# Radical invariant lightlike hypersurfaces of almost product-like statistical manifolds

Ömer Aksu[1], Esra Erkan[2] and Mehmet Gülbahar[3]

[1] Harran University, Faculty of Science and Arts, Department of Mathematics, Şanlıurfa, Turkey
[2]Harran University, Faculty of Science and Arts, Department of Mathematics, Şanlıurfa, Turkey
[3]Harran University, Faculty of Science and Arts, Department of Mathematics, Şanlıurfa, Turkey

E-mail: omeraksu@harran.edu.tr[1], esraerkan@harran.edu.tr[2], mehmetgulbahar@harran.edu.tr[3]

**Abstract**

Introducing a new class of submanifolds is one of the key concepts in submanifold theory of semi-Riemannian manifolds. The purpose of this study is to introduce radical invariant lightlike hypersurfaces of almost product-like Lorentzian manifolds. Some examples of these hypersurfaces are presented herein. The main identities admitting Riemannian curvature tensor fields of these hypersurfaces are established with the aid of statistical structures.

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## 1 Introduction

The study of hypersurfaces furnished with a degenerate metric is a subject of great interest in differential geometry and physics. For a smooth manifold furnished with a Lorentzian metric, there is an inherent existence of lightlike submanifolds. Lightlike hypersurfaces firstly appeared in Einstein's theory of general relativity as a models of black holes. The geometry of lightlike submanifolds was developed by K. L. Duggal, A. Bejancu and B. Sahin in [13, 14, 16]. Later, a large number of manuscripts have been published on these submanifolds involving several smooth structures such as contact and product structures.

The theory of product manifolds contains a significant results in the field of differential geometry and has several physical and geometrical applications. For example, the Minkowski spacetime is a Lorentzian manifold that can be considered a product of 3-dimensional space and 1-dimensional time. There exist also some applications of product manifolds in the Kaluza-Klein theory and the gauge theory (cf.[10, 21], etc.). A semi-Riemannian manifold $(\widetilde{L}, \widetilde{h})$ furnished with a semi-Riemannian metric $\widetilde{h}$ is called an almost product manifold if there exists a tensor field $F$ on $L$ satisfying $F^2 = I$ and

$$\widetilde{h}(FZ_1, FZ_2) = \widetilde{h}(Z_1, Z_2), \tag{1.1}$$

where $I$ indicates the identity transformation [27]. Using differential geometry tools, several submanifolds of product manifolds were examined in [3, 22, 23], etc. As a generalization of product manifolds, the concept of almost product-like semi-Riemannian manifolds was introduced in [17] as follows:

A semi-Riemannian manifold $(\widetilde{L}, \widetilde{h})$ is called an almost product-like semi-Riemannian manifold if it admits two almost product structures $F$ and $F^\star$ satisfying

$$\widetilde{h}(FZ_1, F^\star Z_2) = \widetilde{h}(Z_1, Z_2) \tag{1.2}$$

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for $Z_1, Z_2 \in \Gamma(T\widetilde{L})$. In this case, the equation

$$\widetilde{h}(FZ_1, Z_2) = \widetilde{h}(Z_1, F^\star Z_2) \tag{1.3}$$

is also satisfied and we obtain $(F^\star)^\star = F$. An almost product-like semi-Riemannian manifold is indicated by $(\widetilde{L}, \widetilde{g}, F)$.

Considering the above-mentioned concepts, the following issues arise on almost product-like semi-Riemannian manifolds:

- The problem of investigating existence of degenerate and non-degenerate hypersurfaces of these manifolds.
- The problem of introducing and examining these hypersurfaces that satisfy specific geometric conditions.

In the degenerate case, the existence of lightlike submanifolds of an almost product-like statistical manifold was shown in [1]. The main purpose of this study is to determine the primary identities and properties of radical invariant lightlike hypersurfaces on these manifolds.

## 2 Statistical manifolds

Let $\widetilde{\nabla}$ be a linear connection with zero torsion on a Lorentzian manifold $(\widetilde{L}, \widetilde{h})$. The triple $(\widetilde{L}, \widetilde{h}, \widetilde{\nabla})$ is said to be a statistical manifold if there exists another linear connection $\widetilde{\nabla}^\star$ such that

$$Z_3\widetilde{h}(Z_1, Z_2) = \widetilde{h}(\widetilde{\nabla}_{Z_3}X_1, Z_2) + \widetilde{h}(Z_1, \widetilde{\nabla}^\star_{Z_3}Z_2) \tag{2.1}$$

holds for tangent vector fields $Z_1$, $Z_2$ and $Z_3$ on $\widetilde{L}$. The linear connection $\widetilde{\nabla}^\star$ is called the dual connection of $\widetilde{\nabla}$. We note that $\widetilde{\nabla}^\star$ is torsion-free and $\widetilde{\nabla}^\star\widetilde{h}$ is symmetric. Moreover, we find $(\widetilde{\nabla}^\star)^\star = \widetilde{\nabla}$ and

$$\widetilde{\nabla}^0_{Z_1}Z_2 = \frac{1}{2}(\widetilde{\nabla}_{Z_1}Z_2 + \widetilde{\nabla}^\star_{Z_1}Z_2), \tag{2.2}$$

where $\widetilde{\nabla}^0$ is the Levi-Civita connection [2]. The pair $(\widetilde{\nabla}, \widetilde{\nabla}^\star)$ is said to be the statistical structure.

Let $(\widetilde{L}, \widetilde{h}, \widetilde{\nabla})$ be a statistical manifold. We indicate the curvature tensors defined with the statistical structure as $\widetilde{R}$ and $\widetilde{R}^\star$. Subsequently, it follows that

$$\widetilde{h}(\widetilde{R}(Z_1, Z_2)Z_3, Z_4) = -\widetilde{h}(Z_3, \widetilde{R}^\star(Z_1, Z_2)Z_4). \tag{2.3}$$

The manifold is said to be of constant curvature $\vartheta$ if

$$\widetilde{R}(Z_1, Z_2)Z_3 = \vartheta\left\{\widetilde{h}(Z_2, Z_3)Z_1 - \widetilde{h}(Z_1, Z_3)Z_2\right\} \tag{2.4}$$

is satisfied for any $Z_1, Z_2, Z_3 \in \Gamma(T\widetilde{L})$. We note that from (2.3) and (2.4), the triple $(\widetilde{L}, \widetilde{h}, \widetilde{\nabla})$ is of constant curvature if and only if $(\widetilde{L}, \widetilde{h}, \widetilde{\nabla}^\star)$ is also of constant curvature. For more details related to statistical manifolds of constant curvature, we refer to [18].

From (2.3), we see that $\widetilde{R}$ and $\widetilde{R}^\star$ are not symmetric. Thus, these tensors lose their geometric importance. For this reason, B. Opozda [24] defined the notion of $K-$sectional curvature for statistical manifolds as follows:

Let $\Pi$ be a non-degenerate plane section on $\Gamma(T\widetilde{L})$ and $\{Z_1, Z_2\}$ be an orthonormal frame field on $\Pi$. The $K-$sectional curvature of $\Pi$ is denoted by $\widetilde{K}^s(\Pi) \equiv \widetilde{K}^s(Z_1, Z_2)$ and defined by

$$\widetilde{K}^s(\Pi) = \frac{1}{2}\varepsilon(Z_1)\varepsilon(Z_2)\left\{\widetilde{h}(\widetilde{R}(Z_1, Z_2)Z_2, Z_1) + \widetilde{h}(\widetilde{R}^\star(Z_1, Z_2)Z_2, Z_1) - 2\widetilde{h}(\widetilde{R}^0(Z_1, Z_2)Z_2, Z_1)\right\}, \tag{2.5}$$

where $\widetilde{R}^0$ is the Riemannian curvature defined by $\widetilde{\nabla}^0$, $\varepsilon(Z_1) = \widetilde{h}(Z_1, Z_1) = \mp 1$ and $\varepsilon(Z_2) = \widetilde{h}(Z_2, Z_2) = \mp 1$ [24]. It is clear that the $K-$sectional curvature is symmetric on $\Pi$. The sectional curvatures with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^\star$ are formulated as

$$\widetilde{K}(\Pi) = \varepsilon(Z_1)\varepsilon(Z_2)\widetilde{h}(\widetilde{R}(Z_1, Z_2)Z_2, Z_1) \text{ and } \widetilde{K}^\star(\Pi) = \varepsilon(Z_1)\varepsilon(Z_2)\widetilde{h}(\widetilde{R}^\star(Z_1, Z_2)Z_2, Z_1), \tag{2.6}$$

respectively. Thus, we find

$$\widetilde{K}^s(\Pi) = \frac{1}{2}\left\{\widetilde{K}(\Pi) + \widetilde{K}^\star(\Pi) - 2\widetilde{K}^0(\Pi)\right\}, \tag{2.7}$$

where $\widetilde{K}^0$ is the sectional curvature map defined by $\widetilde{\nabla}^0$.

Let $\dim\widetilde{L} = n+1$ and $\{E_1, \ldots, E_{n+1}\}$ be an orthonormal frame field of $\Gamma(T\widetilde{L})$. In this manner, the Ricci tensors can be defined by

$$\widetilde{S}(Z_1, Z_2) = \sum_{j=1}^{n+1}\varepsilon(Z_1)\varepsilon(Z_2)\widetilde{h}(\widetilde{R}(Z_1, E_j)E_j, Z_2), \tag{2.8}$$

$$\widetilde{S}^\star(Z_1, Z_2) = \sum_{j=1}^{n+1}\varepsilon(Z_1)\varepsilon(Z_2)\widetilde{h}(\widetilde{R}^\star(Z_1, E_j)E_j, Z_2) \tag{2.9}$$

and

$$\widetilde{S}^s(Z_1, Z_2) = \frac{1}{2}\sum_{j=1}^{n+1}\varepsilon(Z_1)\varepsilon(Z_2)\left\{\widetilde{h}(\widetilde{R}(Z_1, E_j)E_j, Z_2) + \widetilde{h}(\widetilde{R}^\star(Z_1, E_j)E_j, Z_2) - 2\widetilde{h}(\widetilde{R}^0(Z_1, E_j)E_j, Z_2\right\}. \tag{2.10}$$

Thus, the Ricci curvatures of $E_i$ for $i \in \{1, \ldots, n+1\}$ are given by

$$\widetilde{\mathrm{Ric}}(E_i) = \sum_{j=1}^{n+1}\widetilde{K}(E_i, E_j), \quad \widetilde{\mathrm{Ric}}^\star(E_i) = \sum_{j=1}^{n+1}\widetilde{K}^\star(E_i, E_j) \text{ and } \widetilde{\mathrm{Ric}}^s(E_i) = \sum_{j=1}^{n+1}\widetilde{K}^s(E_i, E_j). \tag{2.11}$$

$\widetilde{\mathrm{Ric}}^s$ is said to be the $K-$Ricci curvature. Furthermore, the scalar curvatures at $p \in \widetilde{L}$ are formulated as

$$\widetilde{\tau}(p) = \sum_{i,j=1}^{n+1}\widetilde{K}(E_i, E_j), \quad \widetilde{\tau}^\star(p) = \sum_{i,j=1}^{n+1}\widetilde{K}^\star(E_i, E_j) \text{ and } \widetilde{\tau}^s(p) = \sum_{i,j=1}^{n+1}\widetilde{K}^s(E_i, E_j). \tag{2.12}$$

$\widetilde{\tau}^s$ is said to be the $K-$scalar curvature.

For more relations dealing with curvatures of statistical manifolds, we refer to [5, 6, 7, 12, 25], etc.

## 3 Lightlike hypersurfaces

Since the metric $h$ is degenerate for a lightlike hypersurface $(L,h)$ of $(\widetilde{L},\widetilde{h})$, we write

$$\mathrm{Rad}(TL)=TL\cap TL^{\perp}\neq\{0\},$$

where $TL^{\perp}$ indicates the normal bundle. A complementary vector bundle $\mathrm{S}(TL)$ in $TL$ is called a screen distribution and it is usually denoted by $\mathrm{S}(TL)$ [13]. For any lightlike hypersurface $(L,h,\mathrm{S}(TL))$, there exists a unique null section $N$ of the transversal space $\mathrm{ltr}(TL)$ that satisfies

$$\widetilde{h}(\xi,N)=1,\ \widetilde{h}(N,N)=\widetilde{h}(N,Z)=0,\ \forall Z\in\Gamma(\mathrm{S}(TL)). \tag{3.1}$$

The Gauss and Weingarten formulas for the Levi-Civita connection $\widetilde{\nabla}^0$ are given by

$$\widetilde{\nabla}^0_{Z_1}Z_2=\nabla^0_{Z_1}Z_2+B^0(Z_1,Z_2)N \tag{3.2}$$

and

$$\widetilde{\nabla}^0_{Z_1}N=-A^0_NZ_1+\tau^0(Z_1)N, \tag{3.3}$$

where $A^0_N$ is the shape operator and $\tau^0$ is a $1-$form. A lightlike hypersurface $(L,h,\mathrm{S}(TL))$ is called totally geodesic if $B^0=0$ and it is called totally umbilical if there exists a differentiable function $\mu$ such that $B^0(Z_1,Z_2)=\mu h(Z_1,Z_2)$ for any $Z_1,Z_2\in\Gamma(TL)$ [15]. Furthermore, the hypersurface is called minimal if $\mathrm{trace}_{\mathrm{S}(TM)}B^0=0$, where we denote the trace with respect to $\mathrm{S}(TL)$ as $\mathrm{trace}_{\mathrm{S}(TL)}$ [11].

Let $\{e_1,e_2,\ldots,e_n\}$ be an orthonormal frame field on $\Gamma(\mathrm{S}(TL))$. Then, the induced Ricci-type tensor $R^{(0,2)}$ of $L$ is defined by

$$R^{(0,2)}(Z_1,Z_2)=\sum_{j=1}^{n}h(R^0(e_j,Z_1)Z_2,e_j)+\widetilde{h}(R^0(\xi,Z_1)Z_2,N). \tag{3.4}$$

We note that $R^{(0,2)}$ is not symmetric. If $\mathrm{S}(TL)$ is integrable, then $R^{(0,2)}$ is symmetric [13].

The screen Ricci curvature and screen scalar curvature on $(M,g,\mathrm{S}(TM))$ are defined by

$$\mathrm{Ric}^0_{\mathrm{S}(TL)}(e_i)=\sum_{j=1}^{n}h(R^0(e_i,e_j)e_j,e_i) \tag{3.5}$$

and

$$\tau^0_{S(TL)}(p)=\sum_{i,j=1}^{n}h(R^0(e_i,e_j)e_j,e_i), \tag{3.6}$$

respectively [20].

## 4 Almost product-like manifolds with statistical structures

We begin this section with the following example:

**Example 4.1.** Let $\mathbb{R}^4_1$ be a smooth manifold with the local coordinate system $(z_1, z_2, z_3, z_4)$, which admits the following almost product structure $F$:

$$F = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then the pair $(\mathbb{R}^4_1, F)$ is an almost product manifold. If we set

$$\widetilde{h} = \begin{bmatrix} -e^{z_1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{z_1} \end{bmatrix},$$

then we find

$$F^\star = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

which yields $\widetilde{h}(F\partial_i, F^\star\partial_j) = \widetilde{h}(\partial_i, \partial_j)$ for $i, j \in \{1, 2, 3, 4\}$. Here, $\{\partial_1, \partial_2, \partial_3, \partial_4\}$ is the natural frame field on $\mathbb{R}^4_1$. Thus, $(\mathbb{R}^4_1, \widetilde{h}, F)$ and $(\mathbb{R}^4_1, \widetilde{h}, F^\star)$ are almost product-like Lorentzian manifolds.

**Definition 4.2.** [17] If (2.1) and (2.2) are satisfied on $(\widetilde{L}, \widetilde{h}, F)$, then the manifold is called an almost product-like statistical manifold and it is indicated by $(\widetilde{L}, \widetilde{h}, \widetilde{\nabla}, F)$.

**Definition 4.3.** [17] An almost product-like statistical manifold is said to be a locally product-like statistical manifold if $\widetilde{\nabla}F = 0$.

Let $(\widetilde{L}, \widetilde{h}, \widetilde{\nabla}, F)$ be a locally product-like statistical manifold. Considering (1.3), we arrive at

$$\widetilde{h}((\widetilde{\nabla}_{Z_1}F)Z_2, Z_3) = \widetilde{h}(Z_2, (\widetilde{\nabla}^\star_{Z_1}F^\star)Z_3), \tag{4.1}$$

which implies that $\widetilde{\nabla}F = 0 \Leftrightarrow \widetilde{\nabla}^\star F^\star = 0$.

**Example 4.4.** Let $(\mathbb{R}^4_1, \widetilde{h}, F)$ be an almost product-like Lorentzian manifold of Example 4.1 and $\{\partial_1, \partial_2, \partial_3, \partial_4\}$ be the standard frame field of $\Gamma(\mathbb{R}^4_1)$. By a straightforward computation, we obtain

$$\begin{aligned}
&\widetilde{\nabla}_{\partial_1}\partial_1 = \widetilde{\nabla}_{\partial_4}\partial_4 = \frac{1}{2}\partial_1, \quad \widetilde{\nabla}_{\partial_1}\partial_4 = \widetilde{\nabla}_{\partial_4}\partial_1 = \frac{1}{2}\partial_4, \\
&\widetilde{\nabla}_{\partial_1}\partial_2 = \widetilde{\nabla}_{\partial_2}\partial_1 = \widetilde{\nabla}_{\partial_1}\partial_3 = \widetilde{\nabla}_{\partial_3}\partial_1 = 0, \\
&\widetilde{\nabla}_{\partial_2}\partial_3 = \widetilde{\nabla}_{\partial_3}\partial_2 = \widetilde{\nabla}_{\partial_2}\partial_4 = \widetilde{\nabla}_{\partial_4}\partial_2 = \widetilde{\nabla}_{\partial_3}\partial_4 = \widetilde{\nabla}_{\partial_4}\partial_3 = 0
\end{aligned}$$

and

$$\widetilde{\nabla}^{\star}_{\partial_1}\partial_1 = \widetilde{\nabla}^{\star}_{\partial_4}\partial_4 = \frac{1}{2}\partial_1, \quad \widetilde{\nabla}^{\star}_{\partial_1}\partial_4 = \widetilde{\nabla}^{\star}_{\partial_4}\partial_1 = \frac{1}{2}\partial_4,$$
$$\widetilde{\nabla}^{\star}_{\partial_1}\partial_2 = \widetilde{\nabla}^{\star}_{\partial_2}\partial_1 = \widetilde{\nabla}^{\star}_{\partial_1}\partial_3 = \widetilde{\nabla}^{\star}_{\partial_3}\partial_1 = 0,$$
$$\widetilde{\nabla}^{\star}_{\partial_2}\partial_3 = \widetilde{\nabla}^{\star}_{\partial_3}\partial_2 = \widetilde{\nabla}^{\star}_{\partial_2}\partial_4 = \widetilde{\nabla}^{\star}_{\partial_4}\partial_2 = \widetilde{\nabla}^{\star}_{\partial_3}\partial_4 = \widetilde{\nabla}_{\partial_4}\partial_3 = 0.$$

Thus, $(\mathbb{R}^4_1, \widetilde{h}, \widetilde{\nabla}, F)$ is a locally product-like statistical manifold and so is $(\mathbb{R}^4_1, \widetilde{h}, \widetilde{\nabla}^{\star}, F^{\star})$.

Further examples can be derived.

For any lightlike hypersurface $(L, h, \mathrm{S}(TM))$ of $(\widetilde{L}, \widetilde{h}, \widetilde{\nabla}, F)$, the Gauss and Weingarten formulas defined on the statistical structure $(\widetilde{\nabla}, \widetilde{\nabla}^{\star})$ could be represented by

$$\widetilde{\nabla}_{Z_1} Z_2 = \nabla_{Z_1} Z_2 + B(Z_1, Z_2)N, \tag{4.2}$$
$$\widetilde{\nabla}_{Z_1} N = -A^{\star}_N Z_1 + \omega^{\star}(Z_1)N \tag{4.3}$$

and

$$\widetilde{\nabla}^{\star}_{Z_1} Y = \nabla^{\star}_{Z_1} Z_2 + B^{\star}(Z_1, Z_2)N, \tag{4.4}$$
$$\widetilde{\nabla}^{\star}_{Z_1} N = -A_N Z_1 + \omega(Z_1)N \tag{4.5}$$

for any $Z_1, Z_2 \in \Gamma(TL)$, where $\nabla$ and $\nabla^{\star}$ are induced linear connections, $A_N$ and $A^{\star}_N$ are shape operators with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^{\star}$, $\omega$ and $\omega^{\star}$ are $1-$forms.

The Gauss and Weingarten-type formulas on $\mathrm{S}(TL)$ are given by

$$\nabla_{Z_1} PZ_2 = \overline{\nabla}_{Z_1} PZ_2 + C(Z_1, PZ_2)\xi, \tag{4.6}$$
$$\nabla_{Z_1}\xi = -\overline{A}_{\xi} Z_1 - \omega(Z_1)\xi \tag{4.7}$$

and

$$\nabla^{\star}_{Z_1} PZ_2 = \overline{\nabla}^{\star}_{Z_1} PZ_2 + C^{\star}(Z_1, PZ_2)\xi, \tag{4.8}$$
$$\nabla^{\star}_{Z_1}\xi = -\overline{A}^{\star}_{\xi} Z_1 - \omega^{\star}(Z_1)\xi, \tag{4.9}$$

where $\overline{\nabla}_{Z_1} PZ_2$ and $\overline{\nabla}^{\star}_{Z_1} PZ_2$ belong to $\Gamma(\mathrm{S}(TL))$, $\overline{A}_{\xi}$ and $\overline{A}^{\star}_{\xi}$ are local shape operators defined on $\Gamma(\mathrm{S}(TL))$, $P$ denotes the projection of $\Gamma(TL)$ onto $\Gamma(\mathrm{S}(TL))$.

It is known from [8, 9] that the followings hold for any $Z_1, Z_2 \in \Gamma(TL)$:

$$B(Z_1, \xi) + B^{\star}(Z_1, \xi) = 0, \quad h(A_N Z_1 + A^{\star}_N Z_1, Z_2) = 0, \tag{4.10}$$
$$C(Z_1, PZ_2) = h(A_N Z_1, PZ_2), \quad C^{\star}(Z_1, PZ_2) = h(A^{\star}_N Z_1, PZ_2) \tag{4.11}$$

and

$$B(Z_1, Z_2) = g(\overline{A}^{\star}_{\xi} Z_1, Z_2) + B^{\star}(Z_1, \xi)\widetilde{h}(Z_2, N), \tag{4.12}$$
$$B^{\star}(Z_1, Z_2) = h(\overline{A}_{\xi} Z_1, Z_2) + B(Z_1, \xi)\widetilde{h}(Z_2, N). \tag{4.13}$$

A lightlike hypersurface $(L, h, \mathrm{S}(TM))$ of a statistical manifold is called screen conformal with respect to $\widetilde{\nabla}$ (or $\widetilde{\nabla}^{\star}$) if $A_N = \alpha A_{\xi}$ (or $A^{\star}_N = \alpha \overline{A}^{\star}_{\xi}$) holds where $\alpha$ is a differentiable function. Furthermore, $(L, h, \mathrm{S}(TL))$ is called totally geodesic with respect to $\widetilde{\nabla}$ if $B = 0$, totally geodesic with respect to $\widetilde{\nabla}^{\star}$ if $B^{\star} = 0$, $S(TL)-$geodesic with respect to $\widetilde{\nabla}$ if $C = 0$ and $S(TL)-$geodesic with respect to $\widetilde{\nabla}^{\star}$ if $C^{\star} = 0$ [8, 19].

## 5 Radical invariant lightlike hypersurfaces

**Definition 5.1.** Let $(L, h, \mathrm{S}(TL))$ be a lightlike hypersurface of $(\widetilde{L}, \widetilde{h}, F)$. If $F$ and $F^\star$ are invariants on the radical space $\mathrm{Rad}(TL)$, then $(M, h, \mathrm{S}(TL))$ is called a radical invariant lightlike hypersurface.

Suppose that $\mathrm{Rad}(TL) = \mathrm{Span}\{\xi\}$ and $\mathrm{ltr}(TM) = \mathrm{Span}\{N\}$. If $(L, g, \mathrm{S}(TL))$ is a radical invariant lightlike hypersurface, then we can write $F\xi = \lambda_1\xi$, where $\lambda_1$ is a smooth function. Considering $F^2 = I$, we have $\lambda_1 = \mp 1$. Similarly, we find $F^\star\xi = \mp\xi$, $FN = \mp N$, $F^\star N = \mp N$ and $FZ, F^\star Z \in \Gamma(\mathrm{S}(TL))$ for any $Z \in \Gamma(\mathrm{S}(TL))$.

Considering the above facts, we have

**Theorem 5.2.** Every radical invariant lightlike hypersurface is screen invariant; that is, $\mathrm{S}(TL)$ is invariant under the actions of $F$ and $F^\star$.

*Proof.* If $Z \in \Gamma(\mathrm{S}(TL))$, then we have

$$\widetilde{h}(FZ, \xi) = \widetilde{h}(Z, F^\star\xi) = 0, \tag{5.1}$$

$$\widetilde{h}(F^\star Z, \xi) = \widetilde{h}(Z, F\xi) = 0 \tag{5.2}$$

and

$$\widetilde{h}(FZ, N) = \widetilde{h}(Z, F^\star N) = 0, \tag{5.3}$$

$$\widetilde{h}(F^\star Z, N) = \widetilde{h}(Z, FN) = 0. \tag{5.4}$$

In view of (5.1), (5.2), (5.3) and (5.4), we obtain that $\mathrm{S}(TL)$ is invariant under the actions of $F$ and $F^\star$. Q.E.D.

We remark that the converse part of Theorem 5.2 is generally not true. That is, a screen invariant lightlike hypersurface does not need to be radical invariant under the actions of $F$ and $F^\star$.

As a result of Definition 5.1 and Theorem 5.2, we obtain

**Corollary 5.3.** Every radical invariant lightlike hypersurface is both $F-$invariant and $F^\star-$invariant.

**Example 5.4.** Consider the following hypersurface of $(\mathbb{R}^4_1, \widetilde{h}, F)$, which is given in Example 4.1:

$$L = \{(z_1, z_2, z_3, z_4) : z_1 = z_4\}$$

In this case, the induced metric $h$ of $L$ is given by

$$h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, we have

$$\mathrm{Rad}(TL) = \mathrm{Span}\{\xi = \partial_1 + \partial_4\}, \quad \mathrm{ltr}(TL) = \mathrm{Span}\{N = \frac{1}{4}(-\partial_1 + \partial_4)\},$$
$$\mathrm{S}(TL) = \mathrm{Span}\{e_1 = \partial_2, e_2 = \partial_3\}.$$

Since $F\xi = \xi$, $F^\star\xi = -\xi$, $FN = -N$ and $F^\star N = N$, we obtain that $(M, h, \mathrm{S}(TL))$ is radical invariant.

Further examples of these lightlike hypersurfaces can be provided.

For any radical invariant lightlike hypersurface, we can write $Z = PZ + \eta(Z)\xi$, where $\eta(Z) = \widetilde{h}(Z, N)$. We find herewith

$$FZ = F(PZ + \eta(Z)\xi) = \varphi PZ + \varepsilon\eta(Z)\xi, \tag{5.5}$$

where $\varphi PZ \in \Gamma(S(TL))$ and $\varepsilon = \mp 1$. From Theorem 5.2, we can see that

$$\varphi PZ = \varphi Z \tag{5.6}$$

is satisfied for any $Z \in \Gamma(TL)$.

**Proposition 5.5.** For any radical invariant lightlike hypersurface $(L, h, \mathrm{S}(TL))$ of a locally product-like Lorentzian manifold, we get

$$\overline{\nabla}_{Z_2}\varphi Z_1 - \eta(Z_1)\overline{A}_\xi Z_2 = \varphi\nabla_{Z_2}Z_1, \tag{5.7}$$

$$C(Z_2, \varphi Z_1) + Z_2\left[\eta(Z_1)\right] - \varepsilon\eta(Z_1)\tau(Z_2) = \eta(\nabla_{Z_2}Z_1) \tag{5.8}$$

and

$$B(Z_2, \varphi Z_1) + \varepsilon\eta(Z_1)B(Z_2, \xi) = \varepsilon B(Z_2, Z_1). \tag{5.9}$$

*Proof.* For any $Z_1, Z_2 \in \Gamma(TL)$, we can write from (4.2), (5.5) and (5.6) that

$$\widetilde{\nabla}_{Z_2}FZ_1 = \nabla_{Z_2}\varphi Z_1 + B(Z_2, \varphi Z_1)N + \varepsilon\widetilde{\nabla}_{Z_2}\eta(Z_1)\xi. \tag{5.10}$$

Besides this fact, we also get

$$\widetilde{\nabla}_{Z_2}\eta(Z_1)\xi = Z_2\left[\eta(Z_1)\right]\xi + \eta(Z_1)\left[\nabla_{Z_2}\xi + B(Z_2, \xi)N\right]. \tag{5.11}$$

Substituting (5.11) into (5.10), we find

$$\widetilde{\nabla}_{Z_2}FZ_1 = \overline{\nabla}_{Z_2}\varphi Z_1 + C(Z_2, \varphi Z_1)\xi - \varepsilon\eta(Z_1)\overline{A}_\xi Z_2 - \varepsilon\eta(Z_1)\tau(Z_2)\xi + \varepsilon\eta(Z_1)B(Z_2, \xi)N. \tag{5.12}$$

On the other hand, we write

$$F\widetilde{\nabla}_{Z_2}Z_1 = \varphi\nabla_{Z_2}Z_1 + \varepsilon\eta(\nabla_{Z_2}Z_1)\xi + B(Z_2, Z_1)FN. \tag{5.13}$$

Using the fact that $\widetilde{\nabla}_{Z_2}FX_1 = F\widetilde{\nabla}_{Z_2}Z_1$ and considering (5.12) and (5.13), we have (5.7), (5.8) and (5.9) immediately. Q.E.D.

**Theorem 5.6.** $(L, h, \mathrm{S}(TL))$ is totally geodesic with respect to $\widetilde{\nabla}$ if and only if $B$ vanishes in $Rad(TL)$.

*Proof.* Assume that $B$ vanishes on $\mathrm{Rad}(TL)$. In view of (5.9), we obtain $B(XZ_2, \varphi Z_1 - Z_1) = 0$. Therefore, we see that either $\varphi Z_1 = Z_1$ is satisfied or $(L, h, \mathrm{S}(TL))$ is totally geodesic. The first case contradicts the fact that $F \neq I$. Thus, we get $(L, h, \mathrm{S}(TL))$ is totally geodesic. The converse part is straightforward. Q.E.D.

**Theorem 5.7.** $\varphi$ is parallel with respect to $\nabla$ if and only if $(L, h, \mathrm{S}(TL))$ is totally geodesic with respect to $\widetilde{\nabla}^{\star}$ and $C$ vanishes identically.

*Proof.* By using the fact that

$$\nabla_{Z_2}\varphi Z_1 = \varphi\nabla_{Z_2}Z_1 + (\nabla_{Z_2}\varphi)Z_1$$

and from (4.6), we obtain

$$\varphi\nabla_{Z_2}Z_1 = \overline{\nabla}_{Z_2}\varphi Z_1 + C(Z_2, \varphi Z_1)\xi - (\nabla_{Z_2}\varphi)Z_1 = \overline{\nabla}_{Z_2}\varphi Z_1 - \eta(Z_1)\overline{A}_{\xi}Z_2. \tag{5.14}$$

The equation (5.14) implies that

$$(\nabla_{Z_2}\varphi)Z_1 = C(Z_2, \varphi Z_1)\xi + \eta(Z_1)\overline{A}_{\xi}Z_2. \tag{5.15}$$

If $\nabla_{Z_2}\varphi = 0$, then $C(Z_2, \varphi Z_1) = 0$ and $\overline{A}_{\xi}Z_2 = 0$. Therefore, we obtain that the hypersurface is totally geodesic with respect to $\widetilde{\nabla}^{\star}$ and $C$ vanishes identically. The converse is straightforward from (5.15). Q.E.D.

**Proposition 5.8.** Let $(L, h, \mathrm{S}(TL))$ be a radical invariant lightlike hypersurface of a locally product-like statistical manifold $(\widetilde{L}, \widetilde{h}, \widetilde{\nabla}, F)$. Then, the following relations hold for any $Z_1, Z_2 \in \Gamma(\mathrm{S}(TL))$:

$$\overline{\nabla}_{Z_1}\varphi Z_2 = \varphi\nabla_{Z_1}Z_2, \qquad \overline{\nabla}^{\star}_{Z_1}\varphi^{\star}Z_2 = \varphi^{\star}\nabla^{\star}_{Z_1}Z_2, \tag{5.16}$$

$$B(Z_1, \varphi Z_2) = \varepsilon B(Z_1, Z_2), \qquad B^{\star}(Z_1, \varphi^{\star}Z_2) = \varepsilon B^{\star}(Z_1, Z_2), \tag{5.17}$$

$$C(Z_1, \varphi Z_2) = \eta(\nabla_{Z_1}Z_2), \qquad C^{\star}(Z_1, \varphi^{\star}Z_2) = \eta(\nabla^{\star}_{Z_1}Z_2), \tag{5.18}$$

$$\varphi^{\star}A_N Z_2 = A_N\varphi Z_2, \qquad \varphi A^{\star}_N Z_2 = A^{\star}_N\varphi^{\star}Z_2. \tag{5.19}$$

*Proof.* Using the fact that $\widetilde{\nabla}_{Z_1}FZ_2 = F\widetilde{\nabla}_{Z_1}Z_2$, we obtain

$$\nabla_{Z_1}\varphi Z_2 + B(Z_1, \varphi Z_2)N = \varphi\nabla_{Z_1}Z_2 + \eta(\nabla_{Z_1}Z_2)\xi + B(Z_1, Z_2)FN. \tag{5.20}$$

In view of (5.20), we find that $B(Z_1, \varphi Z_2) = \varepsilon B(Z_1, Z_2)$ and

$$\nabla_{Z_1}\varphi Z_2 = \varphi\nabla_{Z_1}Z_2 + \eta(\nabla_{Z_1}Z_2)\xi. \tag{5.21}$$

From (4.6) and (5.21), we obtain

$$\overline{\nabla}_{Z_1}\varphi Z_2 + C(Z_1, \varphi Z_2)\xi = \varphi\nabla_{Z_1}Z_2 + \eta(\nabla_{Z_1}Z_2)\xi,$$

which indicates that $\overline{\nabla}_{Z_1}\varphi Z_2 = \varphi\nabla_{Z_1}Z_2$ and $C(Z_1, \varphi Z_2) = \eta(\nabla_{Z_1}Z_2)$.

With similar arguments to (5.20), we obtain

$$\nabla^{\star}_{Z_1}\varphi^{\star}Z_2 = \varphi^{\star}\nabla_{Z_1}Z_2 + \eta(\nabla^{\star}_{Z_1}Z_2)\xi. \tag{5.22}$$

The equation (5.22) implies that $\overline{\nabla}^{\star}_{Z_1}\varphi^{\star}Z_2 = \varphi^{\star}\nabla^{\star}_{Z_1}Z_2$ and $B^{\star}(Z_1, \varphi^{\star}Z_2) = \varepsilon B^{\star}(Z_1, Z_2)$. Substituting (4.11) into (5.18), we obtain (5.19). Q.E.D.

For any lightlike hypersurface, it is known that any screen distribution does not have to be integrable with respect to $\nabla^0$. In [13], K. L. Duggal and A. Bejancu proved the following theorem:

**Theorem 5.9.** Let $(L, h, \mathrm{S}(TL))$ be a lightlike hypersurface. Then, the following situations are equivalent:

i) $\mathrm{S}(TL)$ is integrable.

ii) $C^0$ is symmetric on $\Gamma(\mathrm{S}(TL))$.

iii) $A^0_N$ is self-adjoint with respect to $h$.

Now, we discuss the necessary conditions of $\mathrm{S}(TL)$ to be integrable with respect to $\nabla$ and $\nabla^\star$ as follows:

**Theorem 5.10.** For any radical invariant lightlike hypersurface of locally product-like statistical manifold $(\widetilde{L}, \widetilde{h}, \widetilde{\nabla}, F)$, the following situations are equivalent:

i) $\mathrm{S}(TL)$ is integrable with respect to $\nabla$.

ii) $C(Z_1, \varphi Z_2) = C(\varphi Z_1, Z_2)$ is satisfied for any $Z_1, Z_2 \in \Gamma(\mathrm{S}(TM))$.

iii) $h(A_N \varphi Z_2, Z_1) = h(A_N \varphi Z_1, Z_2)$ or equivalently, $h(A_N Z_1, \varphi Z_2) = h(A_N Z_2, \varphi Z_1)$ is satisfied for any $Z_1, Z_2 \in \Gamma(\mathrm{S}(TM))$.

*Proof.* In view of (5.18), we have

$$C(Z_1, \varphi Z_2) = \eta(\nabla_{Z_1} Z_2) \quad \text{and} \quad C(Z_2, \varphi Z_1) = \eta(\nabla_{Z_2} Z_1). \tag{5.23}$$

If we subtract the equations in (5.23), we arrive at

$$C(Z_1, \varphi Z_2) - C(\varphi Z_1, Z_2) = \eta(\nabla_{Z_2} Z_1 - \nabla_{Z_1} Z_2),$$

which is the proof of i)⇔ii). The proof of ii)⇔iii) is straightforward based on (4.11). Q.E.D.

In analogy to Theorem 5.10, we have

**Theorem 5.11.** For any radical invariant lightlike hypersurface of locally product-like statistical manifold $(\widetilde{L}, \widetilde{h}, \widetilde{\nabla}, F)$, the following situations are equivalent:

i) $\mathrm{S}(TL)$ is integrable with respect to $\nabla^\star$.

ii) $C^\star(Z_1, \varphi^\star Z_2) = C(\varphi^\star Z_1, Z_2)$ is satisfied for any $Z_1, Z_2 \in \Gamma(\mathrm{S}(TL))$.

iii) $h(A^\star_N \varphi^\star Z_2, Z_1) = h(A^\star_N \varphi^\star Z_1, Z_2)$ or equivalently, $h(A^\star_N Z_1, \varphi Z_2) = h(A^\star_N Z_2, \varphi^\star Z_1)$ is satisfied for any $Z_1, Z_2 \in \Gamma(\mathrm{S}(TL))$.

## 6 Identities involving Riemannian curvature invariants

Let $(\widetilde{L},\widetilde{h},\widetilde{\nabla},F)$ be a locally product-like statistical manifold. Considering the definitions of Riemannian curvature tensor fields $\widetilde{R}$ and $\widetilde{R}^{\star}$, we get the following equalities:

$$\widetilde{R}(Z_1,Z_2)FZ_3=F\widetilde{R}(Z_1,Z_2)Z_3,\qquad \widetilde{R}^{\star}(Z_1,Z_2)F^{\star}Z_3=F^{\star}\widetilde{R}^{\star}(Z_1,Z_2)Z_3, \tag{6.1}$$

$$\widetilde{R}(FZ_1,FZ_2)Z_3=\widetilde{R}(Z_1,Z_2)Z_3,\qquad \widetilde{R}^{\star}(F^{\star}Z_1,F^{\star}Z_2)Z_3=\widetilde{R}^{\star}(Z_1,Z_2)Z_3, \tag{6.2}$$

$$S(FZ_1,FZ_2)=S(Z_1,Z_2),\qquad S(F^{\star}Z_1,F^{\star}Z_2)=S(Z_1,Z_2). \tag{6.3}$$

for any $Z_1,Z_2,Z_3\in\Gamma(T\widetilde{L})$. Next, let us suppose that the equality

$$\begin{aligned}\widetilde{R}(Z_1,Z_2)Z_3=\vartheta\Big[&\widetilde{h}(Z_2,Z_3)Z_1-\widetilde{h}(Z_1,Z_3)Z_2+\widetilde{h}(Z_2,FZ_3)FZ_1\\&-\widetilde{h}(Z_1,FZ_3)FZ_2+\widetilde{h}(FZ_1,Z_2)FZ_3-\widetilde{h}(Z_1,FZ_2)FZ_3\Big]\end{aligned} \tag{6.4}$$

is satisfied. From (2.3), it is clear that

$$\begin{aligned}\widetilde{R}^{\star}(Z_1Z_2)Z_3=\vartheta\Big[&\widetilde{h}(Z_2,Z_3)Z_1-\widetilde{h}(Z_1,Z_3)Z_2+\widetilde{h}(Z_2,F^{\star}Z_3)F^{\star}Z_1\\&-\widetilde{h}(Z_1,F^{\star}Z_3)F^{\star}Z_2+\widetilde{h}(F^{\star}Z_1,Z_2)F^{\star}Z_3-\widetilde{h}(Z_1,F^{\star}Z_2)F^{\star}Z_3\Big]\end{aligned} \tag{6.5}$$

is also satisfied.

Inspired by the notion of Riemannian product manifolds with constant curvatures, the following definition can be given:

**Definition 6.1.** A locally product-like statistical manifold $(\widetilde{L},\widetilde{h},\widetilde{\nabla},F)$ is said to be of constant curvature $\vartheta$ if the equation (6.4) is satisfied.

From (2.3), it is clear that $(\widetilde{L},\widetilde{h},\widetilde{\nabla},F)$ is of constant curvature with respect to $\widetilde{R}$ if and only if it is of constant curvature with respect to $\widetilde{R}^{\star}$.

**Example 6.2.** Let $(\mathbb{R}^4_1,\widetilde{h},F)$ be an almost product-like statistical manifold in Example 4.4. By straightforward computation, we find $\widetilde{R}(\partial_i,\partial_j)\partial_j=0$ for any $i,j\in\{1,2,3,4\}$. Then, it is of constant curvature such that $\vartheta=0$.

Following the definition of almost constant curvature manifolds in Riemannian product manifolds (cf. Theorem 2.5, page 422 in [27]), we can state the following definition, which is a generalized version of locally product-like statistical manifolds with constant curvatures:

**Definition 6.3.** A locally product-like Lorentzian manifold $(\widetilde{L},\widetilde{h},\widetilde{\nabla},F)$ is said to be an almost constant curvature-like manifold if the equation

$$\begin{aligned}\widetilde{h}(\widetilde{R}(Z_1,Z_2)Z_3,Z_4)\;=\;&a_1\Big[\widetilde{h}(Z_1,Z_4)\widetilde{h}(Z_2,Z_3)-\widetilde{h}(Z_1,Z_3)\widetilde{h}(Z_2,Z_4)\\&+\widetilde{h}(Z_2,FZ_3)\widetilde{h}(FZ_1,Z_4)-\widetilde{h}(Z_1,FZ_3)\widetilde{h}(FZ_2,Z_4)\\&+\widetilde{h}(FZ_1,Z_2)\widetilde{h}(FZ_3,Z_4)-\widetilde{h}(Z_1,FZ_2)\widetilde{h}(FZ_3,Z_4)\Big]\\&+b_1\Big[\widetilde{h}(Z_1,FZ_4)\widetilde{h}(Z_2,Z_3)-\widetilde{h}(Z_1,FZ_3)\widetilde{h}(Z_2,Z_4)\\&+\widetilde{h}(Z_1,Z_4)\widetilde{h}(Z_2,FZ_3)-\widetilde{h}(Z_1,Z_3)\widetilde{h}(Z_2,FZ_4)\Big]\end{aligned} \tag{6.6}$$

holds for any $Z_1, Z_2, Z_3, Z_4 \in \Gamma(T\widetilde{L})$, where $a_1$ and $b_1$ are constants. For simplicity, we indicate an almost constant curvature-like manifold by $\widetilde{L}(a_1, b_1)$.

Considering the equations (2.3) and (6.6), it is clear that

$$\begin{aligned}\widetilde{h}(\widetilde{R}^{\star}(Z_1,Z_2)Z_3,Z_4) &= a_2\Big[\widetilde{h}(Z_1,Z_4)\widetilde{h}(Z_2,Z_3)-\widetilde{h}(Z_1,Z_3)\widetilde{h}(Z_2,Z_4)\\ &\quad+\widetilde{h}(Z_2,F^{\star}Z_3)\widetilde{h}(F^{\star}Z_1,Z_4)-\widetilde{h}(Z_1,F^{\star}Z_3)\widetilde{h}(F^{\star}Z_2,Z_4)\\ &\quad+\widetilde{h}(F^{\star}Z_1,Z_2)\widetilde{h}(F^{\star}Z_3,Z_4)-\widetilde{h}(Z_1,F^{\star}Z_2)\widetilde{h}(F^{\star}Z_3Z_4)\Big]\\ &\quad+b_2\Big[\widetilde{h}(Z_1,F^{\star}Z_4)\widetilde{h}(Z_2,Z_3)-\widetilde{h}(Z_1,F^{\star}Z_3)\widetilde{h}(Z_2,Z_4)\\ &\quad+\widetilde{h}(Z_1,Z_4)\widetilde{h}(Z_2,F^{\star}Z_3)-\widetilde{h}(Z_1,Z_3)\widetilde{h}(Z_2,F^{\star}Z_4)\Big] \qquad (6.7)\end{aligned}$$

is also satisfied, where $a_2$ and $b_2$ are constants.

Now, we recall the following lemma [8]:

**Lemma 6.4.** Let $(\widetilde{L},\widetilde{h},\widetilde{\nabla})$ be a statistical manifold and $(L,h,\mathrm{S}(TL))$ be a lightlike hypersurface of $(\widetilde{L},\widetilde{h},\widetilde{\nabla})$. Then, we have

$$\begin{aligned}\widetilde{R}(Z_1,Z_2)Z_3 &= R(Z_1,Z_2)Z_3-B(Z_2,Z_3)A^{\star}_NZ_1+B(Z_1,Z_3)A^{\star}_NZ_2\\ &\quad+\left(B(Z_2,Z_3)\tau^{\star}(Z_1)-B(Z_1,Z_3)\tau^{\star}(Z_2)\right)N\\ &\quad+((\nabla_{Z_1}B)(Z_2,Z_3)-(\nabla_{Z_2}B)(Z_1,Z_3))N \qquad (6.8)\end{aligned}$$

and

$$\begin{aligned}\widetilde{R}^{\star}(Z_1,Z_2)Z_3 &= R^{\star}(Z_1,Z_2)Z_3-B^{\star}(Z_2,Z_3)A_NZ_1+B^{\star}(Z_1,Z_3)A_NZ_2\\ &\quad+(B^{\star}(Z_2,Z_3)\tau(Z_1)-B^{\star}(Z_1,Z_3)\tau(Z_2))N\\ &\quad+((\nabla^{\star}_{Z_1}B^{\star})(Z_2,Z_3)-(\nabla^{\star}_{Z_2}B^{\star})(Z_1,Z_3))N \qquad (6.9)\end{aligned}$$

for any $Z_1, Z_2, Z_3, Z_4 \in \Gamma(TL)$.

In view of Lemma 6.4, we get

$$\begin{aligned}\widetilde{h}(\widetilde{R}(Z_1,Z_2)Z_3,Z_4) &= h(R(Z_1,Z_2)Z_3,Z_4)-B(Z_2,Z_3)g(A^{\star}_NZ_1,Z_4)\\ &\quad+B(Z_1,Z_3)h(A^{\star}_NZ_2,Z_4) \qquad (6.10)\end{aligned}$$

and

$$\begin{aligned}\widetilde{h}(\widetilde{R}^{\star}(Z_1,Z_2)Z_3,Z_4) &= h(R^{\star}(Z_1,Z_2)Z_3,Z_4)-B^{\star}(Z_2,Z_3)h(A_NZ_1,Z_4)\\ &\quad+B^{\star}(Z_1,Z_3)h(A_NZ_2,Z_4). \qquad (6.11)\end{aligned}$$

Let us denote the screen Ricci curvatures of $(L,h,\mathrm{S}(TL))$ with respect to $\nabla$ and $\nabla^{\star}$ by $\mathrm{Ric}_{\mathrm{S}(TL)}$ and $\mathrm{Ric}^{\star}_{\mathrm{S}(TL)}$. For any unit vector field $Z \in \Gamma(S(TL))$, we write

$$\mathrm{Ric}_{\mathrm{S}(TL)}(Z)=\sum_{j=1}^{n}h(R(Z,e_j)e_j,Z) \quad \text{and} \quad \mathrm{Ric}^{\star}_{\mathrm{S}(TL)}(Z)=\sum_{j=1}^{n}h(R^{\star}(Z,e_j)e_j,Z). \qquad (6.12)$$

Also, we write the screen Ricci curvatures of $(\widetilde{L}, \widetilde{h}, \widetilde{\nabla})$ with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^\star$ as follows:

$$\widetilde{\mathrm{Ric}}_{\mathrm{S}(TL)}(Z) = \sum_{j=1}^{n} \widetilde{h}(\widetilde{R}(Z, e_j)e_j, Z) \quad \text{and} \quad \widetilde{\mathrm{Ric}}^{\star}_{\mathrm{S}(TL)}(Z) = \sum_{j=1}^{n} h(R^\star(Z, e_j)e_j, Z). \tag{6.13}$$

**Proposition 6.5.** Let $(L, h, \mathrm{S}(TL))$ be a lightlike hypersurface of a statistical manifold $(\widetilde{L}, \widetilde{h}, \widetilde{\nabla})$. Then, we have the following equalities for each unit vector field $Z \in \Gamma(S(TL))$:

$$\widetilde{\mathrm{Ric}}_{\mathrm{S}(TL)}(Z) = \mathrm{Ric}_{\mathrm{S}(TL)}(Z) - \mathrm{trace}(\overline{A}^{\star}_{\xi})h(A^{\star}_{N}Z, Z) + h(\overline{A}^{\star}_{\xi}Z, A^{\star}_{N}Z), \tag{6.14}$$

$$\widetilde{\mathrm{Ric}}^{\star}_{\mathrm{S}(TL)}(Z) = \mathrm{Ric}^{\star}_{\mathrm{S}(TL)}(Z) - \mathrm{trace}(\overline{A}_{\xi})h(A_{N}Z, Z) + h(\overline{A}_{\xi}Z, A_{N}Z). \tag{6.15}$$

*Proof.* Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal frame field of $\Gamma(\mathrm{S}(TL))$. From (6.10), we obtain

$$\begin{aligned}\sum_{j=2}^{n} \widetilde{h}(\widetilde{R}(e_1, e_j)e_j, e_1) = \sum_{j=2}^{n} \widetilde{h}(R(e_1, e_j)e_j, e_1) - \sum_{j=2}^{n} B(e_j, e_j)h(A^{\star}_{N}e_1, e_1) \\ + \sum_{j=2}^{n} B(e_1, e_j)h(A^{\star}_{N}e_j, e_1).\end{aligned} \tag{6.16}$$

Using

$$\begin{aligned}\sum_{j=2}^{n} B(e_j, e_j)h(A^{\star}_{N}e_1, e_1) &= \mathrm{trace}(\overline{A}^{\star}_{\xi})h(A^{\star}_{N}e_1, e_1) - \sum_{j=1}^{n} h(A^{\star}_{N}e_1, e_j)h(A^{\star}_{N}e_1, e_j) \\ &= \mathrm{trace}(\overline{A}^{\star}_{\xi})h(A^{\star}_{N}e_1, e_1) + h(\overline{A}^{\star}_{\xi}e_1, A^{\star}_{N}e_1)\end{aligned} \tag{6.17}$$

in (6.16) and putting $e_1 = Z$, we find (6.14). Using similar arguments to the proof of (6.14), the equation (6.15) can be proved. Q.E.D.

Let us denote the screen scalar curvatures of $(L, h, \mathrm{S}(TL))$ with respect to $\nabla$ and $\nabla^\star$ by $\tau_{\mathrm{S}(TL)}$ and $\tau^{\star}_{\mathrm{S}(TL)}$. We write

$$\tau_{\mathrm{S}(TL)}(p) = \sum_{i=1}^{n} \mathrm{Ric}_{\mathrm{S}(TL)}(e_i) \quad \text{and} \quad \tau^{\star}_{\mathrm{S}(TL)}(p) = \sum_{i=1}^{n} \mathrm{Ric}^{\star}_{\mathrm{S}(TL)}(e_i). \tag{6.18}$$

Also, we write the screen scalar curvatures of $(\widetilde{L}, \widetilde{h}, \widetilde{\nabla})$ with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^\star$ as follows:

$$\widetilde{\tau}_{\mathrm{S}(TL)}(p) = \sum_{i=1}^{n} \widetilde{\mathrm{Ric}}_{\mathrm{S}(TL)}(e_i) \quad \text{and} \quad \widetilde{\tau}^{\star}_{\mathrm{S}(TL)}(p) = \sum_{i=1}^{n} \widetilde{\mathrm{Ric}^{\star}}_{\mathrm{S}(TL)}(e_i). \tag{6.19}$$

**Proposition 6.6.** Let $(L, h, \mathrm{S}(TL))$ be a lightlike hypersurface of $(\widetilde{L}, \widetilde{h}, \widetilde{\nabla})$ and $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal frame field of $\Gamma(\mathrm{S}(TL))$. Then, the following equalities hold for a point $p \in L$:

$$\widetilde{\tau}_{\mathrm{S}(TL)}(p) = \tau_{\mathrm{S}(TL)}(p) - n\ \mathrm{trace}(\overline{A}^{\star}_{\xi})trace(A^{\star}_{N}) + \sum_{i=1}^{n} h(\overline{A}^{\star}_{\xi}e_i, A^{\star}_{N}e_i), \tag{6.20}$$

$$\widetilde{\tau}^{\star}_{\mathrm{S}(TL)}(p) = \tau^{\star}_{\mathrm{S}(TM)}(p) - n\ \mathrm{trace}(\overline{A}_{\xi})trace(A_{N}) + \sum_{i=1}^{n} h(\overline{A}_{\xi}e_i, A_{N}e_i). \tag{6.21}$$

*Proof.* Taking the trace in (6.16) and considering (6.18) and (6.19), we find (6.20). The proof of (6.21) can be obtained in a similar manner. Q.E.D.

**Proposition 6.7.** Let $\widetilde{L}(a_1, b_1)$ be an almost constant curvature-like manifold. Then, the following situations hold:

i) For any unit vector field $Z \in \Gamma(\mathrm{S}(TL))$, we have

$$\begin{aligned}\widetilde{Ric}_{\mathrm{S}(TL)}(Z) &= a_1\left[n + (\mathrm{trace}F)\widetilde{h}(FZ,Z) - 2\left\|F^{\star}Z\right\|^2\right] \\ &\quad + b_1\left[(n-2)\widetilde{g}(FZ,Z) + \mathrm{trace}F - \left\|F^{\star}X\right\|^2 - (\widetilde{h}(FZ,Z))^2\right] \qquad (6.22)\end{aligned}$$

and

$$\begin{aligned}\widetilde{Ric}^{\star}_{\mathrm{S}(TL)}(Z) &= a_2\left[n + (\mathrm{trace}F^{\star})\widetilde{h}(F^{\star}Z,Z) - 2\left\|FZ\right\|^2\right] \\ &\quad + b_2\left[(n-2)\widetilde{h}(F^{\star}Z,Z) + \mathrm{trace}F^{\star} - \left\|FX\right\|^2 - \widetilde{h}(F^{\star}Z,Z))^2\right]. \qquad (6.23)\end{aligned}$$

ii) For any point $p \in \widetilde{L}$, we have

$$\begin{aligned}\widetilde{\tau}_{\mathrm{S}(TL)}(p) &= a_1\left[n^2 + (\mathrm{trace}\ F)^2 - 2\left\|F^{\star}\right\|^2\right] \\ &\quad + b_1\left[(2n-2)\mathrm{trace}F - (\mathrm{trace}\ F)^2 - \left\|F^{\star}\right\|^2\right] \qquad (6.24)\end{aligned}$$

and

$$\begin{aligned}\widetilde{\tau}^{\star}_{\mathrm{S}(TL)}(p) &= a_2\left[n^2 + (traceF^{\star})^2 - 2\left\|F\right\|^2\right] \\ &\quad + b_2\left[(2n-2)traceF^{\star} - (traceF^{\star})^2 - \left\|F\right\|^2\right]. \qquad (6.25)\end{aligned}$$

*Proof.* Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal frame field of $\Gamma(\mathrm{S}(TL))$. From the definition of almost constant curvature manifolds, we get

$$\begin{aligned}\sum_{j=2}^{n}\widetilde{h}(\widetilde{R}(e_1,e_j)e_j,e_1) &= a_1[(n-1) + \sum_{j=1}^{n}\widetilde{h}(Fe_j,e_j)\widetilde{g}(Fe_1,e_1) - (\widetilde{h}(Fe_1,e_1))^2 \\ &\quad -n\sum_{j=1}^{n}\widetilde{h}(Fe_1,e_j)\widetilde{h}(F^{\star}e_1,e_j) + (\widetilde{h}(Fe_1,e_1))^2 \\ &\quad +\sum_{j=1}^{n}(\widetilde{h}(F^{\star}e_j,e_1))^2 - (\widetilde{h}(Fe_1,e_1))^2 \\ &\quad +\sum_{j=1}^{n}(\widetilde{h}(Fe_j,e_1))^2 + (\widetilde{h}(Fe_1,e_1))^2]. \qquad (6.26)\end{aligned}$$

Using

$$\sum_{j=1}^{n}\widetilde{h}(Fe_1,e_j)\widetilde{h}(F^\star e_1,e_j)=\widetilde{h}(Fe_1,F^\star e_1)=\widetilde{h}(e_1,e_1)=1$$

in (6.26), we obtain (6.22). The proof of (6.23) can be obtained in a similar manner to (6.22).

Taking the trace in (6.22) and considering the definitions of following squared norms

$$\|F\|^2=\sum_{i,j=1}^{n}(\widetilde{h}(Fe_i,e_j))^2 \quad \text{and} \quad \|F^\star\|^2=\sum_{i,j=1}^{n}(\widetilde{h}(F^\star e_i,e_j))^2,$$

we find (6.24). The proof of (6.25) can be obtained in a similar manner to (6.24). Q.E.D.

**Lemma 6.8.** Let $(L,h,\mathrm{S}(TL))$ be a lightlike hypersurface of $(\widetilde{L},\widetilde{h},\widetilde{\nabla})$. Then, the following inequality holds for any $Z\in\Gamma(\mathrm{S}(TL))$:

$$h(A_N^\star Z,\overline{A}_\xi^\star Z)\leq\frac{1}{2}(\|\overline{A}_\xi^\star Z\|^2+\|A_N^\star Z\|^2). \tag{6.27}$$

The equality case of (6.27) holds if and only if one of the following statements occurs:

i) $A_N^\star Z=\overline{A}_\xi^\star Z$.

ii) $A_N^\star=0$ and $\overline{A}_\xi^\star=0$.

*Proof.* Let $\{e_1,e_2,\ldots,e_n\}$ be an orthonormal frame field of $\Gamma(\mathrm{S}(TL))$. Then, we can write $A_N^\star Z=\sum\limits_{i=1}^{n}\lambda_i e_i+\lambda_{n+1}\xi$ and $\overline{A}_\xi^\star Z=\sum\limits_{i=1}^{n}\mu_i e_i$. Using the fact that $0\leq\sum\limits_{i=1}^{n}(\lambda_i-\mu_i)^2$, we get

$$\sum_{i=1}^{n}2\lambda_i\mu_i\leq\sum_{i=1}^{n}\lambda_i^2+\mu_i^2. \tag{6.28}$$

Based on (6.28), the proof is easy to follow. Q.E.D.

In the first case of Lemma 6.8, we see that $(L,h,\mathrm{S}(TM))$ is screen conformal with respect to $\widetilde{\nabla}^\star$ for $\alpha^\star=1$. In the second case, we obtain that $(L,h,\mathrm{S}(TL))$ is totally geodesic and $\mathrm{S}(TL)-$geodesic with respect to $\widetilde{\nabla}^\star$.

**Theorem 6.9.** Let $(L,h,\mathrm{S}(TL))$ be a lightlike hypersurface of $(\widetilde{L},\widetilde{h},\widetilde{\nabla})$. Then we have

$$\tau_{\mathrm{S}(TL)}(p)\geq\widetilde{\tau}_{\mathrm{S}(TL)}(p)+n\ \mathrm{trace}(\overline{A}_\xi^\star)trace(A_N^\star)-\frac{1}{2}\left(\|A_\xi^\star\|^2+\|A_N^\star\|^2\right). \tag{6.29}$$

The equality case of (6.29) holds if and only if one of the following situations occurs:

i) $(L,h,\mathrm{S}(TL))$ is screen conformal with respect to $\widetilde{\nabla}^\star$ for $\alpha^\star=1$.

ii) $(L,h,\mathrm{S}(TL))$ is totally geodesic and $\mathrm{S}(TL)-$geodesic with respect to $\widetilde{\nabla}^\star$.

*Proof.* Taking into account (6.20) and Lemma 6.8, the proof is straightforward. Q.E.D.

**Lemma 6.10.** Let $(L, h, \mathrm{S}(TL))$ be a lightlike hypersurface of $(\widetilde{L}, \widetilde{h}, \widetilde{\nabla})$. The following inequality is satisfied for any $Z \in \Gamma(\mathrm{S}(TL))$:

$$h(A_N Z, \overline{A}_\xi Z) \leq \frac{1}{2}\left(\|\overline{A}_\xi Z\|^2 + \|A_N Z\|^2\right). \tag{6.30}$$

The equality case of (6.30) is satisfied if and only if one of the following statements occurs:

i) $A_N Z = \overline{A}_\xi Z$.

ii) $A_N = 0$ and $\overline{A}_\xi = 0$.

In the first case of Lemma 6.10, we see that $(L, h, \mathrm{S}(TL))$ is screen conformal with respect to $\widetilde{\nabla}$ for $\alpha = 1$. In the second case, we obtain that $(L, h, \mathrm{S}(TM))$ is totally geodesic and $\mathrm{S}(TL)-$geodesic with respect to $\widetilde{\nabla}$.

**Theorem 6.11.** Let $(L, h, \mathrm{S}(TL))$ be a lightlike hypersurface of $(\widetilde{L}, \widetilde{h}, \widetilde{\nabla})$. Then we have

$$\tau^\star_{\mathrm{S}(TL)}(p) \geq \widetilde{\tau}^\star_{\mathrm{S}(TL)}(p) + n\ \mathrm{trace}(\overline{A}_\xi) trace(A_N) - \frac{1}{2}(\|A_\xi\|^2 + \|A_N\|^2). \tag{6.31}$$

The equality case of (6.31) holds if and only if one of the following situations occurs:

i) $(L, h, \mathrm{S}(TM))$ is screen conformal with respect to $\widetilde{\nabla}$ for $\alpha = 1$.

ii) $(L, h, \mathrm{S}(TM))$ is totally geodesic and $\mathrm{S}(TM)-$geodesic with respect to $\widetilde{\nabla}$.

*Proof.* Taking into account (6.21) and Lemma 6.10, the proof is straightforward. Q.E.D.

As a result of (6.24) and Theorem 6.9, we find

**Theorem 6.12.** Let $(L, h, \mathrm{S}(TM))$ be a radical invariant lightlike hypersurface of $\widetilde{L}(a_1, b_1)$. Then, we have

$$\begin{aligned}\tau_{\mathrm{S}(TM)}(p) \geq\ & a_1\left[n^2 + (traceF)^2 - 2\|F^\star\|^2\right] + b_1\left[(2n-2)traceF - (traceF)^2 - \|F^\star\|^2\right] \\ & + n trace(\overline{A}^\star_\xi) trace(A^\star_N) - \frac{1}{2}(\|A^\star_\xi\|^2 + \|A^\star_N\|^2).\end{aligned} \tag{6.32}$$

The equality case of (6.32) is true if and only if one of the following situations occurs:

i) $(L, h, \mathrm{S}(TL))$ is screen conformal with respect to $\widetilde{\nabla}^\star$ for $\alpha^\star = 1$.

ii) $(L, h, \mathrm{S}(TL))$ is totally geodesic and $\mathrm{S}(TL)-$geodesic with respect to $\widetilde{\nabla}^\star$.

As a result of (6.25) and Theorem 6.11, we find

**Theorem 6.13.** Let $(L, h, \mathrm{L}(TM))$ be a radical invariant lightlike hypersurface of $\widetilde{L}(a_1, b_1)$. Then, we have

$$\tau^{\star}_{\mathrm{S}(TL)}(p) \geq a_2\left[n^2 + (traceF^{\star})^2 - 2\left\|F\right\|^2\right] + b_2\left[(2n-2)traceF^{\star} - (traceF^{\star})^2 - \left\|F\right\|^2\right] + ntrace(\overline{A}_{\xi})trace(A_N) - \frac{1}{2}(\|A_{\xi}\|^2 + \|A_N\|^2). \tag{6.33}$$

The equality case of (6.33) is true for any $p \in L$ if and only if one of the following situations occurs:

i) $(L, h, \mathrm{S}(TL))$ is screen conformal with respect to $\widetilde{\overline{\nabla}}$ for $\alpha = 1$.

ii) $(L, h, \mathrm{S}(TL))$ is totally geodesic and $\mathrm{S}(TM)-$geodesic with respect to $\widetilde{\overline{\nabla}}$.

In the remaining part of this paper, we provide some relations involving the $K-$sectional, $K-$Ricci and $K-$scalar curvatures:

For any lightlike hypersurface, the Gauss-type equation with respect to the Levi-Civita connection $\widetilde{\nabla}^0$ is given by

$$h(R^0(Z_1, Z_2)Z_3, Z_4) = \widetilde{h}(\widetilde{R}^0(Z_1, Z_2)Z_3, Z_4) - B^0(Z_1, Z_3)h(A^0_N Z_2, Z_4) + B^0(Z_2, Z_3)h(A^0_N Z_1, Z_4), \tag{6.34}$$

where $Z_1, Z_2, Z_3, Z_4 \in \Gamma(\mathrm{S}(TL))$. Using (6.10), (6.11) and (6.34) in (2.5), we find

$$\begin{aligned}\widetilde{K}^s(\Pi) = K^s(\Pi) - \frac{1}{2}\left[-B(Z_2, Z_2)h(A^{\star}_N Z_1, Z_1) + B(Z_1, Z_2)h(A^{\star}_N Z_2, Z_1)\right.\\ \left.-B^{\star}(Z_2, Z_2)h(A_N Z_1, Z_1) + B^{\star}(Z_1, Z_2)h(A_N Z_2, Z_1)\right]\\ - B^0(Z_2, Z_2)h(A^0_N Z_1, Z_1) + B^0(Z_1, Z_2)h(A^0_N Z_2, Z_1)\end{aligned} \tag{6.35}$$

such that

$$K^s(\Pi) = \frac{1}{2}\left\{h(R(Z_1, Z_2)Z_2, Z_1) + h(R^{\star}(Z_1, Z_2)Z_2, Z_1) - 2h(R^0(Z_1, Z_2)Z_2, Z_1)\right\}. \tag{6.36}$$

Let us denote the screen Ricci curvatures in terms of the $K-$sectional curvature by $Ric^s_{\mathrm{S}(TL)}$ and $\widetilde{Ric}^s_{\mathrm{S}(TL)}$. Then, we write

$$Ric^s_{\mathrm{S}(TL)}(e_i) = \sum_{j=1}^{n} K^s(e_i, e_j) \quad \text{and} \quad \widetilde{\mathrm{Ric}}^s_{\mathrm{S}(TL)}(e_i) = \sum_{j=1}^{n} \widetilde{K}^s(e_i, e_j). \tag{6.37}$$

Furthermore, the screen scalar curvatures in terms of the $K-$sectional curvature are defined by

$$\tau^s_{\mathrm{S}(TL)}(p) = \sum_{i=1}^{n} \mathrm{Ric}^s(e_i) \quad \text{and} \quad \widetilde{\tau}^s_{\mathrm{S}(TL)}(p) = \sum_{i=1}^{n} \widetilde{\mathrm{Ric}}^s(e_i). \tag{6.38}$$

**Lemma 6.14.** Let $(L, h, \mathrm{S}(TL))$ be a lightlike hypersurface of $(\widetilde{L}, \widetilde{h}, \widetilde{\nabla})$. Then we have

$$\begin{aligned}\widetilde{Ric}^{s}_{\mathrm{S}(TL)}(Z) = Ric^{s}_{\mathrm{S}(TM)}(Z) &- trace(\overline{A}^{\star}_{\xi})h(A^{\star}_{N}Z, Z) + h(\overline{A}^{\star}_{\xi}Z, A^{\star}_{N}Z)\\ &- trace(\overline{A}_{\xi})h(A_{N}Z, Z) + h(\overline{A}_{\xi}Z, A_{N}Z)\\ &- trace(A^{0}_{N})h(A^{0}_{\xi}Z, Z) + h(A^{\star 0}_{\xi}Z, A^{0}_{N}Z) \end{aligned} \tag{6.39}$$

for any unit vector field $Z \in \Gamma(\mathrm{S}(TL))$ and

$$\begin{aligned}\widetilde{\tau}^{s}_{\mathrm{S}(TL)}(p) = \tau^{s}_{\mathrm{S}(TM)}(p) &- trace(\overline{A}^{\star}_{\xi})trace(A^{\star}_{N}) + \sum_{i=1}^{n} h(\overline{A}^{\star}_{\xi}e_i, A^{\star}_{N}e_i)\\ &- trace(\overline{A}_{\xi})trace(A_{N}) + \sum_{i=1}^{n} h(\overline{A}_{\xi}e_i, A_{N}e_i)\\ &- trace(A^{0}_{N})trace(A^{0}_{\xi}) + \sum_{i=1}^{n} h(A^{\star 0}_{\xi}e_i, A^{0}_{N}e_i). \end{aligned} \tag{6.40}$$

*Proof.* Taking the trace in (6.35) and using (6.37) and (6.38), the proofs of (6.39) and (6.40) are straightforward. Q.E.D.

**Proposition 6.15.** Let $(L, h, \mathrm{S}(TL))$ be a radical invariant lightlike hypersurface of $\widetilde{L}(a_1, b_1)$. Then we have

$$\begin{aligned}&\frac{1}{2}a_1\left[n^2 + (traceF)^2 - 2\left\|F^{\star}\right\|^2\right] + \frac{1}{2}b_1\left[(2n-2)traceF - (traceF)^2 - \left\|F^{\star}\right\|^2\right]\\ &+\frac{1}{2}a_2\left[n^2 + (traceF^{\star})^2 - 2\left\|F\right\|^2\right] + \frac{1}{2}b_2\left[(2n-2)traceF^{\star} - (traceF^{\star})^2 - \left\|F\right\|^2\right]\\ &= \tau^{0}_{\mathrm{S}(TM)}(p) + \tau^{s}_{\mathrm{S}(TM)}(p) - trace(\overline{A}^{\star}_{\xi})trace(A^{\star}_{N}) + \sum_{i=1}^{n} h(\overline{A}^{\star}_{\xi}e_i, A^{\star}_{N}e_i)\\ &\quad -trace(\overline{A}_{\xi})trace(A_{N}) + \sum_{i=1}^{n} h(\overline{A}_{\xi}e_i, A_{N}e_i) + \sum_{i=1}^{n} h(A^{\star 0}_{\xi}e_i, A^{0}_{N}e_i)\\ &\quad -trace(A^{0}_{N})trace(A^{0}_{\xi}). \end{aligned} \tag{6.41}$$

*Proof.* By considering (6.24) and (6.25) in Lemma 6.14, we have (6.41). Q.E.D.

**Corollary 6.16.** Let $(L, h, \mathrm{S}(TM))$ be a totally geodesic radical invariant lightlike hypersurface with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^{\star}$. Then we have

$$\begin{aligned}\tau^{0}_{\mathrm{S}(TL)}(p) + \tau^{s}_{\mathrm{S}(TL)}(p) = &\frac{1}{2}a_1\left[n^2 + (traceF)^2 - 2\left\|F^{\star}\right\|^2\right]\\ &+ \frac{1}{2}b_1\left[(2n-2)traceF - (traceF)^2 - \left\|F^{\star}\right\|^2\right]\\ &+ \frac{1}{2}a_2\left[n^2 + (traceF^{\star})^2 - 2\left\|F\right\|^2\right]\\ &+ \frac{1}{2}b_2\left[(2n-2)traceF^{\star} - (traceF^{\star})^2 - \left\|F\right\|^2\right]. \end{aligned} \tag{6.42}$$

**Corollary 6.17.** Let $(L, h, \mathrm{S}(TL))$ be a totally geodesic radical invariant lightlike hypersurface of $\widetilde{L}(a_1, b_1)$ such that $a_1 = b_1 = 0$. Then we have

$$\tau^0_{\mathrm{S}(TL)}(p) = -\tau^s_{\mathrm{S}(TL)}(p). \tag{6.43}$$

## 7 Conclusions and future works

In this study, the geometry of radical invariant lightlike hypersurfaces, where almost product structures $F$ and $F^\star$ are invariant in the radical space, is examined. Considering these facts, new types of lightlike hypersurfaces can be defined by defining the different effects of $F$ and $F^\star$ on the radical space, screen space and transversal space. For example, several cases in which almost product structures $F$ and $F^\star$ are anti-invariant in the radical space or invariant and anti-invariant on the screen space can be examined. Thus, the problem of the existence of new types of lightlike hypersurfaces of an almost product-like statistical manifold arises.

It is known that any screen distribution of a lightlike hypersurface does not need to be integrable. This makes it topologically impossible to examine geometric structures on subdistributions that can be defined on any screen distribution. The conditions required for the screen distribution of any lightlike hypersurface to be integrable were examined (cf. Theorem 5.9). It was proven that every lightlike hypersurface whose screen distribution is integrable is locally isometric to $\beta \times L^*$, where $\beta$ is a null curve on the normal space and $L^*$ is a leaf on the screen distribution. We present the required integrable conditions on any screen distribution of a radical invariant lightlike hypersurface of an almost product-like statistical manifold (cf. Theorem 5.10 and Theorem 5.11). The conditions for the screen distribution to be integrable can be investigated on new types of lightlike hypersurfaces of an almost product-like statistical manifold.

In the sixth section of this study, we compute the Riemannian curvatures of radical invariant lightlike hypersurfaces in terms of statistical structures. With the aid of obtained relations, Chen-like inequalities can be derived in the future studies.

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