Spectral expansion formula for a singular Sturm-Liouville problem

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Abstract

In the space $L_2(0,\infty)$ we investigate Sturm-Liouville operator with a nonlinear spectral parameter in the boundary condition. For the problem the scattering data is defined, the resolvent operator is constructed, and using Titchmarsh's method the two-fold expansion formulas in terms of scattering data are obtained.

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1 Introduction

We consider the boundary value problem for the differential equation

$$l(y) \equiv -y'' + V(x)y = \rho^2 y, \tag{1.1}$$

on the half line $[0,\infty)$ with the boundary condition nonlinear dependence on the spectral parameter

$$(\alpha_0 + \alpha_2\rho^2)y'(0) - (\beta_0 + \beta_1\rho + \beta_2\rho^2)y(0) = 0. \tag{1.2}$$

Here ρ is a spectral parameter, the function $V(x)$ is a real valued function and $(1+x)|V(x)| \in L_1(0,\infty)$, α_j, β_j $(j = 0,1,2)$ are real numbers.

The Sturm-Liouville equations with a nonlinear spectral parameter in the boundary condition arise in various problems of mathematical physics. Spectral analysis for the boundary value problem when the spectral parameter appearing in the boundary condition was examined in [2, 3]. In the special case of the boundary value problem (1.1), (1.2) is given in [1] with application to heat transmission problem.

In this paper, we study the spectral analysis of the boundary value problem (1.1), (1.2). Using Jost solution of the equation (1.1) and Titchmarsh's method [7, 18] we constructed the resolvent operator, and two-fold expansion formulas with respect to eigenfunctions in terms of scattering data are obtained. For the classical Sturm-Liouville operator, a similar problem has been studied completely (see [7, 10, 14, 18] and the references therein). The physical application of the expansions problem for the differential equation on the half line $[0,\infty)$ with the boundary condition depending on the spectral parameter is considered in T. Regge's works [15, 16]. Regge problem on the half line was studied in [4, 5, 6].

Expansion formulas according to the eigenfunctions for boundary value problems generated by various boundary conditions for the equation (1.1) are obtained in [11, 12, 13, 17] and other works.

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We note that some aspects of the scattering theory for the boundary value problem (1.1), (1.2) were studied in [12]. In this case, the boundary value problem (1.1), (1.2) is not selfadjoint and it may have complex eigenvalue (see [8] and [14]). For this reason, the scattering data of the problem (1.1), (1.2) is defined differently.

Denote by D_ρ the set consisting of the functions $y(x) \in L_2(0,\infty)$ that satisfies the following conditions:

1) the functions $y(x)$ and $y'(x)$ are absolutely continuous in $[a,b]$ (for every fixed $b > 0$);

2) $l(y) \in L_2(0,\infty)$;

3) for every fixed ρ satisfies the boundary condition (1.2).

Let L_ρ be an operator with domain $D_\rho = D(L_\rho)$ such that for $y \in D_\rho$ satisfies $L_\rho y = l(y)$. If ρ runs through the set of all points of ρ - plane, then we obtain a family of nonselfadjoint singular operators L_k depending on the parameter ρ (see [9]).

Denote by $f(x,\rho)$ the solution of the equation (1.1) possessing the asymptotics $f(x,\rho) \to e^{i\rho x}$ as $x \to \infty$ and for any ρ, $\operatorname{Im}\rho \geqslant 0$ the solution $f(x,\rho)$ can be represented in the form

$$f(x,\rho) = e^{i\rho x} + \int_x^\infty L(x,t)e^{i\rho t}dt. \tag{1.3}$$

The kernel $L(x,t)$ satisfies the inequality

$$|L(x,t)| \leqslant \frac{1}{2}\sigma\left(\frac{x+t}{2}\right)\exp\left\{\sigma_1(x) - \sigma_1\left(\frac{x+t}{2}\right)\right\},$$

where

$$\sigma(x) \equiv \int_x^\infty |V(t)|dt, \quad \sigma_1(x) \equiv \int_x^\infty \sigma(t)dt.$$

The solution $f(x,\rho)$ is an analytic function of ρ in the upper half plane and is continuous on the real line. The following estimates hold through the half plane $\operatorname{Im}\rho \geqslant 0$:

$$|f(x,\rho)| \leqslant \exp\{-\operatorname{Im}\,\rho x + \sigma_1(x)\},$$

$$|f(x,\rho) - e^{i\rho x}| \leqslant \left\{\sigma_1(x) - \sigma_1\left(x + \frac{1}{|\rho|}\right)\right\}\exp\{-\operatorname{Im}\,\rho x + \sigma_1(x)\},$$

$$|f'(x,\rho) - i\rho e^{i\rho x}| \leqslant \sigma(x)\exp\{-\operatorname{Im}\,\rho x + \sigma_1(x)\}.$$

For real $\rho \neq 0$ the functions $f(x,\rho)$ and $f(x,-\rho)$ form a fundamental system of solutions of the equation (1.1) and their Wronskian is

$$W\{f(x,\rho), f(x,-\rho)\} = f'(x,\rho)f(x,-\rho) - f(x,\rho)f'(x,-\rho) = 2i\rho.$$

Similarly, we denote by $\hat{f}(x,\rho)$ the solution of the equation (1.1) in half plane $\operatorname{Im}\rho \geqslant 0$ possessing the asymptotics:

$$\hat{f}(x,\rho) = e^{-i\rho x}(1 + \overline{o}(1)), \quad \hat{f}'(x,\rho) = e^{-i\rho x}(i\rho + \overline{o}(1)), \quad \text{as} \quad x \to \infty;$$

$$\hat{f}(x,\rho) = e^{-i\rho x}\left(1 + O\left(\frac{1}{|\rho|}\right)\right), \quad \hat{f}'(x,\rho) = -i\rho e^{-i\rho x}\left(1 + O\left(\frac{1}{|\rho|}\right)\right), \quad \text{as} \quad |\rho| \to \infty.$$

The existence and properties of these solutions are studied in [14] (p. 299).

Let $\varphi(x,\rho)$ be the solution of the equation (1.1) satisfying the following initial-value conditions:

$$\varphi(0,\rho) = \alpha_0 + \alpha_2\rho^2, \quad \varphi'(0,\rho) = \beta_0 + \beta_1\rho + \beta_2\rho^2.$$

Analogously to the Lemma 1 in [12] is just like that

$$\frac{2i\rho\varphi(x,\rho)}{F(\rho)} = f(x,-\rho) - S(\rho)f(x,\rho), \tag{1.4}$$

for any real number $\rho \neq 0$, the identity is valid

$$S(\rho) = \frac{(\alpha_0 + \alpha_2\rho^2)f'(0,-\rho) - (\beta_0 + \beta_1\rho + \beta_2\rho^2)f(0,-\rho)}{(\alpha_0 + \alpha_2\rho^2)f'(0,\rho) - (\beta_0 + \beta_1\rho + \beta_2\rho^2)f(0,\rho)}.$$

Denote

$$F_1(\rho) \equiv (\alpha_0 + \alpha_2\rho^2)f'(0,-\rho) - (\beta_0 + \beta_1\rho + \beta_2\rho^2)f(0,-\rho),$$
$$F(\rho) \equiv (\alpha_0 + \alpha_2\rho^2)f'(0,\rho) - (\beta_0 + \beta_1\rho + \beta_2\rho^2)f(0,\rho).$$

We shall call the function $S(\rho)$ scattering function for the boundary value problem (1.1), (1.2). The scattering function $S(\rho)$ determines the asymptotics for $x \to \infty$ of normalized eigenfunctions of the operators L_ρ.

For the solution $\hat{f}(x,\rho)$ we have

$$\frac{2i\rho\varphi(x,\rho)}{F(\rho)} = \hat{f}(x,\rho) - \frac{\hat{F}(\rho)}{F(\rho)}f(x,\rho),$$

where

$$\hat{F}(\rho) = (\alpha_0 + \alpha_2\rho^2)\hat{f}'(0,\rho) - (\beta_0 + \beta_1\rho + \beta_2\rho^2)\hat{f}(0,\rho).$$

The functions $F(\rho)$ and $F_1(\rho)$ are analytics in the upper half plane and the asymptotic formulas

$$F(\rho) = i\rho^3\left[\alpha_2 + O\left(\frac{1}{\rho}\right)\right], \quad F_1(\rho) = -i\rho^3\left[\alpha_2 + O\left(\frac{1}{\rho}\right)\right], \quad \text{holds as} \quad |\rho| \to \infty.$$

The roots of the equation $F(\rho) = 0$ in the half plane $\operatorname{Im}\rho > 0$ form a finite set of complex numbers and nonpure imaginary [11]. The multiplicity m_j of a root ρ_j $(j = 1, 2, \ldots, n)$ of the equation $F(\rho) = 0$ is called the multiplicity of the singular value ρ_j.

We have

$$S(\rho) = -1 + O\left(\frac{1}{\rho}\right), \quad \text{as} \quad |\rho| \to \infty.$$

Denote

$$p_j(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\hat{F}(\rho)}{F(\rho)} - 1\right]e^{i\rho x}d\rho = i\operatorname*{Res}_{\rho=\rho_j}\frac{\hat{F}(\rho)}{F(\rho)}e^{i\rho x},$$

where $j = 1, 2, \ldots, n$. We shall call the polynomial

$$P_j(x) = e^{-i\rho_j x} p_j(x), \quad j = 1, 2, \ldots, n.$$

The functions $p_j(x)$, $j = 1, 2, \ldots, n$ are the characteristic of the operator L_ρ on the discrete spectrum.

The set of values $\{S(\rho),\ \rho_j,\ P_j(x)\}$ $(j = 1, 2, \ldots, n)$ is called the scattering data of the boundary value problem (1.1), (1.2). The set provides a complete description of the behavior of infinite of all eigenfunctions of the problem (1.1), (1.2).

2 The construction of resolvent operator

Using the result introduced above it is possible to construct the resolvent operator.

Assume that is not a spectrum point of the operator L_ρ. Then there exists a resolvent operator $R_\rho = (L - \rho I)^{-1}$. Let's find the expression of the operator R_ρ.

Theorem 2.1. All these numbers ρ, Im $\rho \geqslant 0$, $F(\rho) \neq 0$ belong to the resolvent set of operators R_ρ. The resolvent R_ρ is the integral operator

$$R_\rho g = \int_0^\infty K(x, t, \rho) g(t) dt, \tag{2.1}$$

with the kernel

$$K(x, t, \rho) = -\frac{1}{F(\rho)} \begin{cases} f(x, \rho)\varphi(t, \rho), & 0 \leqslant t \leqslant x, \\ \varphi(x, \rho) f(t, \rho), & x \leqslant t < \infty. \end{cases} \tag{2.2}$$

Moreover

$$|K(x, t, \rho)| \leqslant \frac{C(x)}{|F(\rho)|} \exp\{\text{Im } \rho |x - t|\}, \tag{2.3}$$

where

$$C(x) = C \exp\{x\sigma(0) + \sigma_1(0)\}.$$

$C > 0$ is an arbitrary constant.

Proof. Let $g(x) \in D_\rho$ is a finite function at infinity. To construct the resolvent operator, we aim to solve the following boundary value problem:

$$-y'' + V(x)y = \rho^2 y + g(x),$$

$$(\alpha_0 + \alpha_2\rho^2)y'(0) - (\beta_0 + \beta_1\rho + \beta_2\rho^2)y(0) = 0.$$

If we apply Lagrange's method by using properties solutions of the equation (1.1) we find (2.1), where $K(x, t, \rho)$ has the form (2.2). Using the estimates for the functions $f(x, \rho)$ and $\varphi(x, \rho)$, we have the inequality (2.3) for the kernel $K(x, t, \rho)$. Theorem 2.1 is proved. Q.E.D.

Lemma 2.2. Let the function $g(x)$ be continuously differentiable two times and finite. Then the following equality is valid:

$$\int_0^\infty K(x, t, \rho) g(t) dt = -\frac{g(x)}{\rho^2} + \frac{1}{\rho^2} K_1(x, \rho), \tag{2.4}$$

here

$$K_1(x,\rho)=\int\limits_0^\infty K(x,t,\rho)\{-g''(t)+V(t)g(t)\}dt.$$

Proof. The below equality is valid:

$$-K''(x,t,\rho)+V(x)K(x,t,\rho)-\rho^2K(x,t,\rho)=\delta(x-t),$$

$\delta(x)$ is Dirac-delta function. Let's multiply both sides of this equation by $g(x)$ and integrate from 0 to ∞:

$$-\int\limits_0^\infty K''(x,t,\rho)g(t)dt+\int\limits_0^\infty V(t)K(x,t,\rho)g(t)dt-\rho^2\int\limits_0^\infty K(x,t,\rho)g(t)dt=\int\limits_0^\infty \delta(x-t)g(t)dt.$$

From this equation (2.4) obtained. Lemma is proved. Q.E.D.

3 The expansion formula by eigenfunctions

Let us Γ_R denotes the circle of radius R and center at the origin, which contour is positive oriented. Let us $\Gamma_{R,\varepsilon}^{(1)}$ denotes contour be half arc of Γ_R that does not include points z satisfying the condition Im $z>\varepsilon$. Let $\Gamma_{R,\varepsilon}^{(2)}$ be a half arc that does not include Im $z<-\varepsilon$ points of Γ_R and $\Gamma_{R,\varepsilon}=\Gamma_{R,\varepsilon}^{(1)}\bigcup\Gamma_{R,\varepsilon}^{(2)}$ is positive oriented. Let $\Gamma_{R,\varepsilon}^{(3)}$ be negative oriented curve formed with Im $z=\pm\varepsilon$ lines and be arcs including points z satisfying the conditions $|\text{Im } z|<\varepsilon$. According to this, $\Gamma_{R,\varepsilon}=\Gamma_R\bigcup\Gamma_{R,\varepsilon}^{(3)}$. Then we can use the property of the integration

$$\int\limits_{\Gamma_{R,\varepsilon}}=\int\limits_{\Gamma_R}+\int\limits_{\Gamma_{R,\varepsilon}^{(3)}}. \tag{3.1}$$

Let us call

$$\Phi(x,\rho)=\int\limits_0^\infty K(x,t,\rho)g(t)dt.$$

Then from (2.4)

$$\Phi(x,\rho)=-\frac{g(x)}{\rho^2}+\frac{1}{\rho^2}\int\limits_0^\infty K(x,t,\rho)\tilde{g}(t)dt, \tag{3.2}$$

where

$$\tilde{g}(t)=-g''(t)+V(t)g(t).$$

By multiplying both sides of the equality (3.2) by $\frac{1}{2\pi i}\rho$ and integrating the contour $\Gamma_{R,\varepsilon}$, we obtain

$$\frac{1}{2\pi i}\int\limits_{\Gamma_{R,\varepsilon}}\rho\Phi(x,\rho)d\rho=-\frac{1}{2\pi i}\int\limits_{\Gamma_{R,\varepsilon}}\frac{g(x)}{\rho}d\rho+\frac{1}{2\pi i}\int\limits_{\Gamma_{R,\varepsilon}}\frac{1}{\rho}\left\{\int\limits_0^\infty K(x,t,\rho)\tilde{g}(t)dt\right\}d\rho. \tag{3.3}$$

Using the residue calculus we get

$$\frac{1}{2\pi i}\int\limits_{\Gamma_{R,\varepsilon}} \rho\Phi(x,\rho)d\rho = \sum_{D} \text{Res } [\rho\Phi(x,\rho)] + \sum_{\overline{D}} \text{Res } [\rho\Phi(x,\rho)], \tag{3.4}$$

$$\frac{1}{2\pi i}\int\limits_{\Gamma_{R,\varepsilon}} \rho\Phi(x,\rho)d\rho = \frac{1}{2\pi i}\int\limits_{\Gamma_R} \rho\Phi(x,\rho)d\rho + \frac{1}{2\pi i}\int\limits_{\Gamma^{(3)}_{R,\varepsilon}} \rho\Phi(x,\rho)d\rho. \tag{3.5}$$

Using (3.2) calculate the integrals on the right-hand side of (3.5)

$$\frac{1}{2\pi i}\int\limits_{\Gamma_R} \rho\Phi(x,\rho)d\rho = -\frac{1}{2\pi i}\int\limits_{\Gamma_R} \frac{g(x)}{\rho}d\rho + \frac{1}{2\pi i}\int\limits_{\Gamma_R} \frac{1}{\rho}\left\{\int\limits_0^\infty K(x,t,\rho)\tilde{g}(t)dt\right\}d\rho =$$

$$= -g(x) + \frac{1}{2\pi i}\int\limits_{\Gamma_R} O\left(\frac{1}{\rho_2}\right)d\rho = -g(x), \tag{3.6}$$

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \frac{1}{2\pi i}\int\limits_{\Gamma^{(3)}_{R,\varepsilon}} \rho\Phi(x,\rho)d\rho = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \rho[\Phi(x,\rho+i0) - \Phi(x,\rho-i0)]d\rho. \tag{3.7}$$

Substituting (3.7) and (3.6) in (3.5), we have

$$\frac{1}{2\pi i}\int\limits_{\Gamma_{R,\varepsilon}} \rho\Phi(x,\rho)d\rho = -g(x) + \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \rho[\Phi(x,\rho+i0) - \Phi(x,\rho-i0)]d\rho.$$

Implying this equality in (3.4), we get

$$g(x) = -\sum_{D} \text{Res } [\rho\Phi(x,\rho)] + \sum_{\overline{D}} \text{Res } [\rho\Phi(x,\rho)] + \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \rho[\Phi(x,\rho+i0) - \Phi(x,\rho-i0)]d\rho. \tag{3.8}$$

Now multiplying both sides of equality (3.2) by $\frac{1}{2\pi i}$ and integrating over ρ the contour $\Gamma_{R,\varepsilon}$ we obtain

$$\frac{1}{2\pi i}\int\limits_{\Gamma_{R,\varepsilon}} \Phi(x,\rho)d\rho = -\frac{1}{2\pi i}\int\limits_{\Gamma_{R,\varepsilon}} \frac{g(x)}{\rho^2}d\rho + \frac{1}{2\pi i}\int\limits_{\Gamma_{R,\varepsilon}} \left\{\int\limits_0^\infty \frac{1}{\rho^2}K(x,t,\rho)\tilde{g}(t)dt\right\}d\rho.$$

By similar calculations, we obtain

$$-\sum_{D} \text{Res } \Phi(x,\rho) - \sum_{\overline{D}} \text{Res } \Phi(x,\rho) + \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} [\Phi(x,\rho+i0) - \Phi(x,\rho-i0)]d\rho = 0.$$

Thus, the following two-fold spectral expansion we have

$$g(x) = -\sum_{D} \operatorname{Res}\,[\rho\Phi(x,\rho)] - \sum_{\overline{D}} \operatorname{Res}\,[\rho\Phi(x,\rho)] + \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \rho[\Phi(x,\rho+i0) - \Phi(x,\rho-i0)]d\rho, \quad (3.9)$$

$$0 = -\sum_{D} \operatorname{Res}\,\Phi(x,\rho) - \sum_{\overline{D}} \operatorname{Res}\,\Phi(x,\rho) + \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} [\Phi(x,\rho+i0) - \Phi(x,\rho-i0)]d\rho. \quad (3.10)$$

Now, let's convert formulas (3.9) and (3.10).

Let $\psi(0,\rho)$ be the solution of the equation (1.1) satisfying the initial conditions

$$\psi(0,\rho) = \alpha_2, \quad \psi'(0,\rho) = \beta_2.$$

It is clear that

$$W\{\varphi(x,\rho),\psi(x,\rho)\} = \varphi'(0,\rho)\psi(0,\rho) - \varphi(0,\rho)\psi'(0,\rho) =$$
$$= (\beta_0 + \beta_1\rho + \beta_2\rho^2)\alpha_2 - (\alpha_0 + \alpha_2\rho^2)\beta_2 =$$
$$= \alpha_2\beta_0 - \alpha_0\beta_2 + \alpha_2\beta_1\rho = \sigma_2 + \alpha_2\beta_1\rho = \sigma_3 \neq 0$$

Then for $\operatorname{Im}\rho > 0$ we have

$$f(x,\rho) = c_1\varphi(x,\rho) + c_2\psi(x,\rho),$$

where

$$c_2 = \frac{F(\rho)}{\sigma_3}, \quad c_1 = \frac{\beta_2 f(0,\rho) - \alpha_2 f'(0,\rho)}{\sigma_3}.$$

and

$$f(x,\rho) = \frac{\beta_2 f(0,\rho) - \alpha_2 f'(0,\rho)}{\sigma_3}\varphi(x,\rho) + \frac{F(\rho)}{\sigma_3}\psi(x,\rho).$$

Then according to (2.2)

$$K(x,t,\rho) = \frac{\beta_2 f(0,\rho) - \alpha_2 f'(0,\rho)}{\sigma_3 F(\rho)}\varphi(x,\rho)\varphi(t,\rho) + \begin{cases}\varphi(t,\rho)\psi(x,\rho), & x \leqslant t,\\ \varphi(x,\rho)\psi(t,\rho), & t \leqslant x.\end{cases}$$

Therefore, for $\operatorname{Im}\rho > 0$ we obtain:

$$\Phi(x,\rho) = \frac{\beta_2 f(0,\rho) - \alpha_2 f'(0,\rho)}{\sigma_3 F(\rho)}\varphi(x,\rho)\int\limits_0^{\infty}\varphi(t,\rho)g(t)dt +$$
$$+ \frac{\psi(x,\rho)}{\sigma_3}\int\limits_0^{x}\varphi(t,\rho)g(t)dt + \frac{\varphi(x,\rho)}{\sigma_3}\int\limits_x^{\infty}\psi(t,\rho)g(t)dt.$$

By definition of function $\varphi(x,\rho)$ and $\psi(x,\rho)$, it is clear that the function

$$\Psi(x,t,\rho) = \begin{cases}\varphi(t,\rho)\psi(x,\rho), & 0 \leqslant t \leqslant x,\\ \varphi(x,\rho)\psi(t,\rho), & 0 \leqslant t \leqslant \infty.\end{cases}$$

is an analytic function. Then we have

$$\operatorname{Res}\left\{\psi(x,\rho)\int\limits_0^x \varphi(t,\rho)g(t)dt+\varphi(x,\rho)\int\limits_x^\infty \psi(t,\rho)g(t)dt\right\}=0.$$

From this, it follows that

$$\operatorname*{Res}_D\left[\rho\Phi(x,\rho)\right]=-\operatorname*{Res}_D\frac{\beta_2 f(0,\rho)-\alpha_2 f'(0,\rho)}{\sigma_3 F(\rho)}\rho\varphi(x,\rho)\int\limits_0^\infty \varphi(t,\rho)g(t)dt.$$

Let

$$\varphi(g,\rho)=\int\limits_0^\infty \varphi(t,\rho)g(t)dt.$$

Then, the estimate

$$\operatorname*{Res}_D\left[\rho\Phi(x,\rho)\right]=-\operatorname*{Res}_D\frac{\beta_2 f(0,\rho)-\alpha_2 f'(0,\rho)}{\sigma_3 F(\rho)}\rho\varphi(x,\rho)\varphi(g,\rho)$$

holds.

We assume that in the half plane $\operatorname{Im}\rho\geqslant 0$ each $\rho_k\in D$ is zeros of the equation $F(\rho)=0$ with multiplicity m_k, $k=1,2,\ldots,n$.

Let

$$M_k(\rho)=\frac{(\rho-\rho_k)^{m_k}}{(m_k-1)!}\cdot\frac{f(0,\rho)}{F(\rho)}.$$

Then for $\rho_k\in D$ by the residue formula we have

$$\operatorname*{Res}_D\left[\rho\Phi(x,\rho)\right]=-\sum_{k=1}^n\left\{\left(\frac{d}{d\rho}\right)^{m_k-1}\rho M_k(\rho)\varphi(x,\rho)\varphi(g,\rho)\right\}\Bigg|_{\rho=\rho_k}. \tag{3.11}$$

Further, we can obtain an analogous formula for $\operatorname{Im}\rho\leqslant 0$ each $\overline{\rho}_k\in\overline{D}$ in the following form:

$$\operatorname*{Res}_D\left[\rho\Phi(x,\rho)\right]=-\sum_{k=1}^n\left\{\left(\frac{d}{d\rho}\right)^{m_k-1}\rho\overline{M}_k(\rho)\varphi(x,\rho)\varphi(g,\rho)\right\}\Bigg|_{\rho=\overline{\rho}_k}, \tag{3.12}$$

where

$$\overline{M}_k(\rho)=\frac{(\rho-\overline{\rho}_k)^{m_k-1}}{(m_k-1)!}\cdot\frac{f(0,\rho)}{F(\rho)},$$

$\overline{\rho}_k\in\overline{D}$ the roots of the equation $F_1(\rho)=0$, with multiplicity m'_k, $k=1,2,\ldots,n'$.

Since

$$\Phi(x,\rho+i0)-\Phi(x,\rho-i0)=\frac{2i\rho}{F(\rho)F_1(\rho)}\varphi(x,\rho)\int\limits_0^\infty \varphi(t,\rho)g(t)dt=$$

$$=\frac{2i\rho}{F(\rho)F_1(\rho)}\varphi(x,\rho)\varphi(g,\rho).$$

It follows that

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\rho[\Phi(x,\rho+i0)-\Phi(x,\rho-i0)]d\rho=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\rho^2}{F(\rho)F_1(\rho)}\varphi(x,\rho)\varphi(g,\rho)d\rho. \tag{3.13}$$

Substituting (3.11), (3.12), and (3.13) in (3.9), (3.10) respectively, we obtain two-fold expansion formulas in the following form:

$$g(x)=-\sum_{k=1}^{n}\left\{\left(\frac{d}{d\rho}\right)^{m_k-1}\rho M_k(\rho)\varphi(x,\rho)\varphi(g,\rho)\right\}\Bigg|_{\rho=\rho_k}-$$
$$-\sum_{k=1}^{n}\left\{\left(\frac{d}{d\rho}\right)^{m_k-1}\rho\overline{M}_k(\rho)\varphi(x,\rho)\varphi(g,\rho)\right\}\Bigg|_{\rho=\overline{\rho}_k}+$$
$$+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\rho^2}{F(\rho)F_1(\rho)}\varphi(x,\rho)\varphi(g,\rho)d\rho$$

and

$$0=-\sum_{k=1}^{n}\left\{\left(\frac{d}{d\rho}\right)^{m_k-1}\rho M_k(\rho)\varphi(x,\rho)\varphi(g,\rho)\right\}\Bigg|_{\rho=\rho_k}-$$
$$-\sum_{k=1}^{n}\left\{\left(\frac{d}{d\rho}\right)^{m_k-1}\rho\overline{M}_k(\rho)\varphi(x,\rho)\varphi(g,\rho)\right\}\Bigg|_{\rho=\overline{\rho}_k}+$$
$$+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\rho^2}{F(\rho)F_1(\rho)}\varphi(x,\rho)\varphi(g,\rho)d\rho$$

for $\rho_k\in D$, $\overline{\rho}_k\in\overline{D}$. It can be shown that the integrals on the right-hand side converge in the metric of the space $L_2(0,\infty)$.

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