The generalized Moore-Penrose inverse of the generalized Fibonacci matrix

Süleyman Aydınyüz[1] and Mustafa Aşçı[2]

[1]Faculty of Science, Mathematics Department, Pamukkale University, 20160, Kınıklı, Denizli, Turkey
[2]Faculty of Science, Mathematics Department, Pamukkale University, 20160, Kınıklı, Denizli, Turkey
E-mail: aydinyuzsuleyman@gmail.com[1], mustafa.asci@yahoo.com[2]

Abstract

In this study, we introduce the concept of generalized Moore-Penrose inverses for rectangular matrices whose elements are derived from generalized Fibonacci numbers. The matrix $U \in M_{m,n}(\mathbb{C})$ represents a rectangular or square matrix formed by generalized Fibonacci numbers. Special numbers are obtained by selecting arbitrary values for the parameter p in the generalized Fibonacci sequence. For instance, choosing $p = 1$ yields the usual Fibonacci matrix, while $p = 2$ corresponds to the Pell matrix. Finally, we provide the Matlab code to compute the generalized Moore-Penrose inverse of such matrices.

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1 Introduction

Fibonacci, Lucas numbers and their various generalizations have many interesting properties and applications to many disciplines. [1, 2, 3] can be examined for details of these applications and curiosities.

Fibonacci numbers are defined by the recurrence relation

$$F_{n+1} = F_n + F_{n-1},$$

where $F_0 = 0$ and $F_1 = 1$ are for $\forall n \geq 1$. Here, F_n is nth Fibonacci number.

Lucas numbers, on the other hand, are defined by the recurrence relation

$$L_{n+1} = L_n + L_{n-1},$$

where $L_0 = 2$ and $L_1 = 1$ are for $\forall n \geq 1$. L_n is nth Lucas number.

Similarly, Pell numbers are defined by the recurrence relation

$$P_{n+1} = 2P_n + P_{n-1},$$

with the initial conditions $P_0 = 0$ and $P_1 = 1$ for each $n \geq 1$ to P_n, nth Pell number.

Pell-Lucas, Jacobsthal and Jacobsthal-Lucas numbers can also be obtained with similar recurrence relations.

Many generalizations of Fibonacci and Lucas sequences, applications of these generalizations and matrix representations have been studied by many authors. A generalization was given by E. Lucas in 1878 in [4] as follows:

$$\begin{aligned} U_n &= pU_{n-1} - qU_{n-2}, \qquad U_0 = 0,\ U_1 = 1 \\ V_n &= pV_{n-1} - qV_{n-2}, \qquad V_0 = 2,\ \ V_2 = p \end{aligned}$$

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where p and q are real numbers with $q\left(p^2-4q\right)\neq 0$. The Binet formulas of U_n and V_n are

$$U_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$$

and

$$V_n=\alpha^n+\beta^n$$

with the different roots of the equation

$$x^2-px+q=0$$

being $\alpha=\frac{p+\sqrt{p^2-4q}}{2}$ and $\beta=\frac{p-\sqrt{p^2-4q}}{2}$.

In this generalization, if $p=1$, $q=-1$ is chosen,

$$U_n=U_{n-1}+U_{n-2},\qquad U_0=0,\ U_1=1$$

and

$$V_n=V_{n-1}+V_{n-2},\qquad V_0=2,\quad V_2=1$$

are obtained, and the known Fibonacci and Lucas numbers are obtained in the order of $U_n=F_n$ and $V_n=L_n$.

If $p=2$, $q=-1$ is chosen, known Pell and Pell-Lucas numbers are obtained. If $p=1$, $q=-2$ is chosen, Jacobsthal and Jacobsthal-Lucas numbers are obtained.

2 The generalized Moore-Penrose inverse

Generalizations are prevalent in the fields of mathematics and engineering, particularly in linear algebra, when it comes to finding the inverse of a matrix A. One of the most commonly employed generalizations for computing the inverse of a matrix is known as the Moore-Penrose inverse or the pseudo-inverse. The Moore-Penrose inverse has been extensively studied in mathematics and various scientific disciplines. The concept of matrix inversion was independently introduced by E. H. Moore in 1920 in [5], Arne Bjerhammar in 1951 in [6], and Roger Penrose in 1955 in [7]. The Moore-Penrose inverse, also known as the pseudo-inverse, is a widely used method for analyzing systems of linear equations and calculating solutions. For further information, refer to in [8]. Kizilateş investigated novel families of Horadam numbers associated with finite operators in [9] and, in conjunction with Fonseca and Terzioglu, defined a second-order difference equation with sign-alternating coefficients in [10].

There are two methods commonly employed for calculating the Moore-Penrose inverse of a matrix. One approach involves the maximal rank factorization of the matrix, while the other method utilizes the singular value decomposition. Detailed information about these methods can be found in references in [11, 12, 13, 14, 15, 16].

Notation 2.1. In the following discussion, the following conventions are adopted.

- We denote by $M_{m,n}(\mathbb{C})$ and $M_n(\mathbb{C})$ the sets of complex $m\times n$ and $n\times n$ matrices, respectively.
- Given a matrix A, the conjugate transpose of A is denoted A^*.

- For any positive integer n, $I_n \in M_n(\mathbb{C})$ denotes the $n \times n$ identity matrix.

The definitions and theorems that will be given in this section can be found in [17].

Definition 2.2. For $R \in M_{n,m}(\mathbb{C})$, then there exist a unique a pseudoinverse of R as a matrix $R^{\dagger} \in M_{m,n}(\mathbb{C})$ that satisfies all the following four criteria, known as the Moore-Penrose conditions:
(1) $RR^{\dagger}R = R$
(2) $R^{\dagger}RR^{\dagger} = R^{\dagger}$
(3) $R^{\dagger}R = \left(R^{\dagger}R\right)^{*}$
(4) $RR^{\dagger} = \left(RR^{\dagger}\right)^{*}$

If R is a nonsingular matrix, it is clear that $R^{\dagger} = R^{-1}$ trivially satisfies the four conditions.

Theorem 2.3. For any $R \in M_{n,m}(\mathbb{C})$ there exists a $R^{\dagger} \in M_{m,n}(\mathbb{C})$ that satisfies the Moore-Penrose conditions.

Theorem 2.4. For a matrix $R \in M_{n,m}(\mathbb{C})$, then there exists a unique $R^{\dagger} \in M_{m,n}(\mathbb{C})$.

Lemma 2.5. If $R \in M_{n,m}(\mathbb{C})$ with rank r and then $R = FE$ where $F \in M_{n,r}(\mathbb{C})$ and $E \in M_{r,m}(\mathbb{C})$ where $r(F) = r(E) = r$.

After the theorems and lemma given above, the Moore-Penrose inverse theorem can be given as follows:

Theorem 2.6. For any $R \in M_{n,m}(\mathbb{C})$ with rank r and $R = FE$ where $F \in M_{n,r}(\mathbb{C})$ and $E \in M_{r,m}(\mathbb{C})$ then we get

$$R^{\dagger} = E^{*}\left(EE^{*}\right)^{-1}\left(FF^{*}\right)^{-1}F^{*}.$$

Now, after this theorem, let us give two important results.

Corollary 2.7. If R has fullrow rank that is $r = n$, then $F = I_n$ and $E \in M_{m,n}(\mathbb{C})$ .Then

$$R = I_n A$$

and the equation for $R^{\dagger}$ reduces to

$$R^{\dagger} = R^{*}\left(RR^{*}\right)^{-1}.$$

This is called full row rank decomposition.

Corollary 2.8. If R has full column rank that is $r = m$, then $E = I_m$ and $F \in M_{m,n}(\mathbb{C})$. Then

$$R = BI_m$$

and the equation for $R^{\dagger}$ reduces to

$$R^{\dagger} = \left(R^{*}R\right)^{-1}R^{*}.$$

This is called full columns rank decomposition.

3 The generalized Moore-Penrose inverse on the generalized Fibonacci matrix

In this section, we find the general form of the generalized Moore-Penrose inverses of the matrix whose elements are the generalized Fibonacci numbers and give examples. Now, we examine the states of the $U \in M_{m,n}(\mathbb{C})$ matrix according to the m and n values.

In this study, we consider

$$U_n = pU_{n-1} - U_{n-2} \qquad , U_0 = 0,\ U_1 = 1$$

generalized Fibonacci numbers and find the generalized Moore-Penrose inverses.

Theorem 3.1. Let the elements in the $U \in M_{m,n}(\mathbb{C})$ matrix be rectangular matrix with generalized Fibonacci numbers. For positive integers m and n:

(1) If $U \in M_{m,n}(\mathbb{C})$, $m \neq n$ has full column rank, that is $rank(U_{m\times n}) = n < m$ then $U \in M_{m,n}(\mathbb{C})$ has left invers as follows:

$$U^{\dagger}_{n\times m} = \left(U^{*}_{n\times m} U_{m\times n}\right)^{-1} U^{*}_{n\times m} \qquad modmodM$$

where $U^{\dagger}_{n\times m}$ defines

$$U^{\dagger}_{n\times m} U_{m\times n} = I_{n\times n}$$

and

$$U_{m\times n} U^{\dagger}_{n\times m} \neq I_{m\times m}.$$

(2) If $U \in M_{m,n}(\mathbb{C})$, $m \neq n$ has full row rank, that is $rank(U_{m\times n}) = m < n$ then $U \in M_{m,n}(\mathbb{C})$ has right invers as follows:

$$U^{\dagger}_{n\times m} = U^{*}_{n\times m} \left(U_{m\times n} U^{*}_{n\times m}\right)^{-1} \qquad modM$$

where $U^{\dagger}_{n\times m}$ defines

$$U_{m\times n} U^{\dagger}_{n\times m} = I_{m\times m}$$

and

$$U^{\dagger}_{n\times m} U_{m\times n} \neq I_{n\times n}.$$

(3) If $U \in M_{m,n}(\mathbb{C})$, $m = n$ has full column rank, that is $rank(U_{m\times n}) = n = m$ then $U \in M_{m,n}(\mathbb{C})$ as follows:

$$U^{\dagger}_{n\times m} = U^{-1}_{m\times n}.$$

Theorem 3.2. Let the elements in the $U \in M_{m,n}(\mathbb{C})$ matrix be rectangular matrix with generalized Fibonacci numbers as follows:

$$U_{m,n} = \begin{bmatrix} U_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ U_2 & U_1 & 0 & 0 & 0 & \cdots & 0 \\ U_3 & U_2 & U_1 & 0 & 0 & \cdots & 0 \\ U_4 & U_3 & U_2 & U_1 & 0 & \cdots & \vdots \\ U_5 & U_4 & U_3 & U_2 & U_1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ U_m & U_{m-1} & U_{m-2} & U_{m-3} & U_{m-4} & \cdots & U_{m-n+1} \end{bmatrix}$$

where U_n is the nth generalized Fibonacci numbers.

For positive integers m and n:

(1) Let $m > n$.

(a) If $m-n$ is odd, we get as:

$$U^{\dagger}_{n\times m} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -p & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & -p & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & -p & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & -p & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & -p & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{pU_{m-n+3}}{U_{m-n+3}} & \frac{pU_{m-n+2}}{U_{m-n+3}} & -\frac{pU_{m-n+1}}{U_{m-n+3}} & \cdots & -\frac{pU_2}{U_{m-n+3}} & \frac{pU_1}{U_{m-n+3}} \\ 0 & 0 & 0 & \cdots & 0 & -\frac{pU_{m-n+1}}{U_{m-n+1}} & \frac{pU_{m-n}}{U_{m-n+1}} & \cdots & \frac{pU_1}{U_{m-n+1}} & -\frac{pU_0}{U_{m-n+1}} \end{bmatrix}$$

where U_n is nth generalized Fibonacci number.

(b) If $m-n$ is even, we get as:

$$U^{\dagger}_{n\times m} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -p & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & -p & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & -p & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & -p & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & -p & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{pU_{m-n+2}}{U_{m-n+2}} & \frac{pU_{m-n+1}}{U_{m-n+2}} & -\frac{pU_{m-n}}{U_{m-n+2}} & \cdots & \frac{pU_1}{U_{m-n+2}} & -\frac{pU_0}{U_{m-n+2}} \\ 0 & 0 & 0 & \cdots & 0 & -\frac{pU_{m-n+2}}{U_{m-n+2}} & \frac{pU_{m-n+1}}{U_{m-n+2}} & \cdots & -\frac{pU_2}{U_{m-n+2}} & \frac{pU_1}{U_{m-n+2}} \end{bmatrix}$$

where U_n is nth generalized Fibonacci number.

(2) Let $m < n$.

(a) If $m-n$ is odd, we get as:

$$U^{\dagger}_{n\times m} = \begin{bmatrix} 1 & -p & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -p & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -p & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & -p & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & \cdots & -\frac{pU_{m-n+3}}{U_{m-n+3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{pU_{m-n+2}}{U_{m-n+3}} & -\frac{pU_{m-n+1}}{U_{m-n+1}} \\ 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{pU_{m-n+1}}{U_{m-n+3}} & \frac{pU_{m-n}}{U_{m-n+1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{pU_2}{U_{m-n+3}} & \frac{pU_1}{U_{m-n+1}} \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{pU_1}{U_{m-n+3}} & -\frac{pU_0}{U_{m-n+1}} \end{bmatrix}$$

where U_n is nth generalized Fibonacci number.

(b) If $m-n$ is even, we get as:

$$U^{\dagger}_{n\times m}=\begin{bmatrix}
1 & -p & -1 & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & 1 & -p & -1 & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & 1 & -p & -1 & 0 & \cdots & 0 & 0\\
0 & 0 & 0 & 1 & -p & -1 & \cdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & 1 & -p & \cdots & -\frac{pU_{m-n+2}}{U_{m-n+2}} & 0\\
0 & 0 & 0 & 0 & 0 & 1 & \cdots & \frac{pU_{m-n+1}}{U_{m-n+2}} & -\frac{pU_{m-n+2}}{U_{m-n+2}}\\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{pU_{m-n}}{U_{m-n+2}} & -\frac{pU_{m-n+2}}{U_{m-n+2}}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & \frac{pU_1}{U_{m-n+2}} & -\frac{pU_2}{U_{m-n+2}}\\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{pU_2}{U_{m-n+2}} & \frac{pU_1}{U_{m-n+2}}
\end{bmatrix}$$

where U_n is nth generalized Fibonacci number.

(3) Let $m=n$. Then we get as:

$$\begin{aligned}
U^{\dagger}_{m\times m} &= U^{-1}_{m\times m}\\
&= \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0\\
-p & 1 & 0 & 0 & \cdots & 0\\
-1 & -p & 1 & 0 & \cdots & 0\\
0 & -1 & -p & 1 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & \cdots & 0 & -1 & -p & 1
\end{bmatrix}
\end{aligned}$$

Corollary 3.3. When we get $p=1$; we get the generalized Moore-Penrose inverse for the Fibonacci numbers in [18] as follows:

(1) Let $m>n$.

(a) If $m-n$ is odd, we get as:

$$F^{\dagger}_{n\times m}=\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
-1 & -1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & -1 & -1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & -1 & -1 & 1 & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & 0 & -1 & -1 & 1 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & -\frac{F_{m-n+3}}{F_{m-n+3}} & \frac{F_{m-n+2}}{F_{m-n+3}} & -\frac{F_{m-n+1}}{F_{m-n+3}} & \cdots & -\frac{F_2}{F_{m-n+3}} & \frac{F_1}{F_{m-n+3}}\\
0 & 0 & 0 & \cdots & 0 & -\frac{F_{m-n+1}}{F_{m-n+1}} & \frac{F_{m-n}}{F_{m-n+1}} & \cdots & \frac{F_1}{F_{m-n+1}} & -\frac{F_0}{F_{m-n+1}}
\end{bmatrix}$$

where F_n is nth Fibonacci number.

(b) If $m-n$ is even, we get as:

$$F^{+}_{n\times m}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\ -1 & -1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & -1 & -1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & -1 & -1 & 1 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & -\frac{F_{m-n+2}}{F_{m-n+2}} & \frac{F_{m-n+1}}{F_{m-n+2}} & -\frac{F_{m-n}}{F_{m-n+2}} & \cdots & \frac{F_1}{F_{m-n+2}} & -\frac{F_0}{F_{m-n+2}}\\ 0 & 0 & 0 & \cdots & 0 & -\frac{F_{m-n+2}}{F_{m-n+2}} & \frac{F_{m-n+1}}{F_{m-n+2}} & \cdots & -\frac{F_2}{F_{m-n+2}} & \frac{F_1}{F_{m-n+2}} \end{bmatrix}$$

where F_n is nth Fibonacci number.

(2) Let $m<n$.

(a) If $m-n$ is odd, we get as:

$$F^{+}_{n\times m}=\begin{bmatrix} 1 & -1 & -1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 1 & -1 & -1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & -1 & -1 & \cdots & 0 & 0\\ 0 & 0 & 0 & 1 & -1 & \cdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 1 & \cdots & -\frac{F_{m-n+3}}{F_{m-n+3}} & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{F_{m-n+2}}{F_{m-n+3}} & -\frac{F_{m-n+1}}{F_{m-n+1}}\\ 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{F_{m-n+1}}{F_{m-n+3}} & \frac{F_{m-n}}{F_{m-n+1}}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{F_2}{F_{m-n+3}} & \frac{F_1}{F_{m-n+1}}\\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{F_1}{F_{m-n+3}} & -\frac{F_0}{F_{m-n+1}} \end{bmatrix}$$

where F_n is nth Fibonacci number.

(b) If $m-n$ is even, we get as:

$$F^{+}_{n\times m}=\begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 1 & -1 & -1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & -1 & -1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & 1 & -1 & -1 & \cdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 1 & -1 & \cdots & -\frac{F_{m-n+2}}{F_{m-n+2}} & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots & \frac{F_{m-n+1}}{F_{m-n+2}} & -\frac{F_{m-n+2}}{F_{m-n+2}}\\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{F_{m-n}}{F_{m-n+2}} & -\frac{F_{m-n+2}}{F_{m-n+2}}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \frac{F_1}{F_{m-n+2}} & -\frac{F_2}{F_{m-n+2}}\\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{F_2}{F_{m-n+2}} & \frac{F_1}{F_{m-n+2}} \end{bmatrix}$$

where F_n is nth Fibonacci number.

(3) Let $m = n$. Then we get as:

$$\begin{aligned} F^{\dagger}_{m\times m} &= F^{-1}_{m\times m} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & -1 & 1 \end{bmatrix} \end{aligned}$$

Example 3.4. Let $p = 1$. So, we get the usual rectangular Fibonacci matrix. For $m = 6$ and $n = 3$, we get the rectangular Fibonacci matrix as follows:

$$F_{6\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \\ 5 & 3 & 2 \\ 8 & 5 & 3 \end{bmatrix}.$$

Now, we get the generalized Moore-Penrose inverse of the rectangular Fibonacci matrix as:

$$\begin{aligned} F^{\dagger}_{3\times 6} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & \frac{F_5}{F_6} & -\frac{F_4}{F_6} & \frac{F_3}{F_6} & -\frac{F_2}{F_6} & \frac{F_1}{F_6} \\ -1 & -1 & \frac{F_3}{F_4} & -\frac{F_2}{F_4} & \frac{F_1}{F_4} & \frac{F_0}{F_4} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & \frac{5}{8} & -\frac{3}{8} & \frac{2}{8} & -\frac{1}{8} & \frac{1}{8} \\ -1 & -1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}. \end{aligned}$$

Example 3.5. Let $p = 1$. So, we get the usual rectangular Fibonacci matrix. For $m = 4$ and $n = 7$, we get the rectangular Fibonacci matrix as follows:

$$F_{4\times 7} = \begin{bmatrix} 1 & 1 & 2 & 3 & 5 & 8 & 13 \\ 0 & 1 & 1 & 2 & 3 & 5 & 8 \\ 0 & 0 & 1 & 1 & 2 & 3 & 5 \\ 0 & 0 & 0 & 1 & 1 & 2 & 3 \end{bmatrix}.$$

Now, we get the generalized Moore-Penrose inverse of the rectangular Fibonacci matrix as:

$$F_{7\times4}^{\dagger} = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & \frac{F_5}{F_6} & -1 \\ 0 & 0 & -\frac{F_4}{F_6} & \frac{F_3}{F_4} \\ 0 & 0 & \frac{F_3}{F_6} & -\frac{F_2}{F_4} \\ 0 & 0 & -\frac{F_2}{F_6} & \frac{F_1}{F_4} \\ 0 & 0 & \frac{F_1}{F_6} & -\frac{F_0}{F_4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & \frac{5}{8} & -1 \\ 0 & 0 & -\frac{3}{8} & \frac{2}{3} \\ 0 & 0 & \frac{2}{8} & -\frac{1}{3} \\ 0 & 0 & -\frac{1}{8} & \frac{1}{3} \\ 0 & 0 & \frac{1}{8} & 0 \end{bmatrix}.$$

Example 3.6. Let $p = 2$. So, we get the usual rectangular Pell matrix. For $m = 6$ and $n = 3$, we get the rectangular Pell matrix as:

$$P_{6\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 1 \\ 12 & 5 & 2 \\ 29 & 12 & 5 \\ 70 & 29 & 12 \end{bmatrix}.$$

Now, we get the generalized Moore-Penrose inverse of the rectangular Pell matrix as:

$$P_{3\times6}^{\dagger} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & \frac{2P_5}{P_6} & -\frac{2P_4}{P_6} & \frac{2P_3}{P_6} & -\frac{2P_2}{P_6} & \frac{2P_1}{P_6} \\ -1 & -2 & \frac{2P_3}{P_4} & -\frac{2P_2}{P_4} & \frac{2P_1}{P_4} & -\frac{2P_0}{P_4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & \frac{29}{35} & -\frac{12}{35} & \frac{5}{35} & -\frac{2}{35} & \frac{1}{35} \\ -1 & -2 & \frac{5}{6} & -\frac{2}{6} & \frac{1}{6} & 0 \end{bmatrix}.$$

We can easily obtain the generalized Moore-Penrose inverses for other special numbers by changing the p-real numbers.

4 Conclusion

In this paper, we present the general form of the generalized Moore-Penrose inverses for matrices whose elements are derived from the generalized Fibonacci numbers. Furthermore, we provide illustrative examples to demonstrate the application of these generalized Moore-Penrose inverses. Now, we examine the states of the $U \in M_{m,n}(\mathbb{C})$ matrix according to the m and n values.

In this study, we considered

$$U_n = pU_{n-1} - U_{n-2}, \qquad U_0 = 0,\ U_1 = 1$$

generalized Fibonacci numbers and find the generalized Moore-Penrose inverses.

Let the elements in the $U \in M_{m,n}(\mathbb{C})$ matrix be rectangular matrix with generalized Fibonacci numbers. For positive integers m and n:

(1) If $U \in M_{m,n}(\mathbb{C})$, $m \neq n$ has full column rank, that is $rank(U_{m\times n}) = n < m$ then $U \in M_{m,n}(\mathbb{C})$ has left invers as follows:

$$U^{\dagger}_{n\times m} = \left(U^{*}_{n\times m}U_{m\times n}\right)^{-1}U^{*}_{n\times m} \qquad modM$$

where $U^{\dagger}_{n\times m}$ defines

$$U^{\dagger}_{n\times m}U_{m\times n} = I_{n\times n}$$

and

$$U_{m\times n}U^{\dagger}_{n\times m} \neq I_{m\times m}.$$

(2) If $U \in M_{m,n}(\mathbb{C})$, $m \neq n$ has full row rank, that is $rank(U_{m\times n}) = m < n$ then $U \in M_{m,n}(\mathbb{C})$ has right invers as follows:

$$U^{\dagger}_{n\times m} = U^{*}_{n\times m}\left(U_{m\times n}U^{*}_{n\times m}\right)^{-1} \qquad modM$$

where $U^{\dagger}_{n\times m}$ defines

$$U_{m\times n}U^{\dagger}_{n\times m} = I_{m\times m}$$

and

$$U^{\dagger}_{n\times m}U_{m\times n} \neq I_{n\times n}.$$

(3) If $U \in M_{m,n}(\mathbb{C})$, $m = n$ has full column rank, that is $rank(U_{m\times n}) = n = m$ then $U \in M_{m,n}(\mathbb{C})$ as follows:

$$U^{\dagger}_{n\times m} = U^{-1}_{m\times n}.$$

As a result of our investigation, we derive the generalized Moore-Penrose inverses for the special numbers corresponding to the chosen arbitrary values of p. Specifically, we obtain the generalized Moore-Penrose inverse for Fibonacci numbers by setting $p = 1$, and similarly, we determine the generalized Moore-Penrose inverse for Pell numbers when $p = 2$.

4.1 Appendix

Let's give the matlab code to find this generalized Moore-Penrose inverse.

Matlab code for the generalized Moore-Penrose Inverse

```
function Y = geninv(G)
format bank
format rational
% G is the generalized Fibonacci matrix for which we want to get the generalized Moore-Penrose inverse.
G=[1 1 2 3 5 8 13;0 1 1 2 3 5 8;0 0 1 1 2 3 5;0 0 0 1 1 2 3];
```

```
% Transpose if m < n
[m,n]=size(G); transpose=false;
if m <n
transpose=true;
A=G*G'
n=m;
else
A=G'*G
end
dA=diag(A); tol= min(dA(dA>0))*1e-9;
L=zeros(size(A));
r=0;
for k=1:n
r=r+1;
L(k:n,r)=A(k:n,k)-L(k:n,1:(r-1))*L(k,1:(r-1))';
if L(k,r)>tol
L(k,r)=sqrt(L(k,r));
if k<n
L((k+1):n,r)=L((k+1):n,r)/L(k,r);
end
else
r=r-1;
end
end
L=L(:,1:r);
M=inv(L'*L)
if transpose
Y=G'*L*M*M*L'
else
Y=L*M*M*L'*G'
end
```

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