Some results for rational type $F-$contraction with an applications to differential equations

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Abstract

In this paper, we consider rational type F-contraction on orthogonal metric space and we establish existence of fixed point of such mapping. At the end, we give an example and an applications.

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1 Introduction and preliminaries

Fixed point problem is parts of the famous and traditional theories in mathematics and have a broad set of applications. In this problem, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Banach's contraction principle which gives an answer to the existence and uniqueness of a solution of an operator equation $Tx = x$, is the most widely used fixed point theory in all of analysis. Afterwards, the crucial role of the principle in existence and uniqueness problems arising in mathematics has been realized which fact directed the researchers to extend and generalize the principle in many ways. No doubtly, one of those is done by Ran and Reurings [2]. They established fixed point results in partially ordered metric spaces with application to solving matrix equations.

Recently, for the first time Gordji et al. [6] introduced the new concept of an orthogonality in metric spaces and proved the fixed point result for contraction mappings in metric spaces endowed with this new notion. Furthermore, they gave the application of this results for the existence and uniqueness of the solution of a first-order ordinary differential equation, while the Banach contraction principle cannot be applied in this situation. Then Sawangsup et al. [11] introduced the new concept of an orthogonal $F-$contraction mappings and proved the fixed point theorems on orthogonal complete metric space. Also Singh et al. [3] prove Boyd-Wong and Matkowski type fixed point theorems in orthogonal metric space which was defined by M.E. Gordji in 2017 and is an extension of the metric space. Alsasi et al. [1] prove fixed point results for a Meir–Keeler type contraction due to orthogonal $M-$metric spaces. Also we refer to [7, 8, 9, 12].
On the other hand, an interesting generalization of Banach Contraction Principle was given by Wardowski [4], who introduced the new concept of F-contraction. Let $\mathcal{F}$ be the set of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

$(F1)$ F is strictly increasing, i.e., for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta, F(\alpha) < F(\beta)$,

$(F2)$ for each sequence $\{a_n\}$ of positive numbers,

$$\lim_{n\to\infty} a_n = 0 \text{ if and only if } \lim_{n\to\infty} F(a_n) = -\infty,$$

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$(F3)$ there exists $k \in (0,1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$,

$(F4)$ for all $A \subset (0,\infty)$ sets are $F(\inf A) = \inf F(A)$ such that $\inf A > 0$.

We consider by $\mathcal{F}$ be the set of all functions F satisfying $(F1)$-$(F3)$ and $\mathcal{F}_*$ be the set of all functions F satisfying $(F1)$-$(F4)$. Some examples of the functions belonging $\mathcal{F}_*$ are $F_1(\alpha) = \ln \alpha$, $F_2(\alpha) = \alpha + \ln \alpha$, $F_3(\alpha) = -\frac{1}{\sqrt{\alpha}}$ and $F_4(\alpha) = \ln\left(\alpha^2 + \alpha\right)$. If we define $F_5(\alpha) = \ln \alpha$ for $\alpha \leq 1$ and $F_5(\alpha) = 2\alpha$ for $\alpha > 1$, then $F_5 \in \mathcal{F} \diagdown \mathcal{F}_*$.

Definition 1.1 ([4])**.** Let (M,d) be a metric space and $T : M \to M$ be a mapping. Given $F \in \mathcal{F}$, T is called as F-contraction if there exists $\tau > 0$ such that

$$\varsigma, \omega \in M,\ d(T\varsigma, T\omega) > 0 \Rightarrow \tau + F(d(T\varsigma, T\omega)) \leq F(d(\varsigma, \omega)).$$

Then Wardowski give a convenient generalization of Banach fixed point theorem as:

Theorem 1.2 ([4])**.** Let (M,d) be a complete metric space and let $T : M \to M$ be an F-contraction. Then T has a unique fixed point in M.

In this paper, we consider orthogonally complete metric spaces and present some fixed point theorems for rational type $F-$contraction mappings. As a beginning we recall some fundemental concepts of orthogonal metric space which will be needed to prove our results.

Definition 1.3 ([11])**.** (i) Let $M \neq \varnothing$ and $\perp$ is a binary relation on M. If $\perp$ fulfils the criteria:

$$\exists \varsigma_0 (\forall \omega \in M, \omega \perp \varsigma_0) \ \text{ or } \ (\forall \omega \in M, \varsigma_0 \perp \omega),$$

then we say that $(M, \perp)$ is an orthogonal set. Also, ς_0 is an orthogonal element. Orthogonal set is represented with $(M, \perp)$.

(ii) In an orthogonal set (O-set) $(M, \perp)$,we say that any two elements ς and ω are orthogonally related.
if $\varsigma \perp \omega$.

(iii) A sequence $\{\varsigma_n\}$ is called an orthogonal sequence (briefly O-sequence) if

$$(\forall n \in \mathbb{N}, \varsigma_n \perp \varsigma_{n+1}) \ \text{ or } \ (\forall n \in \mathbb{N}, \varsigma_{n+1} \perp \varsigma_n).$$

In like manner, $\{\varsigma_n\}$ is a Cauchy O-sequence if

$$(\forall n \in \mathbb{N}, \varsigma_n \perp \varsigma_{n+1}) \ \text{ or } \ (\forall n \in \mathbb{N}, \varsigma_{n+1} \perp \varsigma_n).$$

(iv) The triplet $(M, \perp, d)$ is called an orthogonal metric space (shortly O-metric space) if $(M, \perp)$ be a O-set and d be a metric.

(v) Let $(M, \curlywedge, d)$ be a O-metric space. Then M is an O-complete if every Cauchy O-sequence is converges in M.

(vi) Let $(M, \curlywedge, d)$ be an orthogonal metric space. Then a mapping $T : M \to M$ is an orthogonally continuous ($\curlywedge$-continuous) at ς if for each O-sequence $\{\varsigma_n\}$ converging to ς implies $T(\varsigma_n) \to T(\varsigma)$ as $n \to \infty$. Also T is $\curlywedge$-continuous on M if T is $\curlywedge$-continuous at every $\varsigma \in M$.

(vii) Let $(M, \curlywedge, d)$ be an orthogonal metric space. A mapping $T : M \to M$ is said to be $\curlywedge$-preserving if $T(\varsigma) \curlywedge T(\omega)$ whenever $\varsigma \curlywedge \omega$ and weakly $\curlywedge$-preserving if $T(\varsigma) \curlywedge T(\omega)$ or $T(\omega) \curlywedge T(\varsigma)$ whenever $\varsigma \curlywedge \omega$.

2 Main results

In this part, we modify definition of rational type $F-$contraction using orthogonality and then give some results for this type contraction on $O-$complete metric space. Also we give an example to illustrate our result.

Definition 2.1. Let $(M, \curlywedge, d)$ be an orthogonal metric space. We say that $T : M \to M$ is an orthogonal rational type $F-$contraction if there are $F \in \mathcal{F}$ and $\tau > 0$ such that the following condition holds:

$$\forall \varsigma, \omega \in M \text{ with } \varsigma \curlywedge \omega;\ \left[d(T\varsigma, T\omega) > 0 \Rightarrow \tau + F(d(T\varsigma, T\omega)) \leq F(M(\varsigma, \omega))\right], \tag{2.1}$$

where

$$M(\varsigma, \omega) = \max \left\{ \begin{array}{c} d(\varsigma, \omega), d(\varsigma, T\varsigma), d(\omega, T\omega), \\ \\ \frac{d(\varsigma, T\varsigma) d(\omega, T\omega)}{1 + d(\varsigma, \omega)}, \frac{d(\varsigma, T\varsigma) d(\omega, T\omega)}{1 + d(T\varsigma, T\omega)} \end{array} \right\}.$$

Theorem 2.2. Let $(M, \curlywedge, d)$ be an $O-$complete orthogonal metric space, ς_0 is an orthogonal element of M and T be a $\curlywedge-$preserving and $\curlywedge-$continuous mapping with satisfying (2.1). Then, T has a unique fixed point in M.

Proof. Using the definition of the orthogonality, we have $\varsigma_0 \curlywedge T(\varsigma_0)$ or $T(\varsigma_0) \curlywedge \varsigma_0$. Let

$$\varsigma_1 := T\varsigma_0, \varsigma_2 := T\varsigma_1 = T^2\varsigma_0, \cdots, \varsigma_n := T\varsigma_{n-1} = T^n\varsigma_0$$

for all $n \in \mathbb{N} \cup \{0\}$. For some $n^* \in \mathbb{N} \cup \{0\}$, if $\varsigma_{n^*} = \varsigma_{n^*+1}$ then ς_{n^*} is a fixed point of T. So, we presume that $\varsigma_n \neq \varsigma_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, we obtain $d(\varsigma_n, \varsigma_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Then we obtain

$$\varsigma_n \curlywedge \varsigma_{n+1} \text{ or } \varsigma_{n+1} \curlywedge \varsigma_n.$$

from $\curlywedge-$preserving of T and then we say that $\{\varsigma_n\}$ is an $O-$sequence. From (2.1), for all $n \in \mathbb{N}$ we

have

$$\begin{aligned} & F(d(\varsigma_n, \varsigma_{n+1})) \\ = \; & F(d(T\varsigma_{n-1}, T\varsigma_n)) \\ \leq \; & F(M(\varsigma_{n-1}, \varsigma_n)) - \tau \\ = \; & F\left(\max\left\{ \begin{array}{l} d(\varsigma_{n-1}, \varsigma_n), d(\varsigma_{n-1}, T\varsigma_{n-1}), d(\varsigma_n, T\varsigma_n), \\ \\ \frac{d(\varsigma_{n-1}, T\varsigma_{n-1})d(\varsigma_n, T\varsigma_n)}{1+d(\varsigma_{n-1}, \varsigma_n)}, \frac{d(\varsigma_{n-1}, T\varsigma_{n-1})d(\varsigma_n, T\varsigma_n)}{1+d(T\varsigma_{n-1}, T\varsigma_n)} \end{array} \right\}\right) - \tau \\ \leq \; & F(d(\varsigma_{n-1}, \varsigma_n)) - \tau. \end{aligned} \tag{2.2}$$

Let $a_n := d(\varsigma_n, \varsigma_{n+1})$ for all $n \in \mathbb{N}$ and from (2.2), we have

$$F(a_n) \leq F(a_{n-1}) - \tau \leq F(a_{n-2}) - 2\tau \leq \cdots \leq F(a_0) - n\tau. \tag{2.3}$$

From (2.3), we get $\lim_{n\to\infty} F(a_n) = -\infty$. Thus, from (F2), we have

$$\lim_{n\to\infty} a_n = 0. \tag{2.4}$$

By the property (F3), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} a_n^k F(a_n) = 0. \tag{2.5}$$

By (2.3), we get

$$a_n^k F(a_n) - a_n^k F(a_0) \leq -a_n^k n\tau \leq 0 \tag{2.6}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.6), we get

$$\lim_{n\to\infty} n a_n^k = 0. \tag{2.7}$$

From (2.7), there exits $n_1 \in \mathbb{N}$ such that $na_n^k \leq 1$ for all $n \geq n_1$. So we have

$$a_n \leq \frac{1}{n^{1/k}}. \tag{2.8}$$

for all $n \geq n_1$. In order to show that $\{\varsigma_n\}$ is a Cauchy $O-$sequence, consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for the metric and from (2.8), we have

$$\begin{aligned} d(\varsigma_n, \varsigma_m) &\leq d(\varsigma_n, \varsigma_{n+1}) + d(\varsigma_{n+1}, \varsigma_{n+2}) + \cdots + d(\varsigma_{m-1}, \varsigma_m) \\ &= a_n + a_{n+1} + \cdots + a_{m-1} \\ &= \sum_{i=n}^{m-1} a_i \\ &\leq \sum_{i=n}^{m-1} \frac{1}{i^{1/k}}. \end{aligned}$$

Hence $\{\varsigma_n\}$ is a Cauchy $O-$sequence in M. Because of the $O-$completeness of M, we have $\varsigma^* \in M$ such that $\varsigma_n \to \varsigma^*$ as $n \to \infty$. Using $\curlywedge-$continuous of T, we have

$$T\varsigma^* = T(\lim_{n\to\infty} \varsigma_n) = \lim_{n\to\infty} \varsigma_{n+1} = \varsigma^*$$

and so ς^* is a fixed point of T. Now we prove the uniqueness of the fixed point. Assume that, ς^* and ω^* are two distinct fixed points . If $\varsigma_n \to \omega^*$ as $n \to \infty$, we get $\varsigma^* = \omega^*$. If ς_n doesn't convergence to ω^* as $n \to \infty$, in this case, there is a subsequence $\{\varsigma_{n_k}\}$ with $T\varsigma_{n_k} \neq \omega^*$ for all $k \in \mathbb{N}$. Then, we have

$$(\varsigma_0 \curlywedge \omega^*) \text{ or } (\omega^* \curlywedge \varsigma_0)$$

by the choice of ς_0 in the beginning of the proof. Since T is $\curlywedge-$preserving and $T^n\omega^* = \omega^*$ for all $n \in \mathbb{N}$, we have

$$(T^n\varsigma_0 \curlywedge \omega^*) \text{ or } (\omega^* \curlywedge T^n\varsigma_0)$$

for all $n \in \mathbb{N}$. From (2.1), we obtain

$$\begin{aligned} F(d(T^{n_k}\varsigma_0, \omega^*)) &= F(d(T^{n_k}\varsigma_0, T^{n_k}\omega^*)) \\ &\leq F(M(T^{n_{k-1}}\varsigma_0, T^{n_{k-1}}\omega^*)) - \tau \\ &= F(d(T^{n_{k-1}}\varsigma_0, T^{n_{k-1}}\omega^*)) - \tau \\ &\vdots \\ &\leq F(d(\varsigma_0, \omega^*)) - n_k\tau \end{aligned}$$

for all $n \in \mathbb{N}$. Thus $F(d(T^{n_k}\varsigma_0, \omega^*)) \to -\infty$ as $k \to \infty$ and so $d(T^{n_k}\varsigma_0, \omega^*) \to 0$ as $k \to \infty$. This yields that $\varsigma_n \to \omega^*$as $n \to \infty$, which is a contradiction. Hence T has a unique fixed point. Q.E.D.

Corollary 2.3. Let $(M, \curlywedge, d)$ be an $O-$complete orthogonal metric space with an orthogonal elements ς_0 and T be a $\curlywedge-$preserving and $\curlywedge-$continuous self mapping on M such that

$$\forall \varsigma, \omega \in M \text{ with } \varsigma \curlywedge \omega; \ [d(T\varsigma, T\omega) > 0 \Rightarrow \tau + F(d(T\varsigma, T\omega)) \leq F(d(\varsigma, \omega))] .$$

Then, T has a unique fixed point in M.

Corollary 2.4. Let (M, d) be a complete metric space and $T : M \to M$ be a mapping such that, for some $\alpha \in (0, 1)$,

$$d(T\varsigma, T\omega) \leq \alpha d(\varsigma, \omega)$$

for all $\varsigma, \omega \in M$. Then T has a unique fixed point in M.

Example 2.5. Let $W = [0, \infty)$ and $d : W \times W \to [0, \infty)$ be a mapping defined by

$$d(\varsigma, \omega) = |\varsigma - \omega|$$

for all $\varsigma, \omega \in W$. Consider the sequence $\{G_p\}_{p\in N}$ defined as

$$G_p = \frac{p(p+1)}{2}, \forall p \in \mathbb{N} \cup \{0\}.$$

Define a relation $\curlywedge$ on W by

$$\varsigma \curlywedge \omega \Longleftrightarrow \varsigma\omega \in \{\varsigma, \omega\} \subseteq \{G_p\}.$$

Thus $(W, \curlywedge, d)$ is an $O-$complete orthogonal metric space. Now, we will define a mapping $T : W \to W$ by

$$T\varsigma = \begin{cases} G_1 & if \quad G_0 \leq \varsigma \leq G_1 \\ G_{p-1} & if \quad G_p \leq \varsigma \leq G_{p+1}, \forall p > 1 \end{cases}.$$

Then T is $\curlywedge-$continuous and T is $\curlywedge-$preserving. Let $F \in \mathcal{F}$ be a function defined by $F(\alpha) = \alpha + \ln \alpha$ for all $\alpha > 0$. We assert that T is an orthogonal rational F-contraction mapping with $\tau = 1$. Indeed, let $\varsigma, \omega \in W$ with $\varsigma \curlywedge \omega$ and $d(T\varsigma, T\omega) > 0$. So, we may suppose that $\varsigma < \omega$. So $\varsigma \in \{G_0, G_1\}$ and $\omega = G_i$ for some $i \in \mathbb{N}\backslash\{1\}$. Then, we get

$$\begin{aligned} & \frac{d(T\varsigma, T\omega)}{M(\varsigma, \omega)} e^{[d(T\varsigma, T\omega) - M(\varsigma, \omega)]} \\ = \quad & \frac{G_{i-1} - 1}{G_i - 1} e^{[G_{i-1} - 1 - G_i + 1)]} < e^{-1}. \end{aligned}$$

Hence, all the conditions of Theorem 2.2 are satisfied and so T has a unique fixed point.

3 Applications

Recall that, for any $1 \leq \rho < \infty$, the space $L^{\rho}(M, F, \mu)$ (or $L^{\rho}(M)$) consists of all complex-valued measurable functions κ on the underlying space M satisfying

$$\int_M |\kappa(\varsigma)|^{\rho} \, d\mu(\varsigma),$$

where F is the σ-algebra of measurable sets and μ is the measure. When $\rho = 1$, the space $L^1(M)$ consists of all integrable functions κ on M and we define the L^1-norm of κ by

$$\|\kappa\|_1 = \int_M |\kappa(\varsigma)| \, d\mu(\varsigma).$$

In the section, using Theorem 2.2, we show the existence of a solution of the following differential equation:

$$\begin{cases} u'(t) = \mathrm{f}(t, u(t)), & a.e.\ t \in I := [0, T] \\ u(0) = a, & a \geq 1, \end{cases} \tag{3.1}$$

where $\mathrm{f} : I \times \mathbb{R} \to \mathbb{R}$ is an integrable function satisfying the following conditions:

(i) $\mathrm{f}(s, \eta) \geq 0$ for all $\eta \geq 0$ and $s \in I$;

(ii) for each $\varsigma, \omega \in L^1(I)$ with $\varsigma(s)\omega(s) \geq \varsigma(s)$ or $\varsigma(s)\omega(s) \geq \omega(s)$ for all $s \in I$, there exist $\kappa \in L^1(I)$ and $\tau > 0$ such that

$$|\mathrm{f}(s, \varsigma(s)) - \mathrm{f}(s, \omega(s))| \leq \frac{\kappa(s)}{\left(1 + \tau\sqrt{\kappa(s)}\right)^2} |\varsigma(s) - \omega(s)| \tag{3.2}$$

and

$$|\varsigma(s) - \omega(s)| \leq \kappa(s) e^{A(s)}$$

for all $s \in I$,where $A(s) := \int_0^s |\kappa(w)| dw$.

Theorem 3.1. Consider the differential Eq. (3.1). If (i) and (ii) are satisfied, then the differential Eq. (3.1) has a unique positive solution.

Proof. Let $X = \{u \in C(I, \mathbb{R}) : u(t) > 0 \text{ for all } t \in I\}$. Define the orthogonality relation $\perp$ on M by

$$\varsigma \perp \omega \Longleftrightarrow \varsigma(s)\omega(s) \geq \varsigma(s) or \varsigma(s)\omega(s) \geq \omega(s) \text{ for all } t \in I.$$

Since $A(t) = \int_0^t |\kappa(s)| ds$, we have $A'(t) = |\kappa(t)|$ for almost everywhere $t \in I$.

Define a mapping

$$\rho(\varsigma, \omega) = \|\varsigma - \omega\|_A = \sup_{t \in I} e^{-A(t)} |\varsigma(s) - \omega(s)|$$

for all $\varsigma, \omega \in M$. Thus, (X, d) is a metric space and also a complete metric space (see, [6] for details). Define a mapping $\mathbb{G} : M \to M$ by

$$(\mathbb{G}\varsigma)(t) = a + \int_0^t \mathrm{f}(s, \varsigma(s)) ds.$$

Then, we see that $\mathbb{G}$ is $\perp$-continuous.

Now, we shot that $\mathbb{G}$ is $\perp$- preserving. For each $\varsigma, \omega \in M$ with $\varsigma \perp \omega$ and $t \in I$, we have

$$(\mathbb{G}\varsigma)(t) = a + \int_0^t \mathrm{f}(s, \varsigma(s)) ds \geq 1.$$

It follows that $[(\mathbb{G}\varsigma)(t)][(\mathbb{G}\omega)(t)] \geq (\mathbb{G}\omega)(t)$ and so $(\mathbb{G}\varsigma)(t) \perp (\mathbb{G}\omega)(t)$. Then $\mathbb{G}$ is $\perp$-preserving.

Now, we can say that $\mathbb{G}$ satisfies Corollary 2.3 with $F(\alpha) = \frac{-1}{\sqrt{\alpha}}$. Hence the differential equation (3.1) has a unique positive solution. Q.E.D.

Conflict of interest The authors declare that they have no competing interests.

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References

[1] A. Alsaadi, B. Singh, V. Singh, I. Uddin, *Meir–Keeler type contraction in orthogonal $M-$metric spaces*, Symmetry **14** (2022) 1–14.

[2] A.C.M. Ran, M.C.B. Reuring, *A fixed point theorem in partially ordered sets and some appllications to matrix equations*, Proc. Am. Math. Soc. **132** (2004) 1435–1443.

[3] B. Singh, V. Singh, I. Uddin, Ö. Acar, *Fixed point theorems on an orthogonal metric space using Matkowski type contraction*, Carpathian Mathematical Publications **14** (2022) 124–134.

[4] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl. **1** (2012) 1–6.

[5] J.J. Nieto, R. Rodriguez Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order **22** (2005) 223–239.

[6] M.E. Gordji, M. Rameani, M. De La Sen, Y.J. Cho, *On orthogonal sets and Banach fixed point theorem*, Fixed Point Theory **18** (2017) 569–578.

[7] Ö. Acar, A. S. Özkapu, *Multivalued rational type $F-$contraction on orthogonal metric space*, Mathematical Foundations of Computing **6** (2023) 303–312.

[8] Ö. Acar, E. Erdoğan, *Some fixed point results for almost contraction on orthogonal metric space*, Creative Mathematics Informatics **31** (2022) 147–153.

[9] Ö. Acar, E. Erdoğan, A. S. Özkapu *Generalized integral type mappings on orthogonal metric spaces*, Carpathian Mathematical Publications **14** (2022) 485–492.

[10] S. Banach, *Sur les operations dans les ensembles abstraits et leurs applications aux equations integrales*, Fund. Math. **3** (1922) 133-181.

[11] S. Konakwan, W. Sintunavarat, Y.J. Cho, *Fixed point theorems for orthogonal $F-$contraction mappings on $O-$complete metric space*, J. Fixed Point Theory Appl. **22** (2020) 1–14.

[12] S. Konakwan, W. Sintunavarat, *Fixed point results for orthogonal $Z-$contraction mappings in $O-$complete metric space*, Int. J. Appl. Phys. Math **10** (2020) 33–40.