A note on global classical solutions to a Cauchy problem

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Abstract

In this paper, we study the Cauchy problem associated with first-order differential inclusion and present some results on approximation and existence of global classical solution to the problem without convexity and compactness assumptions. For this purpose, we make use successive continuously differentiable approximations for Lipschitzian set-valued maps with nonconvex values. Thus we show that the results about continuously differentiable version of the famous Filippov's theorem still hold in the nonconvex case.

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1 Introduction

Differential inclusions have an important place in applied mathematics because of their generalization to a standard form of the ordinary differential equation and their role in the mathematical modeling of many problems in different fields such as social, economics, biology, and engineering. In many modern models, dynamic processes are described by Cauchy problems associated with discontinuous differential inclusions, in which the concepts of Carathéodory-type solution and classical (i.e. continuously differentiable) solution are used. In the literature, there are many existence theorems in the convex (or nonconvex) case for these problems (see, [1, 6, 14, 15]). One of the famous of these is Filippov's theorem as expressed generalized Filippov existence theorem in [15, Theorem 2.4.3]. The theorem provides approximation and stability properties of the solutions and attainable sets for the system. For this reason, there are recent papers in the literature, for example, [2, 5, 7, 8, 13], in which the theorem plays an important role, especially the qualitative and numerical analysis of discontinuous differential inclusions and optimal control problems. In addition, the results on various versions of Filippov's theorem are to be found in [3, 5, 6, 7].

The continuously differentiable version of Filippov's theorem implies that there exists a classical solution to the problem for a continuously differentiable function and a Lipschitzian set-valued map with convex values. It is well known that results of this form holds in the convex case, while results about Carathéodory-type solutions of the theorem hold in the nonconvex case.

In this paper, we deal with continuously differentiable approximations of (global) classical solutions to the Cauchy problem associated with first-order differential inclusion, and present some existence results. Thus we show that even without convexity and compactness hypotheses, the continuously differentiable version of Filippov's theorem is still true for the set-valued maps with closed values.

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2 Preliminaries

In this section, we discuss some basic concepts and results on differential inclusion theory (for more details, refer to [1, 4, 14, 15]). Let Ω be a subset of $\mathbb{R}^n$ and $\Gamma : \Omega \to \mathbb{R}^n$ a set-valued map. For $x \in \mathbb{R}^n$, the projection of x onto Ω is denoted by $P(x, \Omega)$, that is,

$$P(x, \Omega) = \{r \in \Omega : |x - r| = d(x, \Omega)\}$$

where $d(x, \Omega) = \inf\{|x - r| : r \in \Omega\}$ and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$. Here, if $P(x, \Omega)$ is nonempty, each element of it is called the closest point in Ω to x . In addition, it is known that $P(x, \Omega)$ is nonempty compact set if Ω is a closed subset of $\mathbb{R}^n$. A single-valued function $\gamma : \Omega \to \mathbb{R}^n$ is said to be a continuous selection from Γ if γ is continuous and $\gamma(r) \in \Gamma(r)$ for all r in Ω (for details, see [12, 14]).

By $C([0, T])$, $C^1([0, T])$ and $\mathrm{Lip}([0, T])$, we denote the class of all continuous functions from $[0, T]$ to $\mathbb{R}^n$, the class of all continuously differentiable functions from $[0, T]$ to $\mathbb{R}^n$ and the class of all (globally) Lipschitz functions from $[0, T]$ to $\mathbb{R}^n$ respectively. It is known that $C([0, T]) \subset C^1([0, T]) \subset \mathrm{Lip}([0, T])$. By $\|\cdot\|_\infty$ and $\|\cdot\|_{C^1}$, we denote supremum norm on $C([0, T])$ and C^1-norm on $C^1([0, T])$, here for $\rho \in C^1([0, T])$, $\|\rho\|_{C^1} := \|\rho\|_\infty + \|\dot{\rho}\|_\infty$. Note that $(C([0, T]), \|\cdot\|_\infty)$ and $(C^1([0, T]), \|\cdot\|_{C^1})$ are Banach spaces (for further information, see [9, 10]).

Let G be a relatively open subset of $[0, T] \times \mathbb{R}^n$. Let $F : G \to 2^{R^n} \setminus \{\varnothing\}$ be a set-valued map. The Cauchy problem associated with first-order differential inclusion is formulated as follows

$$\dot{x}(t) \in F(t, x(t)), \; t \in [0, T], \; x(0) = x_0 \tag{2.1}$$

where $T > 0$, $x_0 \in \mathbb{R}^n$. Let Ω_0 be a subset of G. The function $F(t, x)$ is called Lipschitz (with ℓ) in x, if there exists $\ell > 0$ such that, for all $(t, x), (s, r) \in \Omega_0$, the inclusion

$$F(s, r) \subseteq F(t, x) + \ell\,|r - x|\,B$$

holds where B denotes the closed unit ball of $\mathbb{R}^n$. Now, define the set $\mathcal{S}_F$ as

$$\mathcal{S}_F = \left\{x \in C^1([0, T]) : x \text{ satisfies } (2.1), Grx \subseteq G\right\}$$

where Grx denotes the graph of the function x. The function x is said to be a classical solution to the problem (2.1) if $x \in \mathcal{S}_F$. Let $\rho \in C^1([0, T])$ and $\lambda > 0$. By $T_\lambda(\rho)$, it is defined the tube (of radius λ) on X, namely

$$T_\lambda(\rho) := \{(t, r) \in [0, T] \times \mathbb{R}^n : t \in [0, T], \; |r - \rho(t)| < \lambda\}.$$

It is said that a function $v \in C^1([0, T])$ lies in $T_\lambda(\rho)$ if $(s, v(s)) \in T_\lambda(\rho)$ for each $s \in [0, T]$. For $v \in C^1([0, T])$ such that $Grv \subseteq G$, define

$$R[v] := \int_0^T d(\dot{v}(\xi), F(\xi, v(\xi)))\, d\xi.$$

This functional is used under hypotheses that guarantee the well-definedness of the integral (for further information, see [1, 4, 15]).

3 Main results

Theorem 3.1. Let G be a relatively open subset of $[0,T]\times\mathbb{R}^n$. Let $F: G\to 2^{R^n}\setminus\{\varnothing\}$ be a set-valued map, $\rho\in C^1([0,T])$ and $\lambda>0$ with $T_\lambda(\rho)\subseteq G$. Suppose that the following are satisfied,

(i) $F(s,r)$ is a closed set for all $(s,r)\in T_\lambda(\rho)$, and $F(s,r)$ is continuous in s on G.

(ii) $F(s,r)$ is Lipschitz (with ℓ) in r on $T_\lambda(\rho)$.

Suppose further that $R[\rho]<\lambda e^{-lT}$. Then there exists $x\in S_F$ satisfying

$$\|x-\rho\|_\infty<\lambda.$$

Proof. To prove the theorem, we construct a Cauchy sequence $\{\rho_i\}$ in the space $(C^1([0,T]),\|\cdot\|_{C^1})$ by choosing $\dot\rho_i(s)$ as the closest point in $F(s,\rho_{i-1}(s))$ to $\dot\rho_{i-1}(s)$, and show the solution is the limit of this sequence. For this, let $\rho_0=\rho\in C^1([0,T])$. Then by using hypotheses, one can see that the function $P_0:[0,T]\to 2^{R^n}\setminus\{\varnothing\}$ defined by $P_0(s)=P(\dot\rho_0(s),F(s,\rho_0(s)))$ is a Lipschitz set-valued map with compact values. According to [11], we can take a continuous selection q_0 from P_0. Thus we can define ρ_1 in $C^1([0,T])$ as

$$\rho_1(s)=x_0+\int_0^s q_0(\xi)d\xi.$$

Note that $\dot\rho_1=q_0$, $d(\dot\rho_1(s),F(s,\rho_0(s)))=0$ and $|\dot\rho_1(s)-\dot\rho_0(s)|=d(\dot\rho_0(s),F(s,\rho_0(s)))$. It follows from these relations that

$$\begin{aligned}|\rho_1(s)-\rho_0(s)| &\le \int_0^s|\dot\rho_1(\xi)-\dot\rho_0(\xi)|\,d\xi=\int_0^s d(\dot\rho_0(\xi),F(\xi,\rho_0(\xi)))\,d\xi\\ &\le R[\rho_0]\end{aligned}\tag{3.1}$$

Hence, we obtain that ρ_1 lies in $T_\lambda(\rho_0)$. Now we claim that one can inductively choose functions $\rho_0,...,\rho_{i+1}$ lying in $T_\lambda(\rho_0)$ and define both $P_i(s)=P(\dot\rho_i(s),F(s,\rho_i(s)))$, and $\rho_{i+1}(s)=x_0+\int_0^s q_i(\xi)d\xi$ (here, q_i is a continuous selection from P_i) such that the following hold for $s\in[0,T]$,

$$q_i(s) = \dot\rho_{i+1}(s)\in F(s,\rho_i(s)) \text{ for } i\ge 0,$$

$$|\rho_{i+1}(s)-\rho_i(s)| \le R[\rho_0]\frac{(\ell s)^i}{i!} \text{ for } i\ge 0,\tag{3.2}$$

$$|\dot\rho_{i+1}(s)-\dot\rho_i(s)| \le R[\rho_0]\frac{\ell(\ell s)^{i-1}}{(i-1)!} \text{ for } i\ge 1.\tag{3.3}$$

Indeed, (3.1) implies that, for $i\ge 1$,

$$\begin{aligned}|\dot\rho_{i+1}(s)-\dot\rho_i(s)| &= d(\dot\rho_i(s),F(s,\rho_i(s)))\\ &\le d(\dot\rho_i(s),F(s,\rho_{i-1}(s)))+\ell|\rho_i(s)-\rho_{i-1}(s)|\\ &= \ell|\rho_i(s)-\rho_{i-1}(s)|.\end{aligned}\tag{3.4}$$

From (3.4),

$$\begin{aligned}|\rho_{i+1}(s)-\rho_i(s)| &\leq \int_0^s |q_i(s_1)-\dot{\rho}_i(s_1)|\,ds_1\\ &\leq \int_0^s \ell\,|\rho_i(s_1)-\rho_{i-1}(s_1)|\,ds_1\\ &\leq \int_0^s \ell\int_0^{s_1}\ell...\int_0^{s_{i-1}}\ell\,|\rho_1(s_i)-\rho_0(s_i)|\,ds_i...ds_1.\end{aligned}$$

Moreover, from (3.2), for $i\geq 0$, we obtain

$$\begin{aligned}|\rho_i(s)-\rho_0(s)| &\leq \sum_{j=0}^{i-1}|\rho_{j+1}(s)-\rho_j(s)|\\ &\leq R\left[\rho_0\right]\sum_{j=0}^{i-1}\frac{(\ell s)^j}{j!}<\lambda.\end{aligned}\tag{3.5}$$

Hence, by using induction on i, it is easy to see that we can get a sequence $\{\rho_i\}$ in $C^1([0,T])$ satisfying the above relations we obtained. It follows from (3.2) and (3.3) that the sequence is Cauchy in $(C^1([0,T]),\|\cdot\|_{C^1})$. The space is being complete, $\{\rho_i\}$ converges uniformly to a function $x\in C^1([0,T])$. Note that $\{\dot{\rho}_i\}$ converges uniformly $\dot{x}$ on $C([0,T])$, and that

$$\dot{x}(s)=\lim_{i\to\infty}\dot{\rho}\left(s\right)\in\lim_{i\to\infty}F\left(s,\rho_{i-1}\left(s\right)\right)=F(s,x(s)).$$

Hence $x\in S_F$. From (3.5), one can get $\|x-\rho\|_\infty<\lambda$. Q.E.D.

Corollary 3.2. For the problem (1), suppose that (i) and (ii) conditions of Theorem 3.1 are satisfied. Suppose further that $h:=\|d\left(\dot{\rho}\left(\cdot\right),F(\cdot,\rho\left(\cdot\right))\right)\|_\infty<\lambda$ and $R\left[\rho\right]<\min\left\{\lambda e^{-lT},(\frac{\lambda-h}{l})e^{-lT}\right\}$. Then there exists $x\in S_F$ satisfying $\|x-\rho\|_{C^1}<\lambda$.

Proof. From (3.3), we obtain for $i\geq 1$ and $s\in[0,T]$,

$$\begin{aligned}|\dot{\rho}_i(s)-\dot{\rho}_0(s)| &\leq \sum_{j=0}^{i-1}|\rho_{j+1}(s)-\rho_j(s)|\\ &\leq d\left(\dot{\rho}_0(s),F(s,\rho_0(s))\right)+R\left[\rho_0\right]\ell\sum_{j=0}^{i-1}\frac{(\ell s)^{j-1}}{(j-1)!}\\ &\leq h+R\left[\rho_0\right]\ell e^T\\ &< \lambda.\end{aligned}$$

Considering the last inequality and (3.5), we have the desired result. Q.E.D.

4 Conclusion

In this paper, the Cauchy problem associated with first-order differential inclusion is considered. The results related to approximation and existence of global classical solution to the problem under weak conditions are obtained. According to the results of this paper, it is shown that the continuously differentiable version of Filippov's theorem still hold without convexity and compactness assumptions. It is known that the form plays a major role in the qualitative-numerical analysis of differential inclusions and in existence theorems for optimal control as well as applications in the different many fields. Therefore, it is thought that the results obtained have a wide application area.

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