k^2}{\mathfrak{f}_k \left(\frac{\mathfrak{g}_k}{\mathfrak{f}_k}\right)'} \cos\varphi = \frac{\mathfrak{f}_k \left(1 + \boldsymbol{\sigma}_k^2\right)}{\boldsymbol{\sigma}_k'} \cos\varphi \end{aligned} \tag{3.4}$$

where $\boldsymbol{\sigma}_k$ is given by (2.12). Hence, it is clear that the axis $\boldsymbol{u}$ has the form of (3.2) from equations (3.1) and (3.4). Q.E.D.

Now, we can give the following characterization of the framed (k+1)-slant helix with respect to the k-alternative adapted frame.

Theorem 3.3. Let $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta$ with the k-alternative adapted frame apparatus $\left\{\widehat{\boldsymbol{\Omega}}_{k-1}, \widehat{\boldsymbol{\Omega}}_k, \boldsymbol{\Omega}_k, (\mathfrak{f}_k, \mathfrak{g}_k, \mathfrak{a})\right\}$ for $k \in \mathbb{N}$. Then, $\boldsymbol{\gamma}$ is a framed (k+1)-slant helix if and only if

$$\boldsymbol{\sigma}_{k+1} = \frac{\boldsymbol{\sigma}_k'}{\mathfrak{f}_k(1+\boldsymbol{\sigma}_k^2)^{\frac{3}{2}}}$$

is a constant function where $\boldsymbol{\sigma}_k$ is given by (2.12)

Proof. Suppose that $\boldsymbol{\gamma}$ is a framed (k+1)-slant helix with axis $\boldsymbol{u}$. Then, its axis $\boldsymbol{u}$ has the form of (3.2) with respect to the k-alternative adapted frame of $\boldsymbol{\gamma}$. Also, since $\boldsymbol{u}$ is a fixed unit vector, we get

$$\boldsymbol{\sigma}_{k+1} = \frac{\boldsymbol{\sigma}_k'}{\mathfrak{f}_k(1+\boldsymbol{\sigma}_k^2)^{\frac{3}{2}}} = \pm\cot\varphi \tag{3.5}$$

where φ is the constant angle between framed type-k co-Darboux vector $\widehat{\boldsymbol{\Omega}}_k$ and $\boldsymbol{u}$. Thus, it is clear that $\boldsymbol{\sigma}_{k+1}$ is a constant function which is given by (3.5).

On the other hand, suppose that $\boldsymbol{\sigma}_{k+1}$ is the constant function which is given by (3.5) and $\boldsymbol{u}$ is a vector field which has the form of (3.2) where φ is a differentiable function. After differentiating of (3.2) by using equations (2.10), we get

$$-\mathfrak{f}_k\left\langle\widehat{\boldsymbol{\Omega}}_{k-1}, \boldsymbol{u}\right\rangle + \mathfrak{g}_k\left\langle\boldsymbol{\Omega}_k, \boldsymbol{u}\right\rangle = -\varphi'\sin\varphi, \tag{3.6}$$

Now, if we take into account the equations $\left\langle\widehat{\boldsymbol{\Omega}}_{k-1}, \boldsymbol{u}\right\rangle$ and $\left\langle\boldsymbol{\Omega}_k, \boldsymbol{u}\right\rangle$ which have the form of (3.4) in equation (3.6), then $\varphi'\sin\varphi = 0$. That is, φ is a constant function. Thus, it leads to that the framed curve $\boldsymbol{\gamma}$ is a framed (k+1)-slant helix. Q.E.D.

Thus, we easily deduce the following result from Definition 3.1 and Theorem 3.3.

Corollary 3.4. Let $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta$ with the k-alternative adapted frame apparatus $\left\{\widehat{\boldsymbol{\Omega}}_{k-1}, \widehat{\boldsymbol{\Omega}}_k, \boldsymbol{\Omega}_k, (\mathfrak{f}_k, \mathfrak{g}_k, \mathfrak{a})\right\}$ for $k \in \mathbb{N}$. Then the following statements hold:

- framed 0-slant helix corresponds to the framed helix,
- framed 1-slant helix corresponds to the framed slant helix,
- framed 2-slant helix corresponds to the framed clad helix,

Especially in the classical sense, when $\boldsymbol{\gamma}$ be a regular framed curve, then the concepts of helix [20] (0-slant helix [1]), slant helix [17] (or 1-slant helix [1]), clad helix [33] (or C-slant helix [36]), g-clad helix [33], and k-slant helix [1] are obtained. Hence, we easily get the following result by using Proposition 2.3.

Corollary 3.5. Let $\boldsymbol{\gamma}$ be a unit speed framed curve (i.e., $\mathfrak{a}(t) = 1$ for all t) in $\mathbb{R}^3$. Then its framed curvature $(\mathfrak{p}, \mathfrak{q}, \mathfrak{a})$ is equal to $(\kappa, \tau, 1)$, and the following statements hold:

- If $\boldsymbol{\gamma}$ is a framed 0-slant helix, then it is a helix (or 0-slant helix) in $\mathbb{R}^3$,
- If $\boldsymbol{\gamma}$ is a framed 1-slant helix, then it is a slant helix (or 1-slant helix) in $\mathbb{R}^3$,
- If $\boldsymbol{\gamma}$ is a framed 2-slant helix, then it is a clad helix (or C-slant helix or 2-slant helix) in $\mathbb{R}^3$,
- If $\boldsymbol{\gamma}$ is a framed 3-slant helix, then it is a g-clad helix (or 3-slant helix) in $\mathbb{R}^3$,
- If $\boldsymbol{\gamma}$ is a framed k-slant helix, then it is a k-slant helix in $\mathbb{R}^3$,

In [26], Mak demonstrated that the vector fields $\boldsymbol{\Omega}_k$ and $\widehat{\boldsymbol{\Omega}}_k$ are framed spherical curves in $\mathbb{S}^2 \times \Delta$ and provided the definitions of framed spherical images of a framed curve with respect to the adapted frame $\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2\}$. Moreover, some framed spherical images of framed slant helices and framed clad helices were examined in [29] and [26], respectively.

Proposition 3.6. $\left(\boldsymbol{\Omega}_k, \widehat{\boldsymbol{\Omega}}_{k+1}, \boldsymbol{\Omega}_{k+1}\right)$, $\left(\widehat{\boldsymbol{\Omega}}_k, \widehat{\boldsymbol{\Omega}}_{k+2}, \boldsymbol{\Omega}_{k+2}\right)$ are framed spherical curves in $\mathbb{S}^2 \times \Delta$ [26].

Definition 3.7. Let $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta$ with adapted frame $\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2\}$ and the unit vectors $\boldsymbol{\Omega}_1, \widehat{\boldsymbol{\Omega}}_1$ be given by (2.5),(2.6), respectively. Then, the following framed spherical curves:

(**i**) $\left(\boldsymbol{\vartheta}, \widehat{\boldsymbol{\Omega}}_1, \boldsymbol{\Omega}_1\right)$, $\left(\boldsymbol{\eta}_1, \widehat{\boldsymbol{\Omega}}_2, \boldsymbol{\Omega}_2\right)$, $\left(\boldsymbol{\eta}_2, \widehat{\boldsymbol{\Omega}}_1, \boldsymbol{\Omega}_1\right)$ in $\mathbb{S}^2 \times \Delta$ are called generalized tangent (or framed $\boldsymbol{\vartheta}-$) indicatrix, generalized principal normal (or framed $\boldsymbol{\eta}_1-$) indicatrix, generalized binormal (or framed $\boldsymbol{\eta}_2-$)indicatrix of framed curve $\boldsymbol{\gamma}$, respectively.

(**ii**) $\left(\boldsymbol{\Omega}_1, \widehat{\boldsymbol{\Omega}}_2, \boldsymbol{\Omega}_2\right)$ and $\left(\widehat{\boldsymbol{\Omega}}_1, \widehat{\boldsymbol{\Omega}}_3, \boldsymbol{\Omega}_3\right)$ in $\mathbb{S}^2 \times \Delta$ are called framed Darboux (or framed $\boldsymbol{\Omega}_1-$) indicatrix and framed co-Darboux (or framed $\widehat{\boldsymbol{\Omega}}_1-$) indicatrix of framed curve $\boldsymbol{\gamma}$, respectively [26].

Proposition 3.8. Let $\left(\boldsymbol{\vartheta}, \widehat{\boldsymbol{\Omega}}_1, \boldsymbol{\Omega}_1\right)$ be framed $\boldsymbol{\vartheta}-$indicatrix of framed curve $\boldsymbol{\gamma}$. Then, the adapted frame apparatus of framed $\boldsymbol{\vartheta}-$indicatrix is given by $\left\{\boldsymbol{\eta}_1, \widehat{\boldsymbol{\Omega}}_1, \boldsymbol{\Omega}_1, (\mathfrak{f}_1, \mathfrak{g}_1, \mathfrak{p})\right\}$ [26].

Proposition 3.9. Let $\left(\boldsymbol{\eta}_1, \widehat{\boldsymbol{\Omega}}_2, \boldsymbol{\Omega}_2\right)$ be framed $\boldsymbol{\eta}_1-$indicatrix of framed curve $\boldsymbol{\gamma}$. Then, the adapted frame apparatus of framed $\boldsymbol{\eta}_1-$indicatrix is given by $\left\{\widehat{\boldsymbol{\Omega}}_1, \widehat{\boldsymbol{\Omega}}_2, \boldsymbol{\Omega}_2, (\mathfrak{f}_2, \mathfrak{g}_2, \mathfrak{f}_1)\right\}$ [26].

Proposition 3.10. Let $\left(\boldsymbol{\eta}_2, \pm\widehat{\boldsymbol{\Omega}}_1, \boldsymbol{\Omega}_1\right)$ be framed $\boldsymbol{\eta}_2-$indicatrix of framed curve $\boldsymbol{\gamma}$. Then, the adapted frame apparatus of framed $\boldsymbol{\eta}_2-$indicatrix is given by $\left\{\pm\boldsymbol{\eta}_1, \pm\widehat{\boldsymbol{\Omega}}_1, \boldsymbol{\Omega}_1, (\mathfrak{f}_1, \pm\mathfrak{g}_1, \mp\mathfrak{q})\right\}$ [26].

Proposition 3.11. Let $\left(\boldsymbol{\Omega}_1, \pm\widehat{\boldsymbol{\Omega}}_2, \boldsymbol{\Omega}_2\right)$ be framed $\boldsymbol{\Omega}_1-$indicatrix of framed curve $\boldsymbol{\gamma}$. Then, the adapted frame apparatus of framed $\boldsymbol{\Omega}_1-$indicatrix is given by $\left\{\pm\widehat{\boldsymbol{\Omega}}_1, \pm\widehat{\boldsymbol{\Omega}}_2, \boldsymbol{\Omega}_2, (\mathfrak{f}_2, \pm\mathfrak{g}_2, \mp\mathfrak{g}_1)\right\}$ [26].

Proposition 3.12. Let $\left(\widehat{\boldsymbol{\Omega}}_1, \widehat{\boldsymbol{\Omega}}_3, \boldsymbol{\Omega}_3\right)$ be framed $\widehat{\boldsymbol{\Omega}}_1-$indicatrix of framed curve $\boldsymbol{\gamma}$. Then, the adapted frame apparatus of framed $\widehat{\boldsymbol{\Omega}}_1-$indicatrix is given by $\left\{\pm\widehat{\boldsymbol{\Omega}}_2, \pm\widehat{\boldsymbol{\Omega}}_3, \boldsymbol{\Omega}_3, (\mathfrak{f}_3, \mathfrak{g}_3, \mathfrak{f}_2)\right\}$ [26].

Now, as a generalization of Theorem 4.1, Theorem 4.2, Theorem 4.3 in [29], and Corollary 3.4 in [26], we give the following theorem:

Theorem 3.13. Let $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ be a framed curve in $\mathbb{R}^3 \times \Delta$ with adapted frame apparatus $\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, (\mathfrak{p}, \mathfrak{q}, \mathfrak{a})\}$. Then $\boldsymbol{\gamma}$ is a framed (k+3)-slant helix if and only if the following statements are hold:

(i) framed $\boldsymbol{\vartheta}-$indicatrix of $\boldsymbol{\gamma}$ is a framed spherical (k+2)-slant helix in $\mathbb{S}^2 \times \Delta$,

(ii) framed $\boldsymbol{\eta}_1-$indicatrix of $\boldsymbol{\gamma}$ is a framed spherical (k+1)-slant helix in $\mathbb{S}^2 \times \Delta$,

(iii) framed $\boldsymbol{\eta}_2-$indicatrix of $\boldsymbol{\gamma}$ is a framed spherical (k+2)-slant helix in $\mathbb{S}^2 \times \Delta$,

(iv) framed $\boldsymbol{\Omega}_1$-indicatrix of $\boldsymbol{\gamma}$ is a framed spherical (k+1)-slant helix in $\mathbb{S}^2 \times \Delta$,

(v) framed $\widehat{\boldsymbol{\Omega}}_1-$indicatrix of $\boldsymbol{\gamma}$ is a framed spherical k-slant helix in $\mathbb{S}^2 \times \Delta$.

Proof. Suppose that $(\boldsymbol{\gamma}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ is a framed (k+3)-slant helix in $\mathbb{R}^3 \times \Delta$ with adapted frame apparatus $\{\boldsymbol{\vartheta}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, (\mathfrak{p}, \mathfrak{q}, \mathfrak{a})\}$. Then, by Theorem 3.3, there exists the constant function

$$\boldsymbol{\sigma}_{k+3} = \frac{\boldsymbol{\sigma}'_{k+2}}{\mathfrak{f}_{k+2}\left(1+\boldsymbol{\sigma}_{k+2}^2\right)^{\frac{3}{2}}}.$$

Now, we give the parts of proof.

Proof of **(i)**: By Proposition 3.8, framed $\boldsymbol{\vartheta}-$indicatrix of $\boldsymbol{\gamma}$ is given by a framed spherical curve $\boldsymbol{\alpha} = \boldsymbol{\vartheta}$ in $\mathbb{S}^2$ with framed curvature $(\mathfrak{p}_{\boldsymbol{\alpha}}, \mathfrak{q}_{\boldsymbol{\alpha}}, \mathfrak{a}_{\boldsymbol{\alpha}}) = (\mathfrak{f}_1, \mathfrak{g}_1, \mathfrak{p})$. Then, by using equations (2.4), (2.7), (2.9), (2.11), (2.12) and (2.13), we have

$$\begin{aligned}
&\boldsymbol{\sigma}_{0\boldsymbol{\alpha}} = \mathfrak{h}_{\boldsymbol{\alpha}} = \frac{\mathfrak{q}_{\boldsymbol{\alpha}}}{\mathfrak{p}_{\boldsymbol{\alpha}}} = \frac{\mathfrak{g}_1}{\mathfrak{f}_1} = \boldsymbol{\sigma}_1,\\
&\boldsymbol{\sigma}_{1\boldsymbol{\alpha}} = \frac{\mathfrak{h}'_{\boldsymbol{\alpha}}}{\mathfrak{p}_{\alpha}(1+\mathfrak{h}_{\boldsymbol{\alpha}}^2)^{\frac{3}{2}}} = \frac{\boldsymbol{\sigma}'_1}{\mathfrak{f}_1(1+\boldsymbol{\sigma}_1^2)^{\frac{3}{2}}} = \boldsymbol{\sigma}_2,\\
&\mathfrak{f}_{1\boldsymbol{\alpha}} = \mathfrak{p}_{\boldsymbol{\alpha}}\sqrt{1+\mathfrak{h}_{\boldsymbol{\alpha}}^2} = \mathfrak{f}_1\sqrt{1+\boldsymbol{\sigma}_1^2} = \mathfrak{f}_2,\\
&\boldsymbol{\sigma}_{2\boldsymbol{\alpha}} = \frac{\boldsymbol{\sigma}'_{1\boldsymbol{\alpha}}}{\mathfrak{f}_{1\boldsymbol{\alpha}}(1+\boldsymbol{\sigma}_{1\boldsymbol{\alpha}}^2)^{\frac{3}{2}}} = \frac{\boldsymbol{\sigma}'_2}{\mathfrak{f}_2(1+\boldsymbol{\sigma}_2^2)^{\frac{3}{2}}} = \boldsymbol{\sigma}_3,\\
&\vdots\\
&\mathfrak{f}_{(k+1)\boldsymbol{\alpha}} = \mathfrak{f}_{k\boldsymbol{\alpha}}\sqrt{1+\boldsymbol{\sigma}_{k\boldsymbol{\alpha}}^2} = \mathfrak{f}_{k+1}\sqrt{1+\boldsymbol{\sigma}_{k+1}^2} = \mathfrak{f}_{k+2},\\
&\boldsymbol{\sigma}_{(k+2)\boldsymbol{\alpha}} = \frac{\boldsymbol{\sigma}'_{(k+1)\boldsymbol{\alpha}}}{\mathfrak{f}_{(k+1)\boldsymbol{\alpha}}\left(1+\boldsymbol{\sigma}_{(k+1)\boldsymbol{\alpha}}^2\right)^{\frac{3}{2}}} = \frac{\boldsymbol{\sigma}'_{k+2}}{\mathfrak{f}_{k+2}\left(1+\boldsymbol{\sigma}_{k+2}^2\right)^{\frac{3}{2}}} = \boldsymbol{\sigma}_{k+3} = constant.
\end{aligned}$$

Thus, the desired result is obtained by Theorem 3.3.

Proof of **(ii)**: By Proposition 3.9, framed $\boldsymbol{\eta}_1-$indicatrix of $\boldsymbol{\gamma}$ is given by a framed spherical curve $\boldsymbol{\beta} = \boldsymbol{\eta}_1$ in $\mathbb{S}^2$ with framed curvature $(\mathfrak{p}_{\boldsymbol{\beta}}, \mathfrak{q}_{\boldsymbol{\beta}}, \mathfrak{a}_{\boldsymbol{\beta}}) = (\mathfrak{f}_2, \mathfrak{g}_2, \mathfrak{f}_1)$. Then, by using equations (2.11), framed harmonic curvature of $\boldsymbol{\beta}$ is

$$\boldsymbol{\sigma}_{0\boldsymbol{\beta}} = \mathfrak{h}_{\boldsymbol{\beta}} = \frac{\mathfrak{q}_{\boldsymbol{\beta}}}{\mathfrak{p}_{\boldsymbol{\beta}}} = \frac{\mathfrak{g}_2}{\mathfrak{f}_2} = \boldsymbol{\sigma}_2,$$

After applying the sequential procedure used in the proof of (i), we get

$$\mathfrak{f}_{k\boldsymbol{\beta}} = \mathfrak{f}_{(k-1)\boldsymbol{\beta}}\sqrt{1+\boldsymbol{\sigma}^2_{(k-1)\boldsymbol{\beta}}} = \mathfrak{f}_{k+1}\sqrt{1+\boldsymbol{\sigma}^2_{k+1}} = \mathfrak{f}_{k+2},$$

$$\boldsymbol{\sigma}_{(k+1)\boldsymbol{\beta}} = \frac{\boldsymbol{\sigma}'_{k\boldsymbol{\beta}}}{\mathfrak{f}_{k\boldsymbol{\beta}}\left(1+\boldsymbol{\sigma}^2_{k\boldsymbol{\beta}}\right)^{\frac{3}{2}}} = \frac{\boldsymbol{\sigma}'_{k+2}}{\mathfrak{f}_{k+2}\left(1+\boldsymbol{\sigma}^2_{k+2}\right)^{\frac{3}{2}}} = \boldsymbol{\sigma}_{k+3} = constant.$$

Thus, the proof is clear by Theorem 3.3.

Proof of **(iii)**: By Proposition 3.10, framed $\boldsymbol{\eta}_2-$indicatrix of $\boldsymbol{\gamma}$ is given by a framed spherical curve $\boldsymbol{\xi} = \boldsymbol{\eta}_2$ in $\mathbb{S}^2$ with framed curvature $(\mathfrak{p}_{\boldsymbol{\xi}}, \mathfrak{q}_{\boldsymbol{\xi}}, \mathfrak{a}_{\boldsymbol{\xi}}) = (\mathfrak{f}_1, \varepsilon\mathfrak{g}_1, -\varepsilon\mathfrak{q})$ such that $\varepsilon = \pm 1$. After applying the sequential procedure used in the proof of (i), we get

$$\mathfrak{f}_{(k+1)\boldsymbol{\xi}} = \mathfrak{f}_{k\boldsymbol{\xi}}\sqrt{1+\boldsymbol{\sigma}^2_{k\boldsymbol{\xi}}} = \mathfrak{f}_{k+1}\sqrt{1+\boldsymbol{\sigma}^2_{k+1}} = \mathfrak{f}_{k+2},$$

$$\boldsymbol{\sigma}_{(k+2)\boldsymbol{\xi}} = \frac{\varepsilon\boldsymbol{\sigma}'_{(k+1)\boldsymbol{\xi}}}{\mathfrak{f}_{(k+1)\boldsymbol{\xi}}\left(1+\boldsymbol{\sigma}^2_{(k+1)\boldsymbol{\xi}}\right)^{\frac{3}{2}}} = \frac{\varepsilon\boldsymbol{\sigma}'_{k+2}}{\mathfrak{f}_{k+2}\left(1+\boldsymbol{\sigma}^2_{k+2}\right)^{\frac{3}{2}}} = \varepsilon\boldsymbol{\sigma}_{k+3} = constant.$$

Thus, it leads to desired result.

Proof of **(iv)**: By Proposition 3.11, framed $\boldsymbol{\Omega}_1$-indicatrix of $\boldsymbol{\gamma}$ is given by a framed spherical curve $\boldsymbol{\omega} = \boldsymbol{\Omega}_1$ in $\mathbb{S}^2$ with framed curvature $(\mathfrak{p}_{\boldsymbol{\omega}}, \mathfrak{q}_{\boldsymbol{\omega}}, \mathfrak{a}_{\boldsymbol{\omega}}) = (\mathfrak{f}_2, \varepsilon\mathfrak{g}_2, -\varepsilon\mathfrak{g}_1)$. After applying the sequential procedure used in the proof of (ii), the proof is clear.

Proof of **(v)**: By Proposition 3.12, framed $\widehat{\boldsymbol{\Omega}}_1-$indicatrix of $\boldsymbol{\gamma}$ is given by a framed spherical curve $\hat{\boldsymbol{\omega}} = \widehat{\boldsymbol{\Omega}}_1$ in $\mathbb{S}^2$ with framed curvature $(\mathfrak{p}_{\hat{\boldsymbol{\omega}}}, \mathfrak{q}_{\hat{\boldsymbol{\omega}}}, \mathfrak{a}_{\hat{\boldsymbol{\omega}}}) = (\mathfrak{f}_3, \mathfrak{g}_3, \mathfrak{f}_2)$. Then, by using equations (2.11), framed harmonic curvature of $\hat{\boldsymbol{\omega}}$ is

$$\boldsymbol{\sigma}_{0\hat{\boldsymbol{\omega}}} = \mathfrak{h}_{\hat{\boldsymbol{\omega}}} = \frac{\mathfrak{q}_{\hat{\boldsymbol{\omega}}}}{\mathfrak{p}_{\hat{\boldsymbol{\omega}}}} = \frac{\mathfrak{g}_3}{\mathfrak{f}_3} = \boldsymbol{\sigma}_3,$$

After applying the sequential procedure used in the proof of (i), we get

$$\mathfrak{f}_{(k-1)\hat{\boldsymbol{\omega}}} = \mathfrak{f}_{(k-2)\hat{\boldsymbol{\omega}}}\sqrt{1+\boldsymbol{\sigma}^2_{(k-2)\hat{\boldsymbol{\omega}}}} = \mathfrak{f}_{k+1}\sqrt{1+\boldsymbol{\sigma}^2_{k+1}} = \mathfrak{f}_{k+2},$$

$$\boldsymbol{\sigma}_{k\hat{\boldsymbol{\omega}}} = \frac{\boldsymbol{\sigma}'_{(k-1)\hat{\boldsymbol{\omega}}}}{\mathfrak{f}_{(k-1)\hat{\boldsymbol{\omega}}}\left(1+\boldsymbol{\sigma}^2_{(k-1)\hat{\boldsymbol{\omega}}}\right)^{\frac{3}{2}}} = \frac{\boldsymbol{\sigma}'_{k+2}}{\mathfrak{f}_{k+2}\left(1+\boldsymbol{\sigma}^2_{k+2}\right)^{\frac{3}{2}}} = \boldsymbol{\sigma}_{k+3} = constant.$$

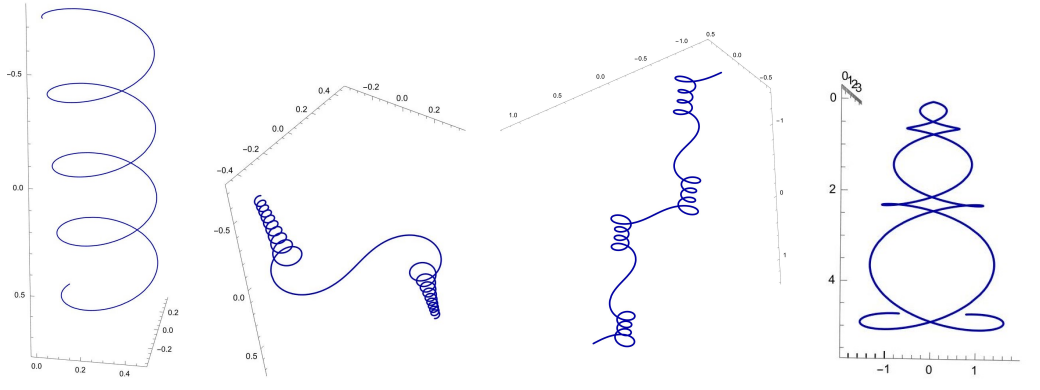
Thus, the proof is clear by Theorem 3.3. Q.E.D.

According to the Existence and Uniqueness Theorems of framed curves in [13], if the framed curvature of a framed curve is given, then we can draw a congruent graphic to the framed curve by applying the numerical solution method to Frenet-type differential equation system (2.1),(2.2),(2.3) with initial conditions. Hence, as an application of Theorem 3.3, we can give the following examples:

Example 3.14. Let $\boldsymbol{\gamma}$ be a framed curve with framed curvature

$$(\mathfrak{p}(t), \mathfrak{q}(t), \mathfrak{a}(t)) = \left(\frac{\lambda\, h(t)}{\mu},\, \lambda\, h(t),\, \mathfrak{a}(t)\right)$$

where λ, μ are nonzero constants, and $\mathfrak{a}(t), h(t)$ are smooth functions. Then, by using Theorem 2.9, we see that $\boldsymbol{\gamma}$ is a framed 0-slant helix (i.e. framed helix) such that $\boldsymbol{\sigma}_0 = \mu$ (see Figure 1).

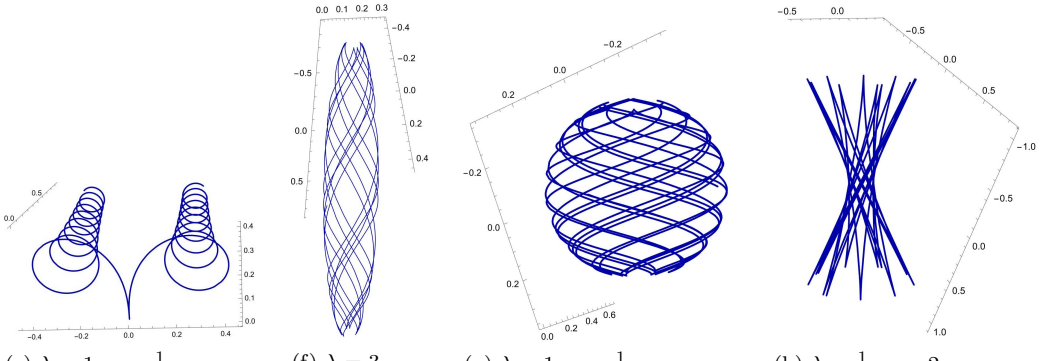


(a) $\lambda = 1,\ \mu = \frac{1}{4},$ $h(t) = 1, \mathfrak{a}(t) = 1$

(b) $\lambda = 3,\ \mu = \frac{1}{4},$ $h(t) = 1, \mathfrak{a}(t) = 1$

(c) $\lambda = 3,\ \mu = \frac{1}{4},$ $h(t) = \sin t, \mathfrak{a}(t) = 1$

(d) $\lambda = 2,\ \mu = 1,$ $h(t) = 1, \mathfrak{a}(t) = t$



(e) $\lambda = 1,\ \mu = \frac{1}{5},$ $h(t) = t^2, \mathfrak{a}(t) = t$

(f) $\lambda = 3,$ $\mu = 2,$ $h(t) = 1,$ $\mathfrak{a}(t) = \cos t$

(g) $\lambda = 1,\ \mu = \frac{1}{3},$ $h(t) = 1, \mathfrak{a}(t) = \cos t$

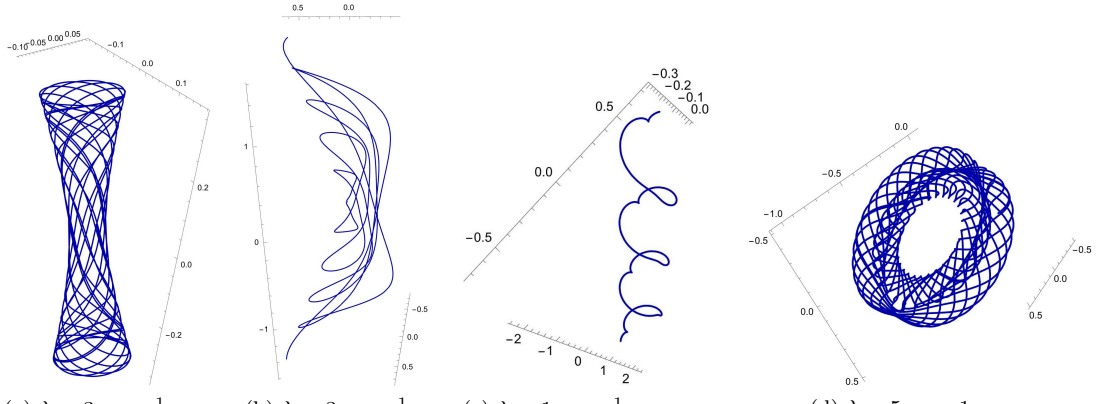
(h) $\lambda = \frac{1}{5},\ \mu = 2,$ $h(t) = 1, \mathfrak{a}(t) = \cos t$

Figure 1: Framed 0-slant helices are generated by some constants λ, μ and some smooth functions $\mathfrak{a}, h$.

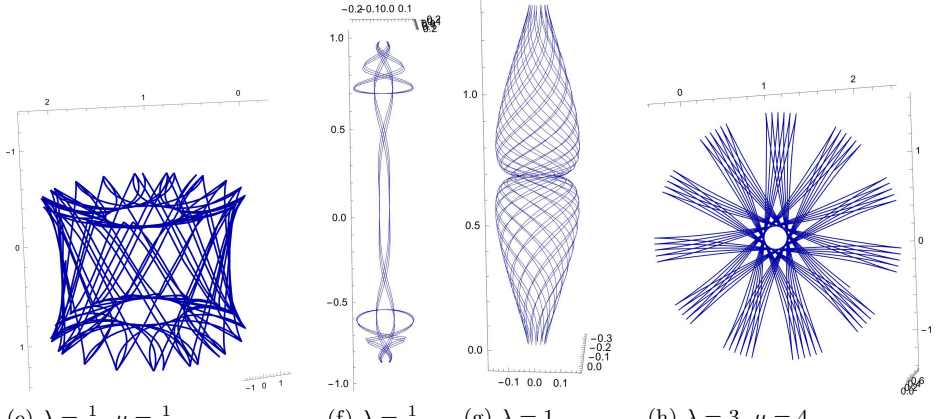
Example 3.15. Let $\boldsymbol{\gamma}$ be a framed curve with framed curvature

$$(\mathfrak{p}(t), \mathfrak{q}(t), \mathfrak{a}(t)) = \left(\frac{\lambda}{\mu} \sin(\lambda t), -\frac{\lambda}{\mu} \cos(\lambda t), \mathfrak{a}(t) \right)$$

where λ, μ are nonzero constants, and $\mathfrak{a}(t)$ are smooth functions. Then, by using Theorem 2.11 and Theorem 3.3, we see that $\boldsymbol{\gamma}$ is a framed 1-slant helix (i.e. framed slant helix) such that $\boldsymbol{\sigma}_1 = \mu$ (see Figure 2).



(a) $\lambda = 3$, $\mu = \frac{1}{4}$, $\mathfrak{a}(t) = 1$ (b) $\lambda = 3$, $\mu = \frac{1}{3}$, $\mathfrak{a}(t) = t$ (c) $\lambda = 1$, $\mu = \frac{1}{4}$, $\mathfrak{a}(t) = \cos t$ (d) $\lambda = 5$, $\mu = 1$, $\mathfrak{a}(t) = \cos t$



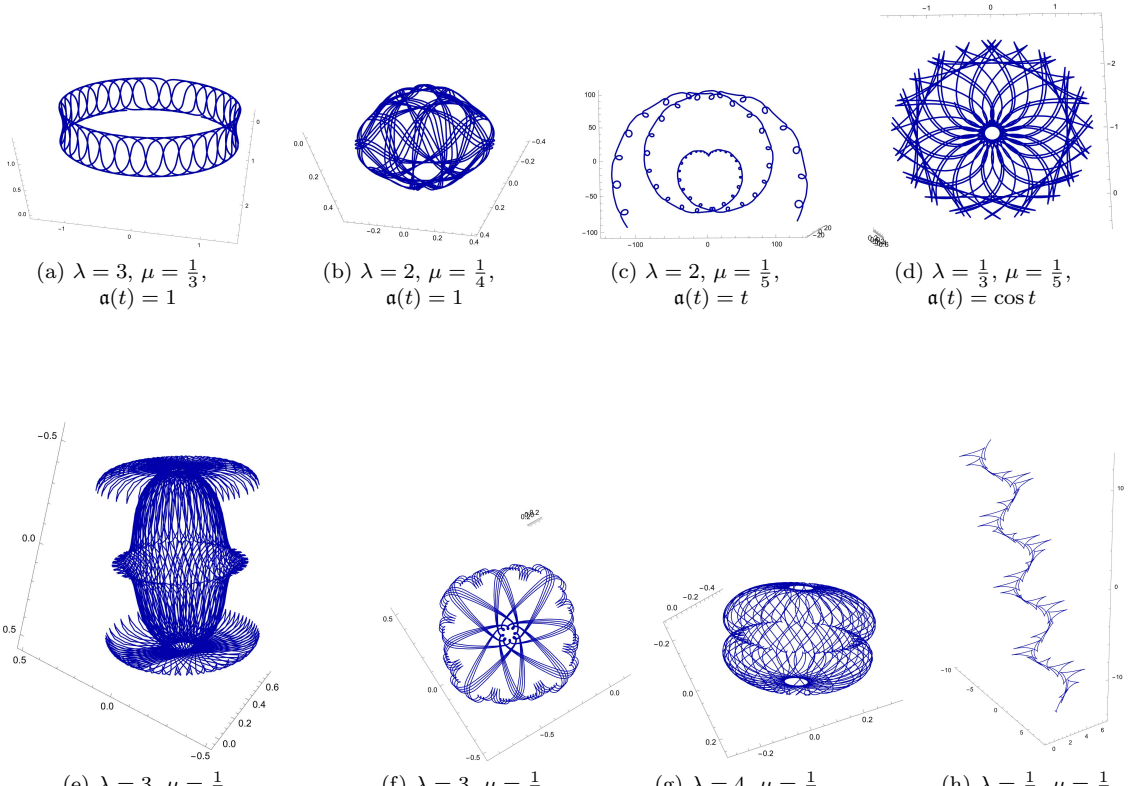
(e) $\lambda = \frac{1}{2}$, $\mu = \frac{1}{2}$, $\mathfrak{a}(t) = \cos t$ (f) $\lambda = \frac{1}{2}$, $\mu = \frac{1}{12}$, $\mathfrak{a}(t) = \cos t$ (g) $\lambda = 1$, $\mu = \frac{1}{6}$, $\mathfrak{a}(t) = \sin 2t$ (h) $\lambda = 3$, $\mu = 4$, $\mathfrak{a}(t) = \cos t + \sin t$

Figure 2: Framed 1-slant helices are generated by some constants λ, μ and some smooth functions $\mathfrak{a}$.

Example 3.16. Let $\boldsymbol{\gamma}$ be a framed curve with framed curvature

$$(\mathfrak{p}(t), \mathfrak{q}(t), \mathfrak{a}(t)) = \left(\frac{\lambda}{\mu}\sin(\lambda t)\sin\left(\frac{1}{\mu}\sin(\lambda t)\right), \frac{\lambda}{\mu}\sin(\lambda t)\cos\left(\frac{1}{\mu}\sin(\lambda t)\right), \mathfrak{a}(t)\right)$$

where λ, μ are nonzero constants, and $\mathfrak{a}(t)$ are smooth functions. Then, by using Theorem 2.15 and Theorem 3.3, we see that $\boldsymbol{\gamma}$ is a framed 2-slant helix (i.e. framed clad helix) such that $\boldsymbol{\sigma}_2 = \mu$ (see Figure 3).



(a) $\lambda = 3$, $\mu = \frac{1}{3}$, $\mathfrak{a}(t) = 1$ (b) $\lambda = 2$, $\mu = \frac{1}{4}$, $\mathfrak{a}(t) = 1$ (c) $\lambda = 2$, $\mu = \frac{1}{5}$, $\mathfrak{a}(t) = t$ (d) $\lambda = \frac{1}{3}$, $\mu = \frac{1}{5}$, $\mathfrak{a}(t) = \cos t$

(e) $\lambda = 3$, $\mu = \frac{1}{3}$, $\mathfrak{a}(t) = \sin 2t$ (f) $\lambda = 3$, $\mu = \frac{1}{6}$, $\mathfrak{a}(t) = \sin 2t$ (g) $\lambda = 4$, $\mu = \frac{1}{2}$, $\mathfrak{a}(t) = \sin 2t$ (h) $\lambda = \frac{1}{3}$, $\mu = \frac{1}{2}$, $\mathfrak{a}(t) = \cos t + \sin t$

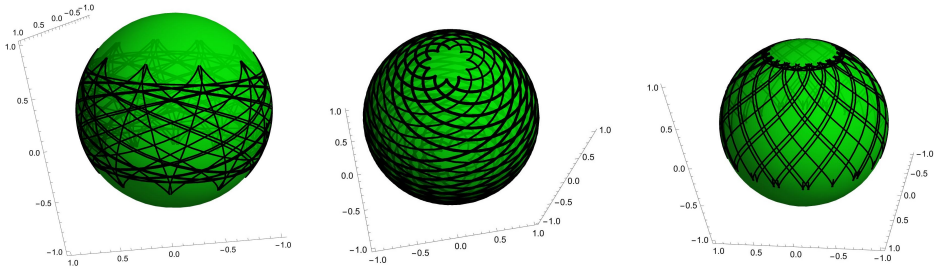
Figure 3: Framed 2-slant helices are generated by some constants λ, μ and some smooth functions $\mathfrak{a}$.

Finally, as an application of Theorem 3.13, we give an example of generalized spherical images of a framed 2-slant helix.

Example 3.17. Let $\boldsymbol{\gamma}$ be a framed 2-slant helix with the framed curvature

$$(\mathfrak{p}(t), \mathfrak{q}(t), \mathfrak{a}(t)) = (10\sin(2t)\sin(5\sin(2t)), 10\sin(2t)\cos(5\sin(2t)), t),$$

such that $\boldsymbol{\sigma}_2 = \frac{1}{5}$ (see Figure 3c). Then, by using Propositions 3.8-3.10, we obtain framed spherical indicatrices of the framed 2-slant helix $\boldsymbol{\gamma}$ (see Figure 4).



(a) Framed $\boldsymbol{\vartheta}-$indicatrix of $\boldsymbol{\gamma}$ which is a framed slant helix such that $\boldsymbol{\sigma}_{1\boldsymbol{\alpha}} = \frac{1}{5}$.

(b) Framed $\boldsymbol{\eta}_1-$indicatrix of $\boldsymbol{\gamma}$ which is a framed helix such that $\mathfrak{h}_{\boldsymbol{\beta}} = \frac{1}{5}$.

(c) Framed $\boldsymbol{\eta}_2-$indicatrix of $\boldsymbol{\gamma}$ which is a framed slant helix such that $\boldsymbol{\sigma}_{1\boldsymbol{\xi}} = \frac{1}{5}$.

Figure 4: Generalized spherical images of the framed 2-slant helix $\boldsymbol{\gamma}$ given in Figure 3c.

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