Geometry of dual umbrella matrices

Mert Çarboğa[1] and Yusuf Yaylı[2]

[1,2]Department of Mathematics, Ankara University, Ankara-06100, Turkey
E-mail: mcarboga@ankara.edu.tr[1], yayli@science.ankara.edu.tr[2]

Abstract

Umbrella matrices and their geometry have been studied in works [1, 2, 3], and the generalized form of [3] is addressed in [4]. In this study, for the first time, we obtained Dual Umbrella matrices using the Cayley Formula for Dual skew-symmetric matrices with zero row sums. We provided the relationship between the skew-symmetric matrix and the Darboux matrix of the motion. We demonstrated that Dual Umbrella matrices represent a screw motion in 3-dimensional Euclidean space and obtained the rotation angle and translation amount associated with this motion. Furthermore, we showed that these matrices are Circulant matrices.

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1 Introduction

In the past, numerous studies have been conducted on the geometry of orthogonal matrices, which find applications in various mathematical fields. Among orthogonal matrices, there exist a more special class, known as umbrella matrices, which preserve the $\mathbf{1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ axis. The geometry of these matrices, first introduced in Ertuğrul Özdamar's doctoral thesis, rotates the entire space around the $\mathbf{1}$ axis [1]. The Cayley Formula can be applied to any skew-symmetric matrix to obtain orthogonal matrices (see [2, 7, 11]).

The concept of the dual number was introduced by Clifford during 1873 and the name was given by Study in 1903. The use of the dual number allows compact fonnulations considerably facilitating the analysis of mechanical systems. While the methodology has been around for many years, it has remained obscure. The basic concepts of dual-number mechanics had been introduced by the beginning of the Twentieth Centwy, but it is only in recent decades that applications and developments have occasionally appeared in the technical literature. In order to introduce students to dual-number methods, the author found it necessary to develop a course featuring application of dual numbers in the displacement, velocity, static and dynamic analysis of mechanisms. This course has been conducted on a more-or-less regular basis over the years, the lectures refined and has been popular with the students. While this course has introduced dual-number methodologies to the students who have taken it, it is realized that such a local effort does little to popularize dual-number methods amongst the academic community and design engineers in industry. [13]

Studies [2, 6, 12] have investigated the geometry of Umbrella matrices and the required properties of the skew-symmetric matrices used to obtain them. Furthermore, the Lie group structure of these matrices has been explored in works [1, 2]. The concept of dual matrices and dual transformations has been examined in [8]. Particularly, this work focuses on dual orthogonal matrices and investigates the conditions for these matrices to be dual umbrella matrices. Additionally, the

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choice of dual skew-symmetric matrices for obtaining the dual umbrella matrices has been examined, emphasizing the relationship between this skew-symmetric matrix and the Darboux matrix of motion.

Circulant matrices represent a specialized class of matrices that play a significant role in the fields of linear algebra and mathematical analysis. These matrices are employed to perform matrix operations more efficiently in various applications (see [5, 7]). Circulant matrices, from the perspective of matrix theory and applications, are subjects of considerable interest.

After presenting the foundational information, our study delves into obtaining dual umbrella matrices in Section 3. In Section 4, we explore the geometry of these matrices and derive certain results related to rotation angles and translation amounts. Section 5, establishes that these matrices are actually Circulant matrices, and the conclusion section summarizes the findings.

The results of this study will contribute to a better understanding of Circulant matrices, which have applications in fields such as mathematical physics, robotics, and computer graphics.

1.1 Preliminaries

In this section, we review some preliminary concepts that will enable us to establish the required notation.

Definition 1.1. A is an orthogonal matrix, for which -1 is not an eigenvalue, may be written as

$$A = (I_n - B)^{-1}(I_n + B)$$

in which B is skew-symmetric matrix; *Cayley Formula.* [9]

Definition 1.2. Let A be an orthogonal matrix. If

$$A\mathbf{1} = \mathbf{1}$$

then A is called an umbrella matrix, where $\mathbf{1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T \in \mathbb{R}^3_1$. (See [1])

Definition 1.3. Let y and x be the position vectors, represented by column matrices, of a point P in the fixed space Σ^n and the moving space Σ^n, respectively. A continuous series of displacements, given by

$$y = Ax + b$$

where the orthogonal matrix A and the translation vector b are functions of a parameter t which may be identified with the time, is called a motion. Now, we consider the rotational motion, given by

$$y(t) = A(t)x.$$

ıf derivative is taken

$$\dot{y} = \dot{A}x$$

and

$$x = A^{-1}y,$$

then we obtain $\dot{y} = \dot{A}A^{-1}y$. The matrix

$$W = \dot{A}A^{-1}$$

is called the *angular velocity* matrix or the *Darboux matrix* of the motion. [9]

Definition 1.4. Let A be an umbrella matrix. The motion generated by the transformation

$$\begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

or

$$y = Ax + b$$

is called an umbrella motion in E^n, where $x, y \in \mathbb{R}_1^n$ and $b \in \mathbb{R}_1^n$. [9]

Definition 1.5. The dual number ε is defined such that

$$\varepsilon \neq 0$$

$$\varepsilon^2 = 0$$

A dual number, which perhaps should be called a "duplex" number in analogy with complex numbers, is written as

$$\widehat{a} = a + \varepsilon a^*$$

where symbol a represents the primary (or real) part of duplex (or dual) number $\widehat{a}$ and symbol a^* represents the dual component of duplex (or dual) number $\widehat{a}$.

In the mathematics of dual numbers, addition is defined as

$$\widehat{a} + \widehat{b} = (a + b) + \varepsilon(a^* + b^*)$$

Multiplication is defined as

$$\begin{aligned} \widehat{a}\widehat{b} &= (a + \varepsilon a^*)(b + \varepsilon b^*) \\ &= ab + \varepsilon(ab^* + a^*b). \text{ [13]} \end{aligned}$$

2 Obtaining dual umbrella matrices

In this section, we investigate the derivation of a Dual Umbrella matrix in the 3-dimensional Dual space $SO(3, \mathbb{D})$. For this purpose, we utilize a Dual skew-symmetric matrix with row sums equal to zero. Subsequently, we establish the relationship between the motion's Darboux matrix and the skew-symmetric matrix.

We can express the space of Dual orthogonal matrices in $O(3, \mathbb{D})$ as

$$O(3, \mathbb{D}) = \{\widehat{A} = A + \varepsilon A^* | \widehat{A}\widehat{A}^T = I_3\}$$

where A represents the real part of the Dual orthogonal matrix and A^* represents the Dual part. Let $\widehat{A} \in O(3, \mathbb{D})$, then we have

$$\widehat{A}\widehat{A}^T = I_3. \tag{2.1}$$

By substituting $\widehat{A} = A + \varepsilon A^*$ for $\widehat{A}$, we obtain

$$\begin{aligned}(A+\varepsilon A^*)(A+\varepsilon A^*)^T &= (A+\varepsilon A^*)(A^T+\varepsilon A^{*^T})\\ &= I+\varepsilon AA^{*^T}+\varepsilon A^*A^T+\varepsilon^2 A^*A^{*^T}\\ &= I+\varepsilon(AA^{*^T}+A^*A^T).\end{aligned}$$

From this, for Eq.(2.1) to hold, we require

$$AA^{*^T}+A^*A^T=0 \tag{2.2}$$

and $A\in O(3)$. Therefore, if we assume that $A^*A^T=B$, we can express Eq.(2.2) as $B^T+B=0$. Thus, the matrix B is a skew-symmetric matrix.

So, when we apply the matrix A from the right side to the equation $A^*A^T=B$, we obtain

$$\begin{aligned}(A^*A^T)A &= BA\\ A^*(A^TA) &= BA\end{aligned}$$

and from there, we can express the space $O(3,\mathbb{D})$ as

$$O(3,\mathbb{D})=\{\widehat{A}=A+\varepsilon BA|\ B^T=-B, A\in O(3)\}$$

Additionally, if we assume $det(\widehat{A})=1$, then we obtain

$$SO(3,\mathbb{D})=\{\widehat{A}=A+\varepsilon BA|\ B^T=-B, A\in SO(3)\}$$

Let $\widehat{A}\in SO(3,\mathbb{D})$. To ensure that $\widehat{A}$ preserves the axis $\mathbf{1}=\begin{bmatrix}1 & 1 & 1\end{bmatrix}^T\in\mathbb{R}^3_1$, we need to find the equation

$$\widehat{A}\mathbf{1}=\mathbf{1}. \tag{2.3}$$

For that, we write $\widehat{A}=A+\varepsilon BA$ in Eq.(2.3) and consider $A\in A_3$ as the umbrella matrix, we get

$$\begin{aligned}(A+\varepsilon BA)\mathbf{1} &= A\mathbf{1}+\varepsilon(BA)\mathbf{1} &(2.4)\\ &= A\mathbf{1}+\varepsilon B(A\mathbf{1}) &(2.5)\\ &= \mathbf{1}+\varepsilon B\mathbf{1}, &(2.6)\end{aligned}$$

where A_3 is the set of reel umbrella matrices of type 3×3.

From this, we can deduce that $B\mathbf{1}=0$ is necessary. Therefore, taking B as an skew-symmetric matrix with row sums equal to zero satisfies Eq.(6). Thus, if we denote the set of Dual Umbrella matrices as $\widehat{A}_3\subset SO(3,\mathbb{D})$, we can write

$$\widehat{A}_3 = \{\widehat{A} = A + \varepsilon BA |\ \widehat{A}\mathbf{1} = \mathbf{1}, B\mathbf{1} = 0, \mathbf{1} = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T \in \mathbb{R}^3_1\},$$

where $B^T = -B$ and $A \in A_3$.

Then, we can make the following definition.

Definition 2.1. Let $\widehat{A} \in SO(3, \mathbb{D})$ be a Dual orthogonal matrix. If

$$\widehat{A}\mathbf{1} = \mathbf{1},$$

then, $\widehat{A}$ is called a dual umbrella matrix, where $\mathbf{1} = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T \in \mathbb{R}^3_1$.

Theorem 2.2. Let $\widehat{C}$ be an skew-symmetric matrix of type 3×3. If row sums of the skew-symmetric matrix $\widehat{C}$ are zero, then orthogonal matrix

$$\widehat{A} = (I_3 - \widehat{C})^{-1}(I_3 + \widehat{C})$$

is a Dual umbrella matrix.

Proof. Suppose that the row sums of the skew-symmetric matrix $\widehat{C}$ be zero. Hence, $\widehat{C}\mathbf{1} = 0$ where $\mathbf{1} = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T \in \mathbb{R}^3_1$. Therefore, we can write that

$$(I_3 - \widehat{C})\mathbf{1} = \mathbf{1} \tag{2.7}$$

Since $det(I_3 - \widehat{C}) \neq 0$, we have

$$(I_3 - \widehat{C})^{-1}\mathbf{1} = \mathbf{1} \tag{2.8}$$

Thus, from Eqs. (2.7) and (2.8), we obtain

$$\begin{aligned}\widehat{A}\mathbf{1} &= (I_3 - \widehat{C})^{-1}(I_3 + \widehat{C})\mathbf{1}\\ &= (I_3 - \widehat{C})^{-1}[I_3\mathbf{1} + \widehat{C}\mathbf{1}]\\ &= (I_3 - B)^{-1}\mathbf{1}\\ &= \mathbf{1}.\end{aligned}$$

So, the dual orthogonal matrix $\widehat{A}$ is a dual umbrella matrix. Q.E.D.

Now, with the following result, we will obtain the real and dual parts of a dual umbrella matrix.

Corollary 2.3. For $\widehat{a} = a + \varepsilon b$, denote the dual skew-symmetric matrix as $\widehat{C}$, given by

$$\widehat{C} = \begin{pmatrix} 0 & \widehat{a} & -\widehat{a} \\ -\widehat{a} & 0 & \widehat{a} \\ \widehat{a} & -\widehat{a} & 0 \end{pmatrix}$$

When we apply the theorem(2.2) to $\widehat{C}$, we obtain

$$\widehat{A} = \frac{1 + 3a^2 - \varepsilon 6ab}{(1 + 3a^2)^2} \begin{pmatrix} 1 - a^2 - \varepsilon 2ab & 2a^2 + 2a + \varepsilon 6b & 2a^2 - 2a + \varepsilon 2b \\ 2a^2 - 2a + \varepsilon 2b & 1 - a^2 - \varepsilon 2ab & 2a^2 + 2a + \varepsilon 6b \\ 2a^2 + 2a + \varepsilon 6b & 2a^2 - 2a + \varepsilon 2b & 1 - a^2 - \varepsilon 2ab \end{pmatrix}$$

dual umbrella matrix. If we arrange this matrix as $\widehat{A} = A + \varepsilon A^*$, where A represents the real part and A^* represents the dual part, we can write

$$\begin{aligned}\widehat{A} &= \frac{1}{1+3a^2}\begin{pmatrix} 1-a^2 & 2(a^2+a) & 2(a^2-a) \\ 2(a^2-a) & 1-a^2 & 2(a^2+a) \\ 2(a^2+a) & 2(a^2-a) & 1-a^2 \end{pmatrix} \\ &\quad +\varepsilon\frac{1}{(1+3a^2)^2}\begin{pmatrix} -8ab & -6a^2b+4ab+2b & 6a^2b+4ab-2b \\ 6a^2b+4ab-2b & -8ab & -6a^2b+4ab+2b \\ -6a^2b+4ab+2b & 6a^2b+4ab-2b & -8ab \end{pmatrix}\end{aligned}$$

Theorem 2.4. Let the Dual skew-symmetric matrix $\widehat{C(t)}$, the row sums of which are zero, and the Dual umbrella matrix corresponding to this matrix be given by

$$\widehat{A(t)} = (I_3 - \widehat{C(t)})^{-1}(I_3 + \widehat{C(t)}). \tag{2.9}$$

Then,

$$W(\widehat{A(t)}) = 2(I_3 - \widehat{C(t)})^{-1}\widehat{C(t)}'(I_3 + \widehat{C(t)})$$

where $W(\widehat{A(t)})$ is the Darboux matrix of the Dual umbrella motion.

Proof. Differentiating Eq.(2.9) with respect to t, we have

$$\widehat{A(t)}' = [(I_3 - \widehat{C(t)})^{-1}]'(I_3 + \widehat{C(t)}) + (I_3 - \widehat{C(t)})^{-1}(I_3 + \widehat{C(t)})' \tag{2.10}$$

For $[I_3 - \widehat{C(t)})^{-1}1]'$, by differantiating $(I_3 - \widehat{C(t)})^{-1}(I_3 - \widehat{C(t)}) = I_3$ with respect to t, we have

$$[(I_3 - \widehat{C(t)})^{-1}]'(I_3 - \widehat{C(t)}) + (I_3 - \widehat{C(t)})^{-1}(I_3 - \widehat{C(t)})' = 0 \tag{2.11}$$

then, considering $(I_3 - \widehat{C(t)})' = -\widehat{C(t)}'$ with Eq.(2.11), we obtain

$$[(I_3 - \widehat{C(t)})^{-1}1]' = (I_3 - \widehat{C(t)})^{-1}\widehat{C(t)}'(I_3 - \widehat{C(t)})^{-1}. \tag{2.12}$$

If we combine Eq.(2.12) with Eq.(2.10) gives us,

$$\begin{aligned}\widehat{A(t)}' &= (I_3 - \widehat{C(t)})^{-1}\widehat{C(t)}'(I_3 - \widehat{C(t)})^{-1}(I_3 + \widehat{C(t)}) + (I_3 - \widehat{C(t)})^{-1}\widehat{C(t)}' \\ &= (I_3 - \widehat{C(t)})^{-1}\widehat{C(t)}'[(I_3 - \widehat{C(t)})^{-1}(I_3 + \widehat{C(t)}) + I_3] \\ &= (I_3 - \widehat{C(t)})^{-1}\widehat{C(t)}'[(I_3 - \widehat{C(t)})^{-1}(I_3 + \widehat{C(t)}) + (I_3 - \widehat{C(t)})^{-1}(I_3 - \widehat{C(t)})] \\ &= (I_3 - \widehat{C(t)})^{-1}\widehat{C(t)}'(I_3 - \widehat{C(t)})^{-1}[I_n + \widehat{C(t)} + I_3 - \widehat{C(t)}] \\ &= 2(I_3 - \widehat{C(t)})^{-1}\widehat{C(t)}'(I_3 - \widehat{C(t)})^{-1}.\end{aligned}$$

Finally, considering $\widehat{A(t)}' = 2(I_3 - \widehat{C(t)})^{-1}\widehat{C(t)}'(I_3 - \widehat{C(t)})^{-1}$ with $W(\widehat{A(t)}) = \widehat{A(t)}'\widehat{A(t)}^T$, we can write,

$$W(\widehat{A(t)}) = 2(I_3 - \widehat{C(t)})^{-1}\widehat{C(t)}'(I_3 + \widehat{C(t)})^{-1}.$$

Q.E.D.

Corollary 2.5. Let $\widehat{C(t)}$ be the Dual skew-symmetric matrix, the row sums of which are zero, and the Dual umbrella matrix corresponding to this matrix can be given by

$$\widehat{A(t)} = (I_3 - \widehat{C(t)})^{-1}(I_3 + \widehat{C(t)}).$$

$W(\widehat{A(t)})$ is the Darboux matrix of the Dual umbrella motion. Then,

$$W(\widehat{A(t)})\mathbf{1} = \mathbf{1}$$

where $\mathbf{1} = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T \in \mathbb{R}^3_1$.

3 Geometry of 1-axis screw motion

In this section, we will demonstrate that the umbrella motion defined by a dual umbrella matrix corresponds to a screw motion.

From Section 2,

$$\widehat{A}_3 = \{\widehat{A} = A + \varepsilon BA |\ \widehat{A}\mathbf{1} = \mathbf{1}, B\mathbf{1} = 0, \mathbf{1} = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T \in \mathbb{R}^3_1\},$$

where $A \in A_3$ and $B^T = -B$, we have defined the set of the dual umbrella matrix, and we have shown that for $\widehat{A} \in \widehat{A}_3$ written as

$$A = \frac{1}{1+3a^2}\begin{pmatrix} 1-a^2 & 2(a^2+a) & 2(a^2-a) \\ 2(a^2-a) & 1-a^2 & 2(a^2+a) \\ 2(a^2+a) & 2(a^2-a) & 1-a^2 \end{pmatrix}$$

and

$$A^* = \frac{1}{(1+3a^2)^2}\begin{pmatrix} -8ab & -6a^2b+4ab+2b & 6a^2b+4ab-2b \\ 6a^2b+4ab-2b & -8ab & -6a^2b+4ab+2b \\ -6a^2b+4ab+2b & 6a^2b+4ab-2b & -8ab \end{pmatrix}.$$

We can analyze the components of the Darboux matrix and their relationships to screw parameters such as translation and rotation.

We can express $\widehat{A}$, where A represents the rotational part of the motion $Y = Ax + \vec{C}$ in the space Y, and $A^* = BA$ implies $B = A^*A^T$. By computing B, we obtain

$$B = \frac{2b}{1+3a^2}\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

and considering $\vec{C}$ is parallel $\mathbf{1}$, we have

$$\vec{C} = \frac{2b}{1+3a^2}(1,1,1)$$

as the translation vector of this motion. Thus, the space motion $Y = Ax + \vec{C}$ (Dual Umbrella motion) represents a 1-axis screw motion. Now, let us calculate the rotation angle and the translation amount along the 1-axis for this screw motion. For the dual skew-symmetric matrix $\widehat{C}$ as follows,

$$\widehat{C} = (a+\varepsilon b)\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix},$$

we can write the equation

$$\vec{u} = (a+\varepsilon b)\,(1,1,1) \cong \widehat{C}$$

as the relationship between $\vec{u}$ and $\widehat{C}$.

From this, $\|\vec{u}\| = \sqrt{3}(a+\varepsilon b)$ can be obtained, and if we define $\widehat{\alpha} = \alpha + \varepsilon\alpha^*$ as the dual angle, we can find

$$\begin{aligned} \tan(\frac{\widehat{\alpha}}{2}) &= \|\vec{u}\| \\ &= \sqrt{3}(a+\varepsilon b). \end{aligned}$$

Therefore, we obtain

$$\widehat{\alpha} = 2\arctan(\sqrt{3})a + \varepsilon\frac{2\sqrt{3}b}{1+3a^2}.$$

From here,

- Rotation angle: $\cos(\alpha) = \frac{1-3a^2}{1+3a^2}$
- Translation amount: $\frac{2\sqrt{3}b}{1+3a^2}$

can be determined.

Example 3.1. Let us consider

$$\widehat{A} = \begin{pmatrix} -2\varepsilon & 2\varepsilon & 1 \\ 1 & -2\varepsilon & 2\varepsilon \\ 2\varepsilon & 1 & -2\varepsilon \end{pmatrix}$$

as the dual umbrella matrix. Then, we can remark about two different interpretations:

- **Interpretation for the space motion:** For $X \in \mathbb{R}^3$ and $\vec{C}$ translation vector, in the space motion $Y = AX + \vec{C}$, we can write

$$\widehat{A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \varepsilon\begin{pmatrix} 0 & -2 & 2 \\ 2 & 0 & -2 \\ -2 & 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and considering $a = \mp 1$, $b = 4$. In this case, $\widehat{\alpha} = \frac{2\pi}{3} + \varepsilon 2\sqrt{3}$. Therefore,

- Rotation angle and axis: $\alpha = \frac{2\pi}{3}$ degree rotation around $\imath = (1,1,1)$-axis.
- Translation vector: $\vec{C} = (2,2,2)$ and translation amounts $2\sqrt{3}$

- **Interpretation for the screw motion:** The matrix

$$\widehat{A} = \begin{bmatrix} A & 0 \\ BA & A \end{bmatrix}$$

transforms the screw $\vec{a} + \varepsilon\vec{a}^*$ into the screw

$$\widehat{A}\begin{bmatrix} \vec{a} \\ a^* \end{bmatrix} = \begin{bmatrix} A & 0 \\ BA & A \end{bmatrix}\begin{bmatrix} \vec{a} \\ a^* \end{bmatrix} = \begin{bmatrix} A\vec{a} \\ BA\vec{a} + Aa^* \end{bmatrix}.$$

4 Dual circulant matrices and dual umbrella matrices

Circulant matrices represent a special class of matrices that play a significant role in mathematical and applied mathematics. These matrices have various properties and applications Circulant matrices are square matrices where each row is created by cyclically shifting the previous row to the right. Let's denote a 3×3 circulant matrix as:

$$\begin{pmatrix} x & y & z \\ z & x & y \\ y & z & x \end{pmatrix}.$$

In this space, the considered umbrella matrix A as follows:

$$A = \frac{1}{1+3a^2}\begin{pmatrix} 1-a^2 & 2(a^2+a) & 2(a^2-a) \\ 2(a^2-a) & 1-a^2 & 2(a^2+a) \\ 2(a^2+a) & 2(a^2-a) & 1-a^2 \end{pmatrix},$$

which is actually a circulant matrix as well. Moreover, the matrix A, possess the property of being orthogonal circulant matrices. Geometrically approaching, the intersection of the unit sphere $\mathbb{S}^2$ defined by

$$x^2 + y^2 + z^2 = 1$$

and the plane $\mathbb{P}$ given by

$$x + y + z = 1$$

for the derivation of these orthogonal Circulant matrices. The algebraic solution of these two equations,

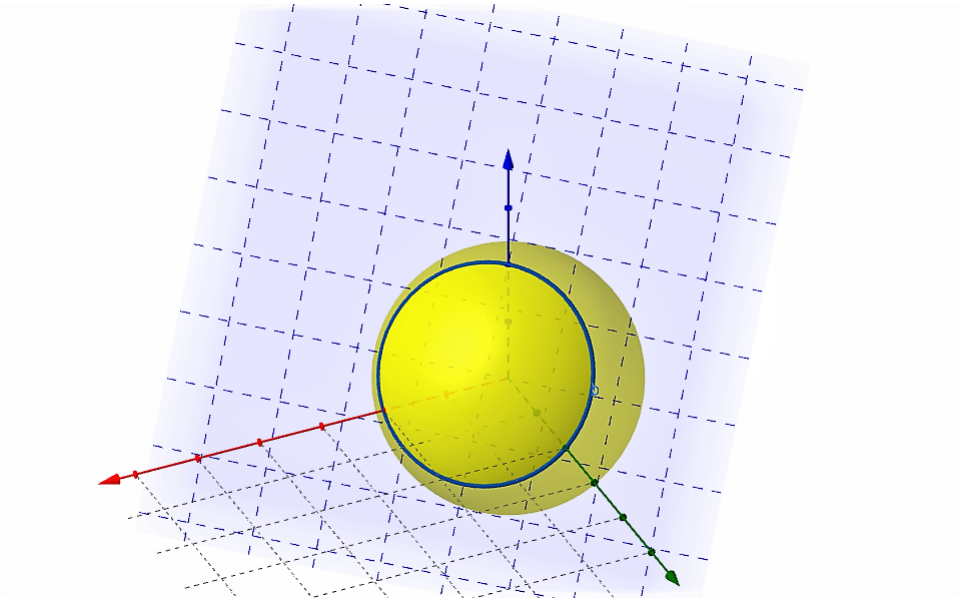


Fig. 1.

$$\mathbb{S}^2 \cap \mathbb{P} = \alpha(t) = (\frac{1}{3} + \frac{2\cos t}{3}, \frac{1}{3} - \frac{\sin t}{\sqrt{3}} - \frac{\cos t}{3}, \frac{1}{3} + \frac{\sin t}{\sqrt{3}} - \frac{\cos t}{3})$$

is a circle curve (See Fig. 1), and if we denote $\alpha(t) = (a, b, c)$, then

$$A(t) = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}.$$

that rotates around the 1-axis by an angle of t is a circulant umbrella matrix. Hence, we can easily evident that if we examine the dual angle $\widehat{t} = t + \varepsilon t^*$ in conjunction with $\alpha(t)$, where angle t represents rotation around 1-axis and t^* denotes unit translation along the same axis, the curve

$$\alpha(\widehat{t}) = (\frac{1}{3} + \frac{2\cos \widehat{t}}{3}, \frac{1}{3} - \frac{\sin \widehat{t}}{\sqrt{3}} - \frac{\cos \widehat{t}}{3}, \frac{1}{3} + \frac{\sin \widehat{t}}{\sqrt{3}} - \frac{\cos \widehat{t}}{3})$$

assumes the form of a circle on the unit dual sphere, and if we denote it as $\alpha(\widehat{t}) = (\widehat{a}, \widehat{b}, \widehat{c})$, then

$$A(\widehat{t}) = \begin{pmatrix} \widehat{a} & \widehat{b} & \widehat{c} \\ \widehat{c} & \widehat{a} & \widehat{b} \\ \widehat{b} & \widehat{c} & \widehat{a} \end{pmatrix}$$

is a dual umbrella matrix, which is a dual circulant matrix.

5 Conclusıon

In this study, it was demonstrated that Dual umbrella matrices correspond to a rotation around the ı-axis in space and translation along the same axis. These matrices transform screws into screws. Furthermore, it has been shown that umbrella matrices can be obtained with the help of the circle formed as a result of the intersection of a sphere and a plane. In this representation, the rotation angle is expressed in terms of the parameter of the circle.

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