# New 2-norms formed on $\ell^p(\mathbb{R})$ by bounded linear functionals

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**Abstract**

The space of $p$-summable sequences of real numbers $\ell^p(\mathbb{R})$, can be equipped with norm or $n$-norm with $n \geq 2$. In this work, we consider two new 2-norms formed on $\ell^p(\mathbb{R})$ based on different choices of bounded linear functionals on it and investigate their relationships between Gähler's and Gunawan's 2-norms. One of these 2-norms is a special case of the 2-norm defined very recently by Konca and Idris [A new 2-norm generated by bounded linear functionals on a normed space, Filomat, 2023.]. They have wondered whether 2-norm defined by them and Gähler's 2-norm are strongly equivalent or not for some special cases of a normed space $X$. A part of this work consists the solution of the open problem given by them for $X = \ell^p(\mathbb{R})$. We define new two 2-norms and show that those are not always equivalent to the usual 2-norm on $\ell^p(\mathbb{R})$. One of the newly defined norms induced from these new 2-norms, is equivalent to the usual norm on $\ell^p(\mathbb{R})$, while the other one is not. These differences are obtained as a result of depending on the chosen bounded linear functionals to form new 2-norms and norms on this space.

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## 1 Introduction and known definitions

Throughout the paper, for brevity, we will use $\ell^p$ instead of $\ell^p(\mathbb{R})$ to demonstrate the space of $p$-summable sequences of real numbers.

A vector space can be equipped with several 2-norms. In such a case, we may have an equivalence relation between them. In [10], it is shown that all 2-norms on a finite dimensional vector space are equivalent. On infinite-dimensional vector spaces there is no quarantee that every two 2-norms are equivalent. There were two known 2-norms defined on the space of $p$-summable sequences of real numbers $\ell^p$. The first 2-norm is a special case of Gähler's formula [1], while the second is due to Gunawan [2]. Konca and Idris [5] defined a new 2-norm on $\ell^p$ and saw that Gahler's and Gunawan's 2-norms are (strongly) equivalent to this new 2-norm on $\ell^p$. Recently, Konca and Idris [6] have introduced a new 2-norm $\|.,.\|^{KI}$ generated by bounded linear functionals on a normed space $X$ with dimension $\dim(X) \geq 2$ and investigated its relationship with the 2-norm defined by Gähler [1]. They had difficulties in proving the strong equivalence between the two 2-norms $\|.,.\|^{KI}$ and $\|.,.\|^{G}$ on $X$. As a matter of fact, they have not known whether the two 2-norms are strongly equivalent or not unless they examined it in detail for the special cases of $X$. This ongoing problem will be continued to research for some special cases of $X$, for example for the space of $p$-summable sequences of real numbers $\ell^p$, the space of $p$-integrable functions $L^p$ $(1 \leq p < \infty)$ and the space $C[a, b]$ of all continuous functions defined on $[a, b]$. These all have remained as open problems to

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explore for the readers. In a part of this work, we revisit the space $\ell^p$ to solve this ongoing problem and consider the 2-norm defined by Konca and Idris in [6] for this special case of $X$.

In this paper, we propose two new 2-norms formed on $\ell^p(\mathbb{R})$ based on different choices of bounded linear functionals on it and investigate their relationships between Gähler's 2-norm [1] and Gunawan's 2-norm [2]. We see that one of these 2-norms is equivalent to the natural 2-norm on this space, while the other one is not equivalent and there is no any equivalence relation among the other known 2-norms on $\ell^p(\mathbb{R})$.

Now, we give the known definitions and preliminaries used in the paper. We use standard notation and terminology as in [9].

Let $X$ be a normed space and $f : X \to \mathbb{R}$ be a bounded linear functional, then $|f(x)| \leq \|f\|\|x\|$. The set of all bounded linear functionals on $X$ called the dual space of $X$ is denoted by $X'$. Now observe that for every $x \in X - \{\theta\}$, $f \in X'$ we obtain $f(x) \leq \|f\|\|x\|$ or $\frac{f(x)}{\|f\|} \leq \|x\|$. So, we conclude that $\sup_{f\neq 0, f\in X'} \frac{f(x)}{\|f\|} = \sup_{\|f\|\leq 1} f(x) \leq \|x\|$.

Let $X$ be a real vector space of dimension $\dim(X) \geq 2$ and $\|.,.\|$ be a real function on $X \times X$ satisfying the following four conditions. The function $\|.,.\|$ is called a 2-norm on $X$ and the pair $(X, \|.,.\|)$ is called a 2-normed space [1].

(1) $\|x, y\| \geq 0$ for every $x, y \in X$; $\|x, y\| = 0$ if and only if $x$ and $y$ are linearly dependent;

(2) $\|x, y\| = \|y, x\|$ for every $x, y \in X$;

(3) $\|\alpha x, y\| = |\alpha|\|x, y\|$ for every $x, y \in X$ and for every $\alpha \in \mathbb{R}$;

(4) $\|x + z, y\| \leq \|x, y\| + \|z, y\|$ for every $x, y, z \in X$.

If $X$ is a normed space, then according to Gähler, the following formula defines a 2-norm on $X$ [1] to compute the area of the parallelogram spanned by two vectors $x$ and $y$ in $X$

$$\|x, y\|^G = \frac{1}{2} \sup_{\substack{f_i\in X',\ \|f_i\|\leq 1 \\ i=1,2}} \begin{vmatrix} f_1(x) & f_2(x) \\ f_1(y) & f_2(y) \end{vmatrix}. \tag{1.1}$$

From the definition of bounded linear functional and equation (1.1) we have $\|x, y\|^G \leq \|x\|\,\|y\| < \infty$.

Recall that a sequence $(x(n))$ in a 2-normed space $(X, \|\cdot,\cdot\|)$ is called a convergent sequence, if there is an $x \in X$ such that $\|x(n) - x, z\| \longrightarrow 0$, as $n \longrightarrow \infty$ for every $z \in X$. Also, $(x(n))$ is said to be Cauchy sequence with respect to the $\|\cdot,\cdot\|$ if $\|x(m) - x(n), y\| \longrightarrow 0$, as $m, n \longrightarrow \infty$ for every $z \in X$ [7]. A linear 2-normed space in which every Cauchy sequence is convergent is called a 2-Banach space [11].

## 2 Main results

In this section, we present our new results.

### 2.1 Bounded linear functionals on $\ell^p$

In this subsection, let us remind the known bounded linear functionals $f_w$ and $\overline{h_\alpha}$ (see, in [9]) given respectively in Case 1 and Case 2 before the construction of new 2-norms on $\ell^p$. Later, in the second subsection, respectively, the sets $F$ and $H$ of bounded linear functionals $f_w$ and $\overline{h_\alpha}$, will be used to redefine Gähler's 2-norm and new 2-norm which has been recently defined by Konca and Idris [6] on a normed space $X$. Here we examine the Gähler's 2-norm and new 2-norm defined in [6] for $X = \ell^p$. Depending on the chosen of bounded linear functionals in both cases we obtained different results. For example in Case 1; from the set $F$, we can define a norm equivalent to the usual norm, while in Case 2; from the set $H$, we can define a norm that is not equivalent to the usual norm. What is the impact if these sets are used to define 2-norms and the norms induced from these 2-norms on $\ell^p$? Next, we will discuss on these two cases. We examine the equivalence relations among these new defined 2-norms with the other already known 2-norms on $\ell^p$. Below we will describe their influence on obtaining the different results and examine equivalence relations.

Case 1:

Let $p \geq 1$ be a fixed real number. By definition, each element in the space $\ell^p$ is a sequence $x = (x_k) = (x_1, x_2, ..., x_k, ...)$ of numbers such that $\sum_{k=1}^{\infty} |x_k|^p < \infty$ $(p \geq 1)$ and the usual norm on $\ell^p$ is defined by $\|x\|_{\ell^p} := (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}}$ for every $x \in \ell^p$. Take $w \in \ell^q$ with $\|w\|_{\ell^q} > 0$ and $\frac{1}{q} = 1 - \frac{1}{p}$. Then for every $x \in \ell^p$, the functional $f_w : \ell^p \to \mathbb{R}$

$$f_w(x) := \sum\nolimits_{k=1}^{\infty} w_k x_k \tag{2.1}$$

is a linear functional and we have the following by Hölder's inequality for every $x \in \ell^p$

$$f_w(x) \leq |f_w(x)| \leq \|w\|_{\ell^q} \|x\|_{\ell^p}. \tag{2.2}$$

So, we conclude that $f_w$ is a bounded linear functional. Now, collect all bounded linear functionals $f_w$ in $F := \{f_w | w \in \ell^q\} = \ell^q$. Here $\ell^q$ denotes the dual of $\ell^p$, which consists of bounded linear functionals on $\ell^p$. See in [9] that $\|f_w\|_F = \|w\|_{\ell^q}$ and $(F, \|\cdot\|_F)$ is a normed space.

Here, we also define

$$\|x\|_* := \sup_{\|f_w\|_F \neq 0} \frac{|f_w(x)|}{\|f_w\|_F}$$

for every $x \in \ell^p$. It is easy to show that $\|\cdot\|_*$ is a norm. By the following proposition we will show that $\|\cdot\|_* = \|\cdot\|_{\ell^p}$.

**Proposition 2.1.** On $\ell^p$, $\|\cdot\|_{\ell^p}$ and $\|\cdot\|_*$ are equivalent.

*Proof.* Firstly, see that for every $x \in \ell^p$,

$$\|x\|_* = \sup_{\|f_w\|_F \neq 0} \frac{|f_w(x)|}{\|f_w\|_F} \leq \sup_{\|f_w\|_F \neq 0} \frac{\|w\|_{\ell^q} \|x\|_{\ell^p}}{\|f_w\|_F}.$$

Since $\|f_w\|_F = \|w\|_{\ell^q}$, then $\|x\|_* \leq \|x\|_{\ell^p}$ holds.

Next, choose $f_w = (w_i)_{i=1}^{\infty}$ where $w_i := |x_i|^{p-1} sign(x_i)$. Now, we can show that

$$\frac{|f_w(x)|}{\|f_w\|_F} = \frac{\sum_{i=1}^{\infty} |x_i|^p}{\|x\|_{\ell^p}^{p-1}} \leq \sup_{\|f_w\|_F \neq 0} \frac{|f_w(x)|}{\|f_w\|_F} = \|x\|_*.$$

Check that $\frac{\sum_{i=1}^{\infty}|x_i|^p}{\|x\|_{\ell^p}^{p-1}} = \|x\|_{\ell^p}$, so we get $\|x\|_{\ell^p} \leq \|x\|_*$. Then $\|x\|_{\ell^p} = \|x\|_*$. Hence we obtain that the norms $\|.\|_{\ell^p}$ and $\|\cdot\|_*$ are equivalent. Q.E.D.

Case 2:

Let $I := [2,3]$. Here we give $H := \left\{\overline{h_\alpha} : \alpha \in I\right\}$. Now, define functional $\overline{h_\alpha} : \ell^p \to \mathbb{R}$

$$\overline{h_\alpha}(x) := \sum_{k=1}^{\infty} \frac{x_k}{k^\alpha} \tag{2.3}$$

for every $x \in \ell^p$. One may show easily that (2.3) is a linear functional. By Hölder's inequality, for every $x \in \ell^p$

$$\overline{h_\alpha}(x) \leq |\overline{h_\alpha}(x)| \leq \left(\sum_{k=1}^{\infty} \frac{1}{k^{q\alpha}}\right)^{\frac{1}{q}} \|x\|_{\ell^p}. \tag{2.4}$$

Obviously, $\sum_{k=1}^{\infty} \frac{1}{k^{q\alpha}} < \infty$. So, we conclude that $\overline{h_\alpha}$ is a bounded linear functional. We also define

$$\|\overline{h_\alpha}\|_H := \sup_{\|x\|_{\ell^p}>0} \frac{|\overline{h_\alpha}(x)|}{\|x\|_{\ell^p}},$$

for every $\overline{h_\alpha} \in H$.

It is true that for each $\overline{h_\alpha} \in H \Rightarrow \overline{h_\alpha} \in \ell^p$ is provided. For each $\overline{h_\alpha} \in H$, $\sum_{k=1}^{\infty} \frac{1}{k^{p\alpha}} < \infty$ can be easily obtained for $p \geq 1$. Hence, obviously $\overline{h_\alpha} \in \ell^p$.

Similarly, in Case 2 we define

$$\|x\|_{\#} := \sup_{\|\overline{h_\alpha}\|_H \neq 0} \frac{|\overline{h_\alpha}(x)|}{\|\overline{h_\alpha}\|_H}$$

for every $x \in \ell^p$. One may show that $\|\cdot\|_{\#}$ is a norm.

**Proposition 2.2.** On $\ell^p$, $\|.\|_{\ell^p}$ and $\|\cdot\|_{\#}$ aren't equivalent.

*Proof.* First step, for every $x \in \ell^p$, we have

$$\begin{aligned}\|x\|_{\#} &= \sup_{\|\overline{h_\alpha}\|_H>0} \frac{|\overline{h_\alpha}(x)|}{\|\overline{h_\alpha}\|_H} = \sup_{\|\overline{h_\alpha}\|_H>0} \frac{1}{\|\overline{h_\alpha}\|_H} \left|\sum_{k=1} \frac{x_k}{k^\alpha}\right| \\ &\leq \sup_{\|\overline{h_\alpha}\|_H>0} \frac{1}{\|\overline{h_\alpha}\|_H} \left(\sum_{k=1}^{\infty} \frac{1}{k^{q\alpha}}\right)^{\frac{1}{q}} \|x\|_{\ell^p} = A\|x\|_{\ell^p},\end{aligned}$$

where

$$\sup_{\|\overline{h_\alpha}\|_H>0} \frac{1}{\|\overline{h_\alpha}\|_H} \left(\sum_{k=1}^{\infty} \frac{1}{k^{q\alpha}}\right)^{\frac{1}{q}} = A.$$

Second step, inspired by [8], here we assume that there is a $B > 0$ such that $B\|x\|_{\ell^p} \leq \|x\|_{\#}$ or

$$B\|x\|_{\ell^p} \leq \sup_{\|\overline{h_\alpha}\|_H>0} \frac{|\overline{h_\alpha}(x)|}{\|\overline{h_\alpha}\|_H} = \sup_{\|\overline{h_\alpha}\|_H>0} \frac{1}{\|\overline{h_\alpha}\|_H} \left|\sum_{k=1} \frac{x_k}{k^\alpha}\right|,$$

for every $x \in \ell^p$. Now take $y(m) \in \ell^p$ with $y_k(m) = 1$ for $k = m$ and $y_k(m) = 0$ for $k \neq m$. Then, we have $\|y(m)\|_{\ell^p} = 1$ and

$$\sup_{\|\overline{h_\alpha}\|_H>0} \frac{1}{\|\overline{h_\alpha}\|_H} \left|\sum_{k=1} \frac{y_k(m)}{k^\alpha}\right| \leq \frac{C}{m^\alpha}$$

with $C > B$. Consequently, $\lim_{m\to\infty} \|y(m)\|_{\ell^p} = 1$, $\lim_{m\to\infty} \frac{C}{m^\alpha} = 0$ and

$$B = \lim_{m\to\infty} B\|y(m)\|_{\ell^p} \leq \lim_{m\to\infty} \sup_{\|\overline{h_\alpha}\|_H>0} \frac{|\overline{h_\alpha}(y(m))|}{\|\overline{h_\alpha}\|_H} \leq \lim_{m\to\infty} \frac{C}{m^\alpha} = 0.$$

This is a contradiction, since $B > 0$. So, we conclude that $\|\cdot\|_{\ell^p}$ and $\|\cdot\|_{\#}$ aren't equivalent. Q.E.D.

The results of the two propositions above strengthen the hypothesis that the choice of the sets of the bounded linear functionals will provide different properties of some 2-norms defined in the next section.

### 2.2 New 2-norms on $\ell^p$ with the set of functional $F$

On $\ell^p$, we use the set of all bounded linear functionals $F = \ell^q$ to show Gähler's definition of 2-norm

$$\begin{aligned}
\|x, y\|^G_{\odot\ell^p} &:= \tfrac{1}{2} \sup_{\substack{0<\|f_v\|_F,\, \|f_w\|_F\leq 1 \\ f_v, f_w \in\, \ell^{p\prime}=F=\ell^q}} \begin{vmatrix} f_v(x) & f_v(y) \\ f_w(x) & f_w(y) \end{vmatrix} \\
&= \tfrac{1}{2} \sup_{\substack{0<\|v\|_{\ell^q},\, \|w\|_{\ell^q}\leq 1 \\ v,w\in\ell^q}} \begin{vmatrix} \sum\limits_{i=1}^{\infty} x_i v_i & \sum\limits_{i=1}^{\infty} y_i v_i \\ \sum\limits_{i=1}^{\infty} x_i w_i & \sum\limits_{i=1}^{\infty} y_i w_i \end{vmatrix} \\
&= \tfrac{1}{4} \sup_{\substack{0<\|v\|_{\ell^q},\, \|w\|_{\ell^q}\leq 1 \\ v,w\in\ell^q}} \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix} \begin{vmatrix} v_i & w_i \\ v_j & w_j \end{vmatrix}.
\end{aligned} \tag{2.5}$$

Meanwhile using $F = \ell^q$, we define the following 2-norm on $\ell^p$

$$\begin{aligned}
\|x, y\|^{KI}_{\odot\ell^p} &:= \sup_{0<\|f_v\|_F\leq 1} \|x f_v(y) - y f_v(x)\|_{\ell^p} \\
&= \sup_{0<\|v\|_{\ell^q}\leq 1} \left\| x \sum_{i=1}^{\infty} y_i v_i - y \sum_{i=1}^{\infty} x_i v_i \right\|_{\ell^p} \\
&= \sup_{0<\|v\|_{\ell^q}\leq 1} \left( \sum_{j=1}^{\infty} \left| \sum_{i=1}^{\infty} (x_j y_i - x_i y_j) v_i \right|^p \right)^{\frac{1}{p}},
\end{aligned} \tag{2.6}$$

for every $x, y \in \ell^p$.

**Lemma 2.3.** For every $x \in \ell^p$, $w \in \ell^q$, $\sup_{0<\|w\|_{\ell^q}\leq 1} \sum_{i=1}^{\infty} x_i w_i = \|x\|_{\ell^p}$.

*Proof.* As it is known that $F = \ell^q$ and from the lemma $\|w\|_{\ell^q} \leq 1$, then $\sum_{i=1}^{\infty} x_i w_i \leq \|x\|_{\ell^p}\|w\|_{\ell^q} \leq \|x\|_{\ell^p}$ holds by Hölder's inequality. Next, choose $w \in \ell^q$ such that $w_i = \frac{|x_i|^{p-1} sign(x_i)}{\|x\|_{\ell^p}^{p-1}}$ with $i \in \mathbb{N}$. Check that $\|w\|_{\ell^q} = 1$. Then calculate the following

$$\sum_{i=1}^{\infty} x_i w_i = \sum_{i=1}^{\infty} x_i \frac{|x_i|^{p-1} sign(x_i)}{\|x\|_{\ell^p}^{p-1}} = \frac{\|x\|_{\ell^p}^{p}}{\|x\|_{\ell^p}^{p-1}} = \|x\|_{\ell^p}.$$

With the above formula, supremum can be attained. Hence $\sup_{0<\|w\|_{\ell^q}\leq 1} \sum_{i=1}^{\infty} x_i w_i = \|x\|_{\ell^p}$ for every $x \in \ell^p$. Q.E.D.

The lemma above makes it easy for us to show the relationship between the two 2-norms contained in the following proposition.

**Proposition 2.4.** For every $x, y \in \ell^p$, we have $2\|x, y\|_{\odot\ell^p}^{G} = \|x, y\|_{\odot\ell^p}^{KI}$.

*Proof.* We can create the following equality.

$$\begin{vmatrix} \sum_{i=1}^{\infty} x_i v_i & \sum_{i=1}^{\infty} y_i v_i \\ \sum_{i=1}^{\infty} x_i w_i & \sum_{i=1}^{\infty} y_i w_i \end{vmatrix} = \sum_{j=1}^{\infty} \left( x_j \sum_{i=1}^{\infty} y_i v_i - y_j \sum_{i=1}^{\infty} x_i v_i \right) w_j = \sum_{j=1}^{\infty} z_j w_j, \tag{2.7}$$

where $z_j = x_j \sum_{i=1}^{\infty} y_i v_i - y_j \sum_{i=1}^{\infty} x_i v_i$ with $j \in \mathbb{N}$. By the following equation, we obtain $z \in \ell^p$

$$\|z\|_{\ell^p} \leq \|x\|_{\ell^p} \left| \sum_{i=1}^{\infty} y_i v_i \right| + \|y\|_{\ell^p} \left| \sum_{i=1}^{\infty} x_i v_i \right| \leq 2\|x\|_{\ell^p}\|y\|_{\ell^p} < \infty.$$

Now, according Lemma 2.3, choose $w \in \ell^q$ such that $w_j = \frac{|z_j|^{p-1} sign(z_j)}{\|z\|_{\ell^p}^{p-1}}$, with $j \in \mathbb{N}$. Thus

$$\sum_{j=1}^{\infty} \left( x_j \sum_{i=1}^{\infty} y_i v_i - y_j \sum_{i=1}^{\infty} x_i v_i \right) w_j = \sum_{j=1}^{\infty} z_j w_j = \|z\|_{\ell^p} = \left\| x \sum_{i=1}^{\infty} y_i v_i - y \sum_{i=1}^{\infty} x_i v_i \right\|_{\ell^p}.$$

By taking the supremum, we have $2\|x, y\|_{\odot\ell^p}^{G} = \|x, y\|_{\odot\ell^p}^{KI}$. Q.E.D.

The above proposition states that $\|\cdot,\cdot\|_{\odot\ell^p}^{KI}$ and $\|\cdot,\cdot\|_{\odot\ell^p}^{G}$ are equivalent. Now recall Gunawan's definition

$$\|x, y\|_{\ell^p}^{H} := \left( \frac{1}{2} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \left( abs \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix} \right)^p \right)^{\frac{1}{p}}, \tag{2.8}$$

for every $x, y \in \ell^p$ and we know that the equations (2.5) and (2.8) are equivalent (see in [12]).

**Corollary 2.5.** Two spaces $(\ell^p, \|\cdot,\cdot\|_{\odot\ell^p}^{G})$ and $(\ell^p, \|\cdot,\cdot\|_{\odot\ell^p}^{KI})$ are 2-Banach spaces.

*Proof.* In [2, 4], $(\ell^p, \|\cdot,\cdot\|^H_{\ell^p})$ is 2-Banach space. Since $\|\cdot,\cdot\|^{KI}_{\odot\ell^p}$, $\|\cdot,\cdot\|^{G}_{\odot\ell^p}$, and $\|\cdot,\cdot\|^{H}_{\ell^p}$ are equivalent, then $(\ell^p, \|\cdot,\cdot\|^{G}_{\odot\ell^p})$ and $(\ell^p, \|\cdot,\cdot\|^{KI}_{\odot\ell^p})$ are 2-Banach spaces. Q.E.D.

Now, take fixed linear independent vectors $a, b \in \ell^p$. For every $x \in \ell^p$, we define the following two new norms induced by 2-norms.

$$\|x\|^G_{\odot\ell^p} := \|x, a\|^G_{\odot\ell^p} + \|x, b\|^G_{\odot\ell^p} \tag{2.9}$$

$$\|x\|^{KI}_{\odot\ell^p} := \|x, a\|^{KI}_{\odot\ell^p} + \|x, b\|^{KI}_{\odot\ell^p}. \tag{2.10}$$

**Proposition 2.6.** The spaces $(\ell^p, \|\cdot\|^G_{\odot\ell^p})$ and $(\ell^p, \|\cdot\|^{KI}_{\odot\ell^p})$ are Banach spaces.

*Proof.* Let $(x(n))$ be a Cauchy sequence in $(\ell^p, \|\cdot\|^G_{\odot\ell^p})$. It means that

$$\|x(m) - x(n)\|^G_{\odot\ell^p} = \|x(m) - x(n), a\|^G_{\odot\ell^p} + \|x(m) - x(n), b\|^G_{\odot\ell^p} \to 0,$$

while $m, n \to \infty$. Consequently $\|x(m) - x(n), a\|^G_{\odot\ell^p} \to 0$ for every $a \in \ell^p$, while $m, n \to \infty$, so $(x(n))$ is a Cauchy sequence with respect to $\|\cdot,\cdot\|^G_{\odot\ell^p}$. By Corollary 2.5, since $(\ell^p, \|\cdot,\cdot\|^G_{\odot\ell^p})$ is 2-Banach space, then $(x(n))$ is a convergent sequence with respect to $\|\cdot,\cdot\|^G_{\odot\ell^p}$. Choose again linearly independent vectors $a, b \in \ell^p$ and use 2.9 to show convergence of $(x(n))$ with respect to $\|\cdot\|^G_{\odot\ell^p}$. Hence $(\ell^p, \|\cdot\|^G_{\odot\ell^p})$ is a Banach space. With similar steps in the above, it is easy to show that $(\ell^p, \|\cdot\|^{KI}_{\odot\ell^p})$ is a Banach space. Q.E.D.

Let $a, b \in \ell^p$ be linearly independent vectors. By [4], we recall definition of norm

$$\|x\|^H_{\circ\ell^p} := \left( \left(\|x, a\|^H_{\ell^p}\right)^p + \left(\|x, b\|^H_{\ell^p}\right)^p \right)^{\frac{1}{p}} \tag{2.11}$$

for every $x \in \ell^p$. The norm $\|\cdot\|^H_{\circ\ell^p}$ is equivalent to usual norm on $\ell^p$.

**Corollary 2.7.** The usual norm on $\ell^p$ is equivalent to $\|\cdot\|^G_{\odot\ell^p}$ and $\|\cdot\|^{KI}_{\odot\ell^p}$.

*Proof.* In [4], equivalence of $\|\cdot\|^H_{\circ\ell^p}$ and usual norm have been proved. Since $\|\cdot,\cdot\|^{KI}_{\odot\ell^p}$, $\|\cdot,\cdot\|^{G}_{\odot\ell^p}$, and $\|\cdot,\cdot\|^{H}_{\ell^p}$ are equivalent, then from the equations 2.9, 2.10, and 2.11 it is easy to see that $\|\cdot\|^G_{\odot\ell^p}$ and $\|\cdot\|^{KI}_{\odot\ell^p}$ are also equivalent with respect to the usual norm on $\ell^p$. Q.E.D.

### 2.3 New 2-norms on $\ell^p$ with the set of functional $H$

In this subsection, the set of bounded linear functionals $H$ will be used to define 2-norm. Gähler's definition of 2-norm will be

$$\begin{aligned}\|x, y\|^G_{\boxdot\ell^p} :=& \frac{1}{2} \sup_{\|\overline{h_{\alpha_1}}\|_H>0, \|\overline{h_{\alpha_2}}\|_H>0} \frac{\begin{vmatrix} \overline{h_{\alpha_1}}(x) & \overline{h_{\alpha_1}}(y) \\ \overline{h_{\alpha_2}}(x) & \overline{h_{\alpha_2}}(y) \end{vmatrix}}{\|\overline{h_{\alpha_1}}\|_H \|\overline{h_{\alpha_2}}\|_H} \\ =& \frac{1}{4} \sup_{\|\overline{h_{\alpha_1}}\|_H>0, \|\overline{h_{\alpha_2}}\|_H>0} \frac{\sum\limits_{j=1}^{\infty}\sum\limits_{i=1}^{\infty} \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix} \begin{vmatrix} \frac{1}{i^{\alpha_1}} & \frac{1}{i^{\alpha_2}} \\ \frac{1}{j^{\alpha_1}} & \frac{1}{j^{\alpha_2}} \end{vmatrix}}{\|\overline{h_{\alpha_1}}\|_H \|\overline{h_{\alpha_2}}\|_H} \end{aligned} \tag{2.12}$$

for every $x, y \in \ell^p$. We use $H$ to define

$$\begin{aligned}\|x,y\|^{KI}_{\square\ell^p} &:= \sup_{\|\overline{h_\alpha}\|_H>0} \frac{\left\|x\overline{h_\alpha}(y) - y\overline{h_\alpha}(x)\right\|_{\ell^p}}{\|\overline{h_\alpha}\|_H} \\ &= \sup_{\|\overline{h_\alpha}\|_H>0} \frac{\left\|x\sum\limits_{i=1}^{\infty}\frac{y_i}{i^\alpha} - y\sum\limits_{i=1}^{\infty}\frac{x_i}{i^\alpha}\right\|_{\ell^p}}{\|\overline{h_\alpha}\|_H} \\ &= \sup_{\|\overline{h_\alpha}\|_H>0} \frac{\left(\sum\limits_{j=1}^{\infty}\left|\sum\limits_{i=1}^{\infty}\frac{(x_jy_i - x_iy_j)}{i^\alpha}\right|^p\right)^{\frac{1}{p}}}{\|\overline{h_\alpha}\|_H}, \end{aligned} \tag{2.13}$$

for every $x, y \in \ell^p$.

The following two propositions explain the relation between $\|\cdot,\cdot\|^{G}_{\square\ell^p}$ and $\|\cdot,\cdot\|^{KI}_{\square\ell^p}$.

**Proposition 2.8.** For every $x, y \in \ell^p$, we have $\|x,y\|^{G}_{\square\ell^p} \le C\|x,y\|^{KI}_{\square\ell^p}$ with $C > 0$.

*Proof.* Take arbitrary $x, y \in \ell^p$ and check that

$$\begin{vmatrix}\overline{h_{\alpha_1}}(x) & \overline{h_{\alpha_1}}(y)\\ \overline{h_{\alpha_2}}(x) & \overline{h_{\alpha_2}}(y)\end{vmatrix} = \overline{h_{\alpha_2}}(y\overline{h_{\alpha_1}}(x) - x\overline{h_{\alpha_1}}(y)) = \sum_{j=1}^{\infty}\frac{y_j\overline{h_{\alpha_1}}(x) - x_j\overline{h_{\alpha_1}}(y)}{j^{\alpha_2}}.$$

We use Hölder's inequality to obtain

$$\left|\sum_{j=1}^{\infty}\frac{y_j\overline{h_{\alpha_1}}(x) - x_j\overline{h_{\alpha_1}}(y)}{j^{\alpha_2}}\right| \le \left(\sum_{j=1}^{\infty}\left|y_j\overline{h_{\alpha_1}}(x) - x_j\overline{h_{\alpha_1}}(y)\right|^p\right)^{\frac{1}{p}}\left(\sum_{j=1}^{\infty}\frac{1}{j^{q\alpha_2}}\right)^{\frac{1}{q}}.$$

Then we have

$$\begin{aligned}\|x,y\|^{G}_{\square\ell^p} =& \frac{1}{2}\sup_{\|\overline{h_{\alpha_1}}\|_H,\|\overline{h_{\alpha_2}}\|_H>0}\frac{\begin{vmatrix}\overline{h_{\alpha_1}}(x) & \overline{h_{\alpha_1}}(y)\\ \overline{h_{\alpha_2}}(x) & \overline{h_{\alpha_2}}(y)\end{vmatrix}}{\|\overline{h_{\alpha_1}}\|_H\|\overline{h_{\alpha_2}}\|_H} \\ =& \frac{1}{2}\sup_{\|\overline{h_{\alpha_1}}\|_H,\|\overline{h_{\alpha_2}}\|_H>0}\frac{\sum\limits_{j=1}^{\infty}\frac{y_j\overline{h_{\alpha_1}}(x) - x_j\overline{h_{\alpha_1}}(y)}{j^{\alpha_2}}}{\|\overline{h_{\alpha_1}}\|_H\|\overline{h_{\alpha_2}}\|_H} \\ \le& \frac{1}{2}\sup_{\|\overline{h_{\alpha_1}}\|_H,\|\overline{h_{\alpha_2}}\|_H>0}\frac{\left(\sum\limits_{j=1}^{\infty}\left|y_j\overline{h_{\alpha_1}}(x) - x_j\overline{h_{\alpha_1}}(y)\right|^p\right)^{\frac{1}{p}}}{\|\overline{h_{\alpha_1}}\|_H}\frac{\left(\sum\limits_{j=1}^{\infty}\frac{1}{j^{q\alpha_2}}\right)^{\frac{1}{q}}}{\|\overline{h_{\alpha_2}}\|_H} \\ \le& C\|x,y\|^{KI}_{\square\ell^p},\end{aligned}$$

where $C = \frac{1}{2}\sup_{\|\overline{h_{\alpha_2}}\|_H>0}\frac{\left(\sum\limits_{j=1}^{\infty}\frac{1}{j^{q\alpha_2}}\right)^{\frac{1}{q}}}{\|\overline{h_{\alpha_2}}\|_H}$. This completes the proof. Q.E.D.

**Proposition 2.9.** Two 2-norms $\|\cdot,\cdot\|^G_{\square\ell^p}$ and $\|\cdot,\cdot\|^{KI}_{\square\ell^p}$ aren't equivalent.

*Proof.* Assume that $\|\cdot,\cdot\|^G_{\square\ell^p}$ and $\|\cdot,\cdot\|^{KI}_{\square\ell^p}$ are equivalent, so there is a $B>0$ such that

$$B\|x,y\|^{KI}_{\square\ell^p}\leq\|x,y\|^G_{\square\ell^p}$$

holds for every $x,y\in\ell^p$. Now choose $y(m)\in\ell^p$ with $y_k(m)=1$ for $k=m$ and $y_k(m)=0$ for $k\neq m$. We have

$$\|y(m),y(m+1)\|^{KI}_{\square\ell^p}=\sup_{\|\overline{h_\alpha}\|_H>0}\frac{\left(\frac{1}{m^{\alpha p}}+\frac{1}{(m+1)^{\alpha p}}\right)^{\frac{1}{p}}}{\|\overline{h_\alpha}\|_H}\geq C_1\frac{1}{(m+1)^3}$$

and

$$\|y(m),y(m+1)\|^G_{\square\ell^p}=\frac{1}{2}\sup_{\|\overline{h_{\alpha_1}}\|_H,\|\overline{h_{\alpha_2}}\|_H>0}\frac{\left(\frac{1}{m^{\alpha_1}(m+1)^{\alpha_2}}-\frac{1}{m^{\alpha_2}(m+1)^{\alpha_1}}\right)}{\|\overline{h_{\alpha_1}}\|_H\|\overline{h_{\alpha_2}}\|_H}\leq C_2\frac{1}{m^4}.$$

So, by assumption $C_3\frac{1}{(m+1)^3}\leq C_4\frac{1}{m^4}$ holds for every $m\in\mathbb{N}$ with $C_3,C_4>0$. Obviously, $\frac{1}{(m+1)^3}$ and $\frac{1}{m^4}$ go to zero while $m\to\infty$, but

$$0<\frac{C_3}{C_4}\leq\frac{\frac{1}{m^4}}{\frac{1}{(m+1)^3}}=\frac{1}{m}\left(1+\frac{1}{m}\right)^3.$$

Consequently, $0<\frac{C_3}{C_4}\leq\lim_{m\to\infty}\frac{1}{m}\left(1+\frac{1}{m}\right)^3=0$ which is a contradiction. Thus we conclude that $\|\cdot,\cdot\|^G_{\square\ell^p}$ and $\|\cdot,\cdot\|^{KI}_{\square\ell^p}$ aren't equivalent. Q.E.D.

Next, we wonder the relation between $\|\cdot,\cdot\|^{KI}_{\square\ell^p}$ and $\|\cdot,\cdot\|^H_{\ell^p}$.

**Proposition 2.10.** For every $x,y\in\ell^p$, we have $\|x,y\|^{KI}_{\square\ell^p}\leq C\|x,y\|^H_{\ell^p}$ with $C>0$.

*Proof.* Take an arbitrary $x,y\in\ell^p$. Using Hölder's inequality, we have

$$\begin{aligned}\left(\sum_{j=1}^\infty\left|y_j\overline{h_\alpha}(x)-x_j\overline{h_\alpha}(y)\right|^p\right)^{\frac{1}{p}}&=\left(\sum_{j=1}^\infty\left|y_j\sum_{i=1}^\infty\frac{x_i}{i^\alpha}-x_j\sum_{i=1}^\infty\frac{y_i}{i^\alpha}\right|^p\right)^{\frac{1}{p}}\\&=\left(\sum_{j=1}^\infty\left|\sum_{i=1}^\infty\frac{(x_iy_j-x_jy_i)}{i^\alpha}\right|^p\right)^{\frac{1}{p}}\\&\leq\left(\sum_{j=1}^\infty\left(\sum_{i=1}^\infty|x_iy_j-x_jy_i|^p\right)\left(\sum_{i=1}^\infty\frac{1}{i^{q\alpha}}\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}\\&=\|x,y\|^H_{\ell^p}\left(\sum_{i=1}^\infty\frac{1}{i^{q\alpha}}\right)^{\frac{1}{q}}.\end{aligned}$$

Consequently,

$$\begin{aligned}\|x,y\|^{KI}_{\boxdot\ell^p} =& \sup_{\|\overline{h_\alpha}\|_H>0} \frac{\left(\sum\limits_{j=1}^{\infty} \left|y_j\overline{h_{\alpha_1}}(x) - x_j\overline{h_\alpha}(y)\right|^p\right)^{\frac{1}{p}}}{\|\overline{h_\alpha}\|_H} \\ \leq& C|x,y\|^H_{\ell^p},\end{aligned}$$

where $C = \sup_{\|\overline{h_\alpha}\|_H>0} \frac{\left(\sum\limits_{j=1}^{\infty} \frac{1}{j^{q\alpha}}\right)^{\frac{1}{q}}}{\overline{h_\alpha}\|_H}$. Q.E.D.

**Proposition 2.11.** $\|\cdot,\cdot\|^{KI}_{\boxdot\ell^p}$ and $\|\cdot,\cdot\|^H_{\ell^p}$ aren't equivalent.

*Proof.* Suppose that $\|\cdot,\cdot\|^{KI}_{\boxdot\ell^p}$ and $\|\cdot,\cdot\|^H_{\ell^p}$ are equivalent. So there is a $B>0$ such that

$$B\|x,y\|^H_{\ell^p} \leq \|x,y\|^{KI}_{\boxdot\ell^p}$$

holds for every $x,y\in\ell^p$. Choose $y(m)\in\ell^p$ with $y_k(m)=1$ for $k=m$ and $y_k(m)=0$ for $k\neq m$. We compute that $\|y(m),y(m+1)\|^H_{\ell^p}=1$ and

$$\|y(m),y(m+1)\|^{KI}_{\boxdot\ell^p} = \sup_{\|\overline{h_\alpha}\|_H>0} \frac{\left(\frac{1}{m^{\alpha p}} + \frac{1}{(m+1)^{\alpha p}}\right)^{\frac{1}{p}}}{\|\overline{h_\alpha}\|_H} \leq C_1\frac{1}{m^3}$$

with $C_1 = \sup_{\|\overline{h_\alpha}\|_H>0} \frac{2^{\frac{1}{p}}}{\|\overline{h_\alpha}\|_H}$. By assumption we obtain an inequality $0<B\leq C_1\frac{1}{m^3}$ for every $m\in\mathbb{N}$. For $m\to\infty$, we have $0<B\leq \lim\limits_{m\to\infty}\frac{1}{m^3}=0$ which is a contradiction. Hence, $\|\cdot,\cdot\|^G_{\boxdot\ell^p}$ and $\|\cdot,\cdot\|^{KI}_{\boxdot\ell^p}$ must not be equivalent. Q.E.D.

By four previous propositions, we show that

$$C_1\|x,y\|^G_{\boxdot\ell^p} \leq \|x,y\|^{KI}_{\boxdot\ell^p} \leq C_2\|x,y\|^H_{\ell^p}$$

for every $x,y\in\ell^p$ with $C_1,C_2>0$ and they aren't equivalent. Now, induction by 2-norm, we define two new norm with following formula. Take fixed linear independent vectors $a,b\in\ell^p$. For every $x\in\ell^p$, two new norms are defined by,

$$\|x\|^G_{\boxdot\ell^p} := \|x,a\|^G_{\boxdot\ell^p} + \|x,b\|^G_{\boxdot\ell^p} \tag{2.14}$$

$$\|x\|^{KI}_{\boxdot\ell^p} := \|x,a\|^{KI}_{\boxdot\ell^p} + \|x,b\|^{KI}_{\boxdot\ell^p} \tag{2.15}$$

What is the relation among $\|\cdot\|_{\ell^p}$ and $\|.\|^G_{\boxdot\ell^p}$, $\|.\|^{KI}_{\boxdot\ell^p}$ given by the equations (2.14) ((2.15), respectively? We will investigate them. We need some lemmas to compare them.

**Lemma 2.12.** There is a $C > 0$, such that

$$\|x\|^{KI}_{\boxdot\ell^p} \leq C\|x\|_{\ell^p},$$

holds for every $x \in \ell^p$.

*Proof.* Now take an arbitrary $x \in \ell^p$. Using triangle inequality

$$\|x, a\|^{KI}_{\boxdot\ell^p} = \sup_{\|\overline{h_\alpha}\|_H>0} \frac{\left\|x\overline{h_\alpha}(a) - a\overline{h_\alpha}(x)\right\|}{\|\overline{h_\alpha}\|_H} \leq \sup_{\|\overline{h_\alpha}\|_H>0} \frac{(\|x\|_{\ell^p}\,|\overline{h_\alpha}(a)| + \|a\|_{\ell^p}\,|\overline{h_\alpha}(x)|)}{\|\overline{h_\alpha}\|_H}.$$

Since $|\overline{h_\alpha}(x)| \leq \left(\sum\limits_{k=1}^{\infty} \frac{1}{k^{q\alpha}}\right)^{\frac{1}{q}} \|x\|_{\ell^p}$, then

$$\|x, a\|^{KI}_{\boxdot\ell^p} \leq C_1\,\|x\|_{\ell^p}$$

with $C_1 = \sup_{\|\overline{h_\alpha}\|_H>0} \frac{|\overline{h_\alpha}(a)|+\|a\|_{\ell^p}\left(\sum\limits_{k=1}^{\infty} \frac{1}{k^{q\alpha}}\right)^{\frac{1}{q}}}{\|\overline{h_\alpha}\|_H}$. Replace $a$ with $b$, so

$$\|x, b\|^{KI}_{\boxdot\ell^p} \leq C_2\,\|x\|_{\ell^p}$$

with $C_2 = \sup_{\|\overline{h_\alpha}\|_H>0} \frac{|\overline{h_\alpha}(b)|+\|b\|_{\ell^p}\left(\sum\limits_{k=1}^{\infty} \frac{1}{k^{q\alpha}}\right)^{\frac{1}{q}}}{\|\overline{h_\alpha}\|_H}$. Combine two inequalities above, then we have

$$\|x\|^{KI}_{\boxdot\ell^p} \leq C\|x\|_{\ell^p},$$

for every $x \in X$ where $C = C_1 + C_2 > 0$. Q.E.D.

**Lemma 2.13.** There is a $C > 0$, such that

$$C\|x\|_{\ell^p} \leq \|x\|^{KI}_{\boxdot\ell^p},$$

holds for every $x \in \ell^p$.

*Proof.* In $\ell^p$, let $a, b \in \ell^p$ be linearly independent vectors. Now, choose $\beta \in I$ such that $\overline{h_\beta}(a) = \sum\limits_{k=1} \frac{a_k}{k^\beta} \neq 0$ and $\overline{h_\beta}(b) = \sum\limits_{k=1} \frac{b_k}{k^\beta} \neq 0$. Next, take an arbitrary $x \in \ell^p$ and observe that

$$\begin{aligned}
\|x\|_{\ell^p}|\overline{h_\beta}(a)| =& \|x\overline{h_\beta}(a) - a\overline{h_\beta}(x) + a\overline{h_\beta}(x)\|_{\ell^p} \\
\leq& \|x\overline{h_\beta}(a) - a\overline{h_\beta}(x)\|_{\ell^p} + \|a\|_{\ell^p}|\overline{h_\beta}(x)| \\
\leq& C_0\|x, a\|^{KI}_{\boxdot\ell^p} + \|a\|_{\ell^p}|\overline{h_\beta}(x)|, \qquad (2.16)
\end{aligned}$$

with $C_0 = \|\overline{h_\beta}\|_H$. Choose $\gamma, \lambda \in I$ such that $\overline{h_\gamma}(a)\overline{h_\lambda}(b) - \overline{h_\gamma}(b)\overline{h_\lambda}(a) \neq 0$. Then

$$\begin{aligned}\left(\overline{h_\gamma}(a)\overline{h_\lambda}(b) - \overline{h_\gamma}(b)\overline{h_\lambda}(a)\right)\overline{h_\beta}(x) =&\overline{h_\beta}(x)\overline{h_\gamma}(a)\overline{h_\lambda}(b) - \overline{h_\beta}(x)\overline{h_\gamma}(b)\overline{h_\lambda}(a)\\ =&\overline{h_\beta}(x)\overline{h_\gamma}(a)\overline{h_\lambda}(b) - \overline{h_\beta}(x)\overline{h_\gamma}(b)\overline{h_\lambda}(a)\\ &+ (\overline{h_\gamma}(x)\overline{h_\beta}(a)\overline{h_\lambda}(b) - \overline{h_\gamma}(x)\overline{h_\beta}(a)\overline{h_\lambda}(b))\\ &+ (\overline{h_\lambda}(x)\overline{h_\beta}(a)\overline{h_\gamma}(b) - \overline{h_\lambda}(x)\overline{h_\beta}(a)\overline{h_\gamma}(b))\\ =&(\overline{h_\beta}(x)\overline{h_\gamma}(a) - \overline{h_\gamma}(x)\overline{h_\beta}(a))\overline{h_\lambda}(b)\\ &+ (\overline{h_\lambda}(x)\overline{h_\beta}(a) - \overline{h_\beta}(x)\overline{h_\lambda}(a))\overline{h_\gamma}(b)\\ &+ (\overline{h_\gamma}(x)\overline{h_\lambda}(b) - \overline{h_\gamma}(b)\overline{h_\lambda}(x))\overline{h_\beta}(a).\end{aligned}$$

We obtain the following by triangle inequality,

$$\begin{aligned}\left|\overline{h_\gamma}(a)\overline{h_\lambda}(b) - \overline{h_\gamma}(b)\overline{h_\lambda}(a)\right|\,|\overline{h_\beta}(x)| =&\left|\overline{h_\beta}(x)\overline{h_\gamma}(a)\overline{h_\lambda}(b) - \overline{h_\beta}(x)\overline{h_\gamma}(b)\overline{h_\lambda}(a)\right|\\ \leq&\left|\overline{h_\beta}(x)\overline{h_\gamma}(a) - \overline{h_\gamma}(x)\overline{h_\beta}(a)\right|\,|\overline{h_\lambda}(b)|\\ &+\left|\overline{h_\lambda}(x)\overline{h_\beta}(a) - \overline{h_\beta}(x)\overline{h_\lambda}(a)\right|\,|\overline{h_\gamma}(b)|\\ &+\left|\overline{h_\gamma}(x)\overline{h_\lambda}(b) - \overline{h_\gamma}(b)\overline{h_\lambda}(x)\right|\,|\overline{h_\beta}(a)|.\end{aligned} \tag{2.17}$$

By Hölder's inequality we have

$$\begin{aligned}\left|\overline{h_\beta}(x)\overline{h_\gamma}(a) - \overline{h_\gamma}(x)\overline{h_\beta}(a)\right| =&\left|\sum_{j=1}^{\infty}\frac{x_j\overline{h_\gamma}(a) - a_j\overline{h_\gamma}(x)}{j^\beta}\right|\\ \leq&\left(\sum_{j=1}^{\infty}|x_j\overline{h_\gamma}(a) - a_j\overline{h_\gamma}(x)|^p\right)^{\frac{1}{p}}\left(\sum_{j=1}^{\infty}\frac{1}{j^{q\beta}}\right)^{\frac{1}{q}}\\ =&\frac{\left(\sum\limits_{j=1}^{\infty}|x_j\overline{h_\gamma}(a) - a_j\overline{h_\gamma}(x)|^p\right)^{\frac{1}{p}}}{\|\overline{h_\gamma}\|_H}\|\overline{h_\gamma}\|_H\left(\sum_{j=1}^{\infty}\frac{1}{j^{q\beta}}\right)^{\frac{1}{q}}\\ \leq&\|x,a\|_{\boxdot\ell^p}^{KI}\|\overline{h_\gamma}\|_H\left(\sum_{j=1}^{\infty}\frac{1}{j^{q\beta}}\right)^{\frac{1}{q}}.\end{aligned}$$

With similar process, we also obtain $\left|\overline{h_\lambda}(x)\overline{h_\beta}(a) - \overline{h_\beta}(x)\overline{h_\lambda}(a)\right|\,| \leq \|x,a\|_{\boxdot\ell^p}^{KI}\|\overline{h_\beta}\|_H\left(\sum\limits_{j=1}^{\infty}\frac{1}{j^{q\lambda}}\right)^{\frac{1}{q}}$ and $\left|\overline{h_\gamma}(x)\overline{h_\lambda}(b) - \overline{h_\gamma}(b)\overline{h_\lambda}(x)\right|\,| \leq \|x,b\|_{\boxdot\ell^p}^{KI}\|\overline{h_\lambda}\|_H\left(\sum\limits_{j=1}^{\infty}\frac{1}{j^{q\gamma}}\right)^{\frac{1}{q}}$. The inequality 2.17 will be

$$\left|\overline{h_\gamma}(a)\overline{h_\lambda}(b) - \overline{h_\gamma}(b)\overline{h_\lambda}(a)\right|\,|\overline{h_\beta}(x)| \leq C_1\|x,a\|_{\boxdot\ell^p}^{KI} + C_2\|x,b\|_{\boxdot\ell^p}^{KI}$$

where $C_1 = |\overline{h_\lambda}(b)|\|\overline{h_\gamma}\|_H \left(\sum_{j=1}^{\infty} \frac{1}{j^{q\beta}}\right)^{\frac{1}{q}} + |\overline{h_\gamma}(b)|\|\overline{h_\beta}\|_H \left(\sum_{j=1}^{\infty} \frac{1}{j^{q\lambda}}\right)^{\frac{1}{q}}$ and $C_2 = |\overline{h_\beta}(a)|\|\overline{h_\lambda}\|_H \left(\sum_{j=1}^{\infty} \frac{1}{j^{q\gamma}}\right)^{\frac{1}{q}}$.
Now, check that

$$\begin{aligned}\|a\|_{\ell^p}|\overline{h_\beta}(x)| =& \frac{\|a\|_{\ell^p}}{\left|\overline{h_\gamma}(a)\overline{h_\lambda}(b) - \overline{h_\gamma}(b)\overline{h_\lambda}(a)\right|} \left|\overline{h_\gamma}(a)\overline{h_\lambda}(b) - \overline{h_\gamma}(b)\overline{h_\lambda}(a)\right| |\overline{h_\beta}(x)|| \\ \leq& \frac{\|a\|_{\ell^p}}{\left|\overline{h_\gamma}(a)\overline{h_\lambda}(b) - \overline{h_\gamma}(b)\overline{h_\lambda}(a)\right|} \left(C_1\|x,a\|_{\boxdot\ell^p}^{KI} + C_2\|x,b\|_{\boxdot\ell^p}^{KI}\right). \end{aligned} \tag{2.18}$$

By (2.16) and (2.18)

$$\begin{aligned}\|x\|_{\ell^p}|\overline{h_\beta}(a)| \leq& C_0\|x,a\|_{\boxdot\ell^p}^{KI} + \|a\|_{\ell^p}|\overline{h_\beta}(x)| \\ \leq& C_0\|x,a\|_{\boxdot\ell^p}^{KI} + \frac{\|a\|_{\ell^p}}{\left|\overline{h_\gamma}(a)\overline{h_\lambda}(b) - \overline{h_\gamma}(b)\overline{h_\lambda}(a)\right|} \left(C_1\|x,a\|_{\boxdot\ell^p}^{KI} + C_2\|x,b\|_{\boxdot\ell^p}^{KI}\right) \\ \leq& C_3\left(\|x,a\|_{\boxdot\ell^p}^{KI} + \|x,b\|_{\boxdot\ell^p}^{KI}\right) = C_3\|x\|_{\boxdot\ell^p}^{KI}, \end{aligned} \tag{2.19}$$

with $C_3 = \max\{C_0 + \frac{C_1\|a\|_{\ell^p}}{\left|\overline{h_\gamma}(a)\overline{h_\lambda}(b) - \overline{h_\gamma}(b)\overline{h_\lambda}(a)\right|}, \frac{C_2\|a\|_{\ell^p}}{\left|\overline{h_\gamma}(a)\overline{h_\lambda}(b) - \overline{h_\gamma}(b)\overline{h_\lambda}(a)\right|}\}$. Finally, $C\|x\|_{\ell^p} \leq \|x\|_{\boxdot\ell^p}^{KI}$ for every $x \in X$ with $C = \frac{|\overline{h_\beta}(a)|}{C_3}$. This completes the proof. Q.E.D.

**Corollary 2.14.** The derived norm $\|\cdot\|_{\boxdot\ell^p}^{KI}$ is equivalent to the usual norm $\|\cdot\|_{\ell^p}$ on $\ell^p$.

*Proof.* Combining both of Lemma 2.12 and Lemma 2.13, we obtain the equivalence of the norms $\|\cdot\|_{\ell^p}$ and $\|\cdot\|_{\boxdot\ell^p}^{KI}$. Q.E.D.

**Theorem 2.15.** In $\ell^p$, a sequence $(x(n))$ converges to $x$ with respect to $\|\cdot\|_{\ell^p}$ if and only if it converges to $x$ with respect to $\|\cdot,\cdot\|_{\boxdot\ell^p}^{KI}$. Similarly, a sequence $(x(n))$ is a Cauchy sequence with respect to $\|\cdot\|_{\ell^p}$ if and only if it is a Cauchy sequence with respect to $\|\cdot,\cdot\|_{\boxdot\ell^p}^{KI}$.

*Proof.* ($\Rightarrow$) Let $(x(n))$ be a sequence converges to $x$ with respect to $\|\cdot\|_{\ell^p}$. By inequality in the proof of Lemma 2.12, for every $y \in \ell^p$, we have $0 \leq \|x(n) - x, y\|_{\boxdot\ell^p}^{KI} \leq C_1\|x(n) - x\|_{\ell^p} \longrightarrow 0$, as $n \longrightarrow \infty$. Thus $(x(n))$ converges to $x$ with respect to $\|\cdot,\cdot\|_{\boxdot\ell^p}^{KI}$.

($\Leftarrow$) Suppose that $(x(n))$ is a sequence converges to $x$ with respect to $\|\cdot,\cdot\|_{\boxdot\ell^p}^{KI}$. So, for every $y \in \ell^p$, we have $\|x(n) - x, y\|_{\boxdot\ell^p}^{KI} \longrightarrow 0$, as $n \longrightarrow \infty$. Now, take arbitrary linearly independent vectors $a, b \in \ell^p$. By Lemma 2.13, we have

$$0 \leq C\|x(n) - x\|_{\ell^p} \leq \|x(n) - x\|_{\boxdot\ell^p}^{KI} = \|x(n) - x, a\|_{\boxdot\ell^p}^{KI} + \|x(n) - x, b\|_{\boxdot\ell^p}^{KI} \longrightarrow 0,$$

as $n \longrightarrow \infty$. Thus $x(n)$ converges to $x$ with respect to $\|\cdot\|_{\ell^p}$.

The second part of the theorem can be proved in a similar way: one only needs to replace the expressions "convergent to $x$" with "Cauchy" and "$x(n) - x$" with "$x(n) - x(m)$" and $n \longrightarrow \infty$ with $m, n \longrightarrow \infty$. Q.E.D.

**Theorem 2.16.** The space $(\ell^p, \|\cdot,\cdot\|_{\boxdot\ell^p}^{KI})$ is a 2-Banach space.

*Proof.* Let $(x(n))$ be a Cauchy sequence in $\ell^p$ with respect to $\|\cdot,\cdot\|^{KI}_{\boxdot\ell^p}$. Then by Theorem 2.15, $(x(n))$ is Cauchy sequence with respect to the usual norm $\|\cdot\|_{\ell^p}$. We know that $(\ell^p, \|\cdot\|_{\ell^p})$ is a Banach space: $\ell^p$ is complete with respect to the usual norm $\|\cdot\|_{\ell^p}$. Then $x(n)$ converges to some $x \in \ell^p$ in $\|\cdot\|_{\ell^p}$. By Theorem 2.15, $x(n)$ also converges to $x$ with respect to $\|\cdot,\cdot\|^{KI}_{\boxdot\ell^p}$. This shows that $X$ is complete with respect to the 2-norm $\|\cdot,\cdot\|^{KI}_{\boxdot\ell^p}$, that is $(\ell^p, \|\cdot,\cdot\|^{KI}_{\boxdot\ell^p})$ is a 2-Banach space. Q.E.D.

**Theorem 2.17.** $\|\cdot\|^{G}_{\boxdot\ell^p}$ and $\|\cdot\|_{\ell^p}$ are not equivalent.

*Proof.* To show that $\|\cdot\|^{G}_{\boxdot\ell^p}$ is not equivalent to $\|\cdot\|_{\ell^p}$; assume that there is a $B > 0$ such that

$$B\|x\|_{\ell^p} \leq \|x\|^{G}_{\boxdot\ell^p},$$

for every $x \in \ell^p$. Choose $y(m) \in \ell^p$ with $y_k(m) = 1$ for $k = m$ and $y_k(m) = 0$ for $k \neq m$. We obtain that $\|y(m)\|_{\ell^p} = 1$, so $\lim_{m\to\infty} \|y(m)\|_{\ell^p} = 1$. On the other hand,

$$\|y(m)\|^{G}_{\boxdot\ell^p} = \|y(m), a\|^{G}_{\boxdot\ell^p} + \|y(m), b\|^{G}_{\boxdot\ell^p} \leq \frac{2(|a_m| + |b_m|)}{m}.$$

Thus $B = \lim_{m\to\infty} B\|y(m)\|_{\ell^p} \leq \lim_{m\to\infty} \|y(m)\|^{G}_{\boxdot\ell^p} \leq \lim_{m\to\infty} \frac{2(|a_m|+|b_m|)}{m} = 0$. Since $B > 0$, this is a contradiction. Q.E.D.

## 3 Concluding remarks

In the definitions of $\|\cdot,\cdot\|^{KI}_{\odot\ell^p}$ and $\|\cdot,\cdot\|^{KI}_{\boxdot\ell^p}$ we use different functionals which aren't equivalent.

**Corollary 3.1.** Two 2-norms $\|\cdot,\cdot\|^{KI}_{\odot\ell^p}$ and $\|\cdot,\cdot\|^{KI}_{\boxdot\ell^p}$ aren't equivalent.

*Proof.* By Proposition 2.4, we have $2\|\cdot,\cdot\|^{G}_{\odot\ell^p} = \|\cdot,\cdot\|^{KI}_{\odot\ell^p}$. One may see in [12] that $\|\cdot,\cdot\|^{G}_{\odot\ell^p}$ is equivalent to $\|\cdot,\cdot\|^{H}_{\ell^p}$. Automatically, $\|\cdot,\cdot\|^{KI}_{\odot\ell^p}$ is equivalent to $\|\cdot,\cdot\|^{H}_{\ell^p}$. Meanwhile Proposition 2.11 states that $\|\cdot,\cdot\|^{KI}_{\boxdot\ell^p}$ is not equivalent to $\|\cdot,\cdot\|^{H}_{\ell^p}$. Hence we conclude that $\|\cdot,\cdot\|^{KI}_{\odot\ell^p}$ and $\|\cdot,\cdot\|^{KI}_{\boxdot\ell^p}$ aren't equivalent. Q.E.D.

However, we have different result which is indeed very interesting for the relation of $\|\cdot\|^{KI}_{\odot\ell^p}$ and $\|\cdot\|^{KI}_{\boxdot\ell^p}$.

**Corollary 3.2.** Two functional $\|\cdot\|^{KI}_{\odot\ell^p}$ and $\|\cdot\|^{KI}_{\boxdot\ell^p}$ are equivalent.

*Proof.* By the Corollary 2.7, $\|\cdot\|^{KI}_{\odot\ell^p}$ is equivalent to the usual norm and Corollary 2.14 states that $\|\cdot\|^{KI}_{\boxdot\ell^p}$ is equivalent to the usual norm. Hence, $\|\cdot\|^{KI}_{\odot\ell^p}$ and $\|\cdot\|^{KI}_{\boxdot\ell^p}$ are equivalent. Q.E.D.

This research may be continued for other spaces. One may interest to investigate some bounded linear functionals in $X = \mathbb{L}^p(\mathbb{R}^n)$. This is still an open problem for the readers.

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