On the scattering problem for a discontinuous boundary value problem

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Abstract

We study Sturm–Liouville Boundary value problems with discontinous interior to interval. When the boundary condition includes the spectral parameters, the scattering problem on the half plane $[0,\infty)$ is considered. For the problem, the scattering data is defined and the properties of the scattering data are investigated.

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1 Introduction

In non-homogeneous case, wave problems under different conditions (mass weight, flexibility effort) are expressed mathematically by discontinuous coefficient or discontinuous boundary condition. For example, vibration problem for a body consisting two objects with different densities or diffusion phenomena are reduced to discontinuous mathematical phsiccs problems in a medium with different density ([1]-[3]). For the physical problems of this form , the direct and inverse problems for scattering keep up to date such as in ([8],[9]). Let's note that, similar conditions are encountered in mathematical physics problems since the boundary condition includes derivatives with respect to time or normal.

In this work , on the half line $[0,\infty)$, the boundary value problem for the equation

$$-u''+p(x)u=\lambda^2 u, \tag{1.1}$$

with the the boundary condition

$$u'(0)+\left(b_1+b_2\lambda^2\right)u(0)=0, \tag{1.2}$$

where discontinuous conditions at point $a\in(0,\infty)$

$$\begin{aligned} &u(a-0)=\alpha u(a+0),\\ &u'(a-0)=\alpha^{-1}u(a+0) \end{aligned} \tag{1.3}$$

is considered. Here $\alpha>0,\alpha\neq 1$, λ is a complex parameter, $p(x)$ is real valued function and the condition

$$\int_x^\infty (1+x)|p(x)|dx<\infty.$$

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is satisfied. In the classical case, the inverse problem of the scattering theory for the (1.1) is completely solved in ([2],[10]). Similar problems for the (1.1) when the boundary conditions dependent on the spectral parameter are examined in ([8],[11]) and spectral analysis on the half line is studied in ([5]). Boundary value problem for the (1.1). Boundary value problems for the equation (1.1) with spectral parameter in boundary condition arise from applied problems such as the study of heat condition in ([6]).

The main aim of this article is to apply Marchenko method to the boundary value problem (1.1)-(1.2) with discontinous conditions (1.3) at the interior point $x = a \in (0, \infty)$. Moreover, similar problem for the boundary condition $u(0) = 0$ is studied in ([7]).

It is not to show that the function is the Jost solution of the equation (1.1) when $q(x) \equiv 0$

$$f_0(x,\lambda) = -\begin{cases} e^{i\lambda x}, & x > a, \\ Ae^{i\lambda x} + Be^{i\lambda(2a-x)}, & 0 < x < a, \end{cases} \tag{1.4}$$

where $A = \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right)$, $B = \frac{1}{2}\left(\alpha - \frac{1}{\alpha}\right)$.

The following (non-triangular) integral representation plays an important role on the Jost solution $f(x,\lambda)$ of the (1.1) ([7]). It is known that, if the conditions (1.4) hold, then the (1.1) has a unique solutions $f(x,\lambda)$ (regular with respect to λ) in the half plane $Im\lambda > 0$ (continuous on the real axis) which can be represented as the form:

$$f(x,\lambda) = f_0(x,\lambda) + \int_x^\infty K(x,t)e^{i\lambda t}dt, \tag{1.5}$$

where the kernel $K(x,t)$ satisfies the inequalities

$$|K(x,t)| \le \frac{c}{2}\sigma\left(\frac{x+t}{2}\right)e^{c\sigma_1(x)}, \quad 0 < |x-a| < t-a,$$

$$|K(x,t)| \le \left\{\frac{c}{2}\sigma\left(\frac{x+t}{2}\right) + \frac{|B|}{2}\sigma\left(\frac{2a+x-t}{2}\right)\right\}e^{c\sigma_1(x)}, \quad |t-a| < a-x,$$

where $C = A + |B|$, $\sigma(x) = \int_x^\infty |q(x)|dx$, $\sigma_1(x) = \int_x^\infty \sigma(x)dx$. Furthermore, the function $K(x,t)$ is continous for $t \neq 2a - x$, $x \neq a$ and possesses the following properties:

$$K(x,x) = \frac{A}{2}\int_x^\infty q(t)dt, \quad x < a,$$

$$K(x,x) = \frac{1}{2}\int_x^\infty q(t)dt, \quad x > a,$$

$$K(x,2a-x+0) - K(x,2a-x-0) = \frac{B}{2}\left(\int_a^\infty q(t)dt - \int_x^\infty q(t)dt\right), \quad x < a.$$

2 The scattering data

For real $\lambda \neq 0$, the functions $f(x,\lambda)$ and $f(x,-\lambda)$ form a fundamental solution system of equation (1.1) and their Wronskian of this system is equal to

$$W\{f(x,\lambda), f(x,-\lambda)\} = f'(x,\lambda)f(x,-\lambda) - f(x,\lambda)f'(x,-\lambda) = 2i\lambda.$$

Let $\varphi(x,\lambda)$ be the special solution of the equation (1.1) satisfying the initial conditions :

$$\varphi(0,\lambda) = 1, \quad \varphi'(0,\lambda) = -\left(b_1 + b_2\lambda^2\right). \tag{2.1}$$

Thus, we have the following assertions.

Lemma 2.1. For all $\lambda \neq 0$, the identity is valid:

$$\frac{2i\lambda\varphi(x,\lambda)}{E(\lambda)} = f(x,-\lambda) - S(\lambda)f(x,\lambda), \tag{2.2}$$

where

$$S(\lambda) = \frac{f'(0,-\lambda) + \left(b_1 + b_2\lambda^2\right) f(0,-\lambda)}{f'(0,\lambda) + \left(b_1 + b_2\lambda^2\right) f(0,\lambda)}, \tag{2.3}$$

with

$$S(\lambda) = \overline{S(-\lambda)} = [S(-\lambda)]^{-1}. \tag{2.4}$$

Proof. Since the function $f(x,\lambda)$ and $f(x,-\lambda)$ form a fundamental system of solutions of (1.1) then for all real $\lambda \neq 0$, we can write

$$\varphi(x,\lambda) = A(\lambda)f(x,\lambda) + B(\lambda)f(x,-\lambda). \tag{2.5}$$

From the condition (2.1), we get

$$A(\lambda)f(0,\lambda) + B(\lambda)f(0,-\lambda) = 1,$$

$$A(\lambda)f'(0,\lambda) + B(\lambda)f'(0,-\lambda) = -(b_1 + b_2\lambda^2).$$

Then we obtain

$$A(\lambda) = \frac{f'(0,-\lambda) + \left(b_1 + b_2\lambda^2\right) f(0,-\lambda)}{-2i\lambda},$$

$$B(\lambda) = \frac{f'(0,\lambda) + \left(b_1 + b_2\lambda^2\right) f(0,\lambda)}{2i\lambda}.$$

Therefore

$$\begin{aligned} w(x,\lambda) \quad &= \frac{f'(0,-\lambda) + \left(b_1 + b_2\lambda^2\right) f(0,-\lambda)}{-2i\lambda} f(x,\lambda) + \\ &+ \frac{f'(0,\lambda) + \left(b_1 + b_2\lambda^2\right) f(0,\lambda)}{2i\lambda} f(x,-\lambda). \end{aligned} \tag{2.6}$$

Now, we prove that $E(\lambda) \neq 0$ for all real $\lambda \neq 0$. Assume the contrary, i.e. $E(\lambda) = 0$ for real $\lambda_0 \neq 0$.

$$f'(0,\lambda_0) = -\left(b_1 + b_2\lambda^2\right) f(0,\lambda),$$

$$\overline{f'(0,\lambda_0)} = -\left(b_1 + b_2\lambda^2\right) \overline{f'(0,\lambda_0)}.$$

Substituting these values in (2.5), according to Wronskian, we find $0 = 2i\lambda_0$ which contradicts the assumption $\lambda_0 \neq 0$. Thus, dividing both sides of (2.6) into $E(\lambda)$,we get

$$\frac{2i\lambda w(x,\lambda)}{f'(0,\lambda) + (b_1 + b_2\lambda^2) f(0,\lambda)} = f(x,-\lambda) - S(\lambda) f(x,\lambda),$$

where

$$S(\lambda) = \frac{f'(0,-\lambda) + \left(b_1 + b_2\lambda^2\right) f(0,-\lambda)}{f'(0,\lambda) + (b_1 + b_2\lambda^2) f(0,\lambda)}. \tag{2.7}$$

Since $q(x)$ is real valued function, it is easy to check that for real λ, the relations

$$E(\lambda) = f'(0,\lambda) + \left(b_1 + b_2\lambda^2\right) f(0,\lambda)$$

and

$$\overline{E(\lambda)} = f'(0,-\lambda) + \left(b_1 + b_2\lambda^2\right) f(0,-\lambda) E(-\lambda)$$

are satisfied. Therefore we get

$$S(-\lambda) = \overline{S(\lambda)} = [S(\lambda)]^{-1}.$$

Lemma is proved. Q.E.D.

The function $S(\lambda)$ is called the scattering function of the boundary value problem (1.1)-(1.2).

Theorem 2.2. The function $E(\lambda)$ may have only a finite number of zeros in half plane $Im\lambda > 0$. All these zeros are simple and lie in the imaginary axis.

Proof. Since $E(\lambda) \neq 0$ for all real $\lambda \neq 0$, the point $\lambda = 0$ can be the only real zero of the function $E(\lambda)$. Using the analyticity of the function $E(\lambda)$ in upper half plane and the properties of the solution (1.1) are obtained the zeros of $E(\lambda)$ form at most countable and bounded set. Thus, the zeros of the function $E(\lambda)$ form a bounded finite or countable set whose unique limiting point can only be zero.

Now let's show that the zeros of the function $E(\lambda)$ lie on the imaginary axis. Suppose that λ_1 and λ_2 are arbitrary zeros of the function of $E(\lambda)$. We consider the following relations:

$$-f''(x,\lambda_1) + q(x) f(x,\lambda_1) = \lambda_1^2 f(x,\lambda_1),$$
$$\overline{-f''(x,\lambda_2)} + \overline{q(x) f(x,\lambda_2)} = \overline{\lambda_2^2 f(x,\lambda_2)}.$$

Multiplying the first of these relations by $\overline{f(x,\lambda_2)}$ substracting the second resulting relation from the first equation and integrating the resulting difference from zero to infinity, we obtain

$$W\left\{f(x,\lambda_1), \overline{f(x,\lambda_2)}\right\}|_{x=0} + \left(\lambda_1^2 - \overline{\lambda_2^2}\right) \int_0^\infty \left(f(\lambda_1,x), \overline{f(\lambda_2,x)}\right) dx = 0. \tag{2.8}$$

On the other hand, according to the definition of the function $E(\lambda)$, the following relations hold:

$$E(\lambda_j) = f'(0,\lambda_j) + \left(b_1 + b_2\lambda^2\right) f(0,\lambda_j), \quad j = 1,2.$$

Therefore,

$$W\left\{f(x,\lambda_1), \overline{f(x,\lambda_2)}\right\}|_{x=0} = f'(0,\lambda_2)\overline{f(0,\lambda_2)} - f(0,\lambda_1)\overline{f'(0,\lambda_2)}$$
$$= f(0,\lambda_1)\overline{f(0,\lambda_2)}\left(-b_1 - b_2\lambda_1^2\right) - f(0,\lambda_1)\overline{f(0,\lambda_2)}\left(-b_1 - b_2\overline{\lambda_2}^2\right).$$

For $\lambda_1 = \lambda_2$, it follows that

$$W\left\{f(x,\lambda_1), \overline{f(x,\lambda_2)}\right\}|_{x=0} = b_2\left|f(0,\lambda_1)\right|^2\left(\overline{\lambda_1}^2 - \lambda_2^2\right).$$

Therefore, the equation (2.8) takes form as:

$$\left(\lambda_1^2 - \overline{\lambda_1^2}\right)\left[\int_0^\infty \left|f(x,\lambda_1)\right|^2 dx + \left|b_2 f(0,\lambda_1)\right|^2\right] = 0. \tag{2.9}$$

Since $b_2 > 0$, it follows from (2.9), $\lambda_1^2 = \overline{\lambda_1^2}$ or $\lambda_1 = i\mu_1$, where $\mu_1 \geq 0$. Therefore, the zeros of the function $E(\lambda)$ can lie only on the imaginary axis.

Let us show that the function $E(\lambda)$ has a finite number of zeros for $Im\lambda \neq 0$. In the general case, since we can give an estimate for the distance between adjacent zeros of the function $E(\lambda)$, it follows that the number of zeros is finite. The estimate can be obtained by the method described in [10] and by using the relation (2.9).

Let's show that the zeros of the function are simple. We denote differentiation with respect to λ with a dot over

$$\dot{f}(x,\lambda) = \frac{d}{d\lambda}(x,\lambda).$$

Now, differentiating the equation (1.1) with respect to λ, the following equation is obtained:

$$-\dot{f}''(x,\lambda) + q(x)\dot{f}(x,\lambda) = \lambda^2 f(x,\lambda) + 2\lambda f(x,\lambda). \tag{2.10}$$

Multiplying the equation (1.1) by $\dot{f}(x,\lambda)$ and the relation (2.10) by $f(x,\lambda)$, substracting the first resulting relation from the second and integrating the resulting difference from zero to infity, we obtain

$$2\lambda\int_0^\infty \left|f(x,\lambda)\right|^2 dx = \left[\dot{E}(\lambda) - 2\lambda b_2 f(0,\lambda)\right] f(0,\lambda) - E(\lambda)\dot{f}(0,\lambda).$$

By substitution $\lambda = i\mu_k$ from the last relation, we obtain

$$2i\mu_k\int_0^\infty \left|f(x,i\mu_k)\right|^2 dx + 2b_2\mu_k\left|f(0,\mu_k)\right|^2 = -i\dot{E}(i\mu_k)f(0,i\mu_k).$$

Therefore $\dot{E}(i\mu_k) \neq 0$, i.e. the zeros of the function $E(\lambda)$ are simple. The lemma is proved. Q.E.D.

We denote the zeros of the function $E(\lambda)$ with $i\mu_1, ..., i\mu_n$ $(0 < \mu_1 < ... < \mu_n)$ and denote

$$m_k^{-2} \equiv \int_0^\infty |f(x, i\mu_k)|^2 \, dx + b_2\mu_k \, |f(0, i\mu_k)|^2 .$$

The numbers m_k (k=1,...,n.) are called the normalized numbers of boundary value problem (1.1) – (1.2).

It is easy to show that

$$u(x, \lambda) = f(x, -\lambda) - S(\lambda)f(x, \lambda), \quad 0 < \lambda < \infty, \tag{2.11}$$

and

$$u_k(x) = m_k f(x, i\mu_k), \quad (k = 1, ..., n), \tag{2.12}$$

are the bounded solutions of the boundary value problem (1.1)-(1.3) and it is prove that they form a complete normalized system. From these formulas (2.11)-(2.12), the normalized eigenfunctions system satisfied the asymptotic formulas as $x \to \infty$,

$$u(x, \lambda) = e^{-i\lambda x} - S(\lambda)e^{i\lambda x} + o(1), \quad -\infty < \lambda < \infty, \tag{2.13}$$

$$u_k(x) = m_k e^{-\mu_k x}(1 + o(1)), \quad k = 1, ..., n. \tag{2.14}$$

The collections $\{S(\lambda), \lambda_k, m_k\}$ $(k = 1, 2, ..., n)$ are called the scattering data of the boundary value problem (1.1)-(1.3).

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