# The Spanne-type result in the vanishing generalized weighted local Morrey spaces

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**Abstract**

In this paper, we prove the Spanne-type boundedness of the generalized fractional maximal operator $M_\rho$ from the vanishing generalized weighted local Morrey spaces $\mathcal{VM}^{\{x_0\}}_{p,\varphi_1}(\mathbb{R}^n, w^p)$ to another one $\mathcal{VM}^{\{x_0\}}_{q,\varphi_2}(\mathbb{R}^n, w^q)$ with $w^q \in A_{1+\frac{q}{p'}}$ for $1 < p < q < \infty$, and from the vanishing generalized weighted local Morrey spaces $\mathcal{VM}^{\{x_0\}}_{1,\varphi_1}(\mathbb{R}^n, w)$ to the vanishing generalized weighted weak local Morrey spaces $\mathcal{VWM}^{\{x_0\}}_{q,\varphi_2}(\mathbb{R}^n, w^q)$ with $w^q \in A_{1,q}$ for $p = 1, 1 < q < \infty$, where $\varphi_1, \varphi_2 \in \mathfrak{M}_{\text{loc}}$ class. Also, we get a new theorem for the Spanne-type result of the generalized fractional maximal operator $M_\rho$ from the vanishing generalized Morrey spaces $\mathcal{VM}_{p,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{VM}_{q,\varphi_2}(\mathbb{R}^n)$, including weak estimates. The all weight functions belong to Muckenhoupt -Weeden classes $A_{p,q}$.

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## 1 Introduction

Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ were introduced by Morrey in [22]. These spaces appeared to be useful in the study of local behavior properties of the solutions of second order elliptic PDEs. Morrey spaces have important applications, for instance see the paper [1, 20, 30].

On the other hand, on the weighted Lebesgue spaces $L_p(\mathbb{R}^n, w)$, the boundedness of some classical operators were obtained by Muckenhoupt [24], Mukenhoupt and Wheeden [23], and Coifman and Fefferman [5]. In [13], Komori et al. defined the weighted Morrey spaces $L_{p,\kappa}(\mathbb{R}^n, w)$ as the following: For $1 \le p \le \infty, 0 < \kappa < 1$ and $w$ be a weight, $f \in L_{p,\kappa}(\mathbb{R}^n, w)$ if $f \in L^{loc}_p(\mathbb{R}^n, w)$ and

$$\|f\|_{L_{p,\kappa}(\mathbb{R}^n,w)} = \sup_{x\in\mathbb{R}^n, r>0} w(B(x,r))^{-\frac{\kappa}{p}} \|f\|_{L_p(B(x,r),w)} < \infty.$$

Weighted inequalities for fractional operators have good applications to potential theory and quantum mechanics. For more detail we refer the book [9].

The vanishing Morrey space $\mathcal{VL}_{p,\lambda}(\mathbb{R}^n)$ of the classical Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ was introduced by Vitanza in [34] (see also [29]) and applied there to obtain a regularity result for elliptic PDEs. Later in [35] Vitanza proved an existence theorem for a Dirichlet problem, under weaker assumptions then those introduced by Miranda in [21], and a $W^{3,2}$ regularity result assuming that the partial derivatives of the coefficients of the highest and lower order terms belong to vanishing Morrey spaces depending on the dimension. Persson et al. [28] showed the commutators of Hardy operators on vanishing Morrey spaces. Also Ragusa [29] obtained a sufficient condition for commutators of fractional integral operators to belong to vanishing Morrey spaces $\mathcal{VL}_{p,\lambda}(\mathbb{R}^n)$.

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The vanishing generalized Morrey space $\mathcal{VM}_{p,\varphi}(\mathbb{R}^n)$ and vanishing generalized local Morrey space $\mathcal{VM}^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n)$ was introduced by Samko (see, [31, 32]). The boundedness of the multi-dimensional Hardy type operators, maximal, potential and singular operators in these spaces were proved in [31, 32]. Inspired by [33], Kucukaslan et al. [14, 19] proved the Spanne-type and Adams-type boundedness of generalized fractional integral operators on vanishing generalized local Morrey spaces.

Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. Then the generalized fractional maximal operator $M_\rho$ and the generalized fractional integral operator $I_\rho$ are defined by

$$M_\rho f(x) = \sup_{t>0} \frac{\rho(t)}{t^n} \int_{B(x,t)} |f(y)|dy, \;\; I_\rho f(x) = \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y)dy$$

for any suitable function $f$ on $\mathbb{R}^n$. If $\rho(t) \equiv t^\alpha$, then $M_\alpha \equiv M_{t^\alpha}$ is the fractional maximal operator and $I_\alpha \equiv I_{t^\alpha}$ is the Riesz potential operator.

The generalized fractional maximal and integral operators $M_\rho$ and $I_\rho$ were initially investigated in [8, 25]. Nakai [25] proved the boundedness of $I_\rho$ from the generalized Morrey spaces $M_{1,\varphi}$ to the spaces $M_{1,\psi}$ for suitable functions $\varphi$ and $\psi$. Nowadays many authors have been culminating important observations about these two-operators $M_\rho$ and $I_\rho$ especially in connection with Morrey spaces (see [6, 12, 15, 16, 17]).

During the last decades, the theory of boundedness of classical operators of the harmonic analysis in the generalized Morrey spaces have been well studied so far [6, 7, 12, 14, 15, 16, 17, 18, 19, 25, 26, 31, 32]. But, Spanne-type boundedness of the generalized fractional maximal operator $M_\rho$ in the vanishing generalized weighted local Morrey spaces $\mathcal{VM}^{\{x_0\}}_{p,\varphi_1}(\mathbb{R}^n, w^p)$ has not been studied, yet.

In this present paper, we prove the Spanne-type boundedness of the generalized fractional maximal operator $M_\rho$ from the vanishing generalized weighted local Morrey spaces $\mathcal{VM}^{\{x_0\}}_{p,\varphi_1}(\mathbb{R}^n, w^p)$ to another one $\mathcal{VM}^{\{x_0\}}_{q,\varphi_2}(\mathbb{R}^n, w^q)$ with $w^q \in A_{1+\frac{q}{p'}}$ for $1 < p < q < \infty$, and from the vanishing generalized weighted local Morrey spaces $\mathcal{VM}^{\{x_0\}}_{1,\varphi_1}$ to the vanishing generalized weighted weak local Morrey spaces $\mathcal{VWM}^{\{x_0\}}_{q,\varphi_2}(\mathbb{R}^n, w^q)$ with $w^q \in A_{1,q}$ for $p = 1, 1 < q < \infty$. Also, we get a new corollary for the Spanne-type result of the generalized fractional maximal operator $M_\rho$ from the vanishing generalized local Morrey spaces $\mathcal{VM}_{p,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{VM}_{q,\varphi_2}(\mathbb{R}^n)$, including weak estimates.

Throughout the paper we use the letter $C$ for a positive constant, independent of appropriate parameters and not necessary the same at each occurrence. By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant $C$.

## 2 Preliminaries

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r) \subset \mathbb{R}^n$ the open ball centered at $x$ of radius $r$. Let $|B(x,r)|$ be the Lebesgue measure of ball $B(x,r)$ and $\mathbb{R}^n$ the Euclidean space. A weight function is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0,\infty)$ almost everywhere. For a weight $w$ and a measurable set $E$, we define $w(E) = \int_E w(x)dx$, in the special case of $w \equiv 1$ we get $w(E) = |E|$. The characteristic function of $E$ by $\chi_E$. If $w$ is a weight function, for all $f \in L_1^{loc}(\mathbb{R}^n)$ we denote by $L_p^{loc}(\mathbb{R}^n, w^p)$ the weighted Lebesgue space defined by the norm

$$\|f\chi_{B(x,r)}\|_{L_p(\mathbb{R}^n,w^p)} = \left(\int_{B(x,r)} (|f(x)|w(x))^p \, dx\right)^{\frac{1}{p}} < \infty,$$

when $1 \leq p < \infty$.

We recall that a weight function $w$ belongs to the Muckenhoupt-Wheeden class $A_{p,q}$ (see [23]) for $1 < p < q < \infty$, if

$$\sup_B \left(\frac{1}{|B|}\int_B w(x)^q dx\right)^{\frac{1}{q}} \left(\frac{1}{|B|}\int_B w(x)^{-p'} dx\right)^{\frac{1}{p'}} \leq C,$$

if $p = 1$, $w$ is in the $A_{1,q}$ with $1 < q < \infty$ then

$$\sup_B \left(\frac{1}{|B|}\int_B w(x)^q dx\right)^{\frac{1}{q}} \left(ess\sup_{x\in B} \frac{1}{w(x)}\right) \leq C,$$

where $C > 0$ and the supremum is taken with respect to all balls $B$.

**Lemma 2.1.** ([9, 10]). If $w \in A_{p,q}$ with $1 < p < q < \infty$, then the following statements are true.
(*i*) $w^p \in A_s$ with $s = 1 + \frac{p}{q'}$.
(*ii*) $w^q \in A_r$ with $r = 1 + \frac{q}{p'}$.

We find it convenient to define the generalized weighted Morrey spaces as the following.

**Definition 2.2.** ([11]). Let $1 \leq p < \infty$, $w$ be a weight function on $\mathbb{R}^n$ and $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$. We denote by $M_{p,\varphi}(\mathbb{R}^n, w)$ the generalized weighted Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n, w)$ with finite norm

$$\|f\|_{M_{p,\varphi}(\mathbb{R}^n,w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x, r)^{-1} w(B(x, r))^{-\frac{1}{p}} \|f\|_{L_p(B(x,r),w)}.$$

Also by $WM_{p,\varphi}(\mathbb{R}^n, w)$ we denote the generalized weighted weak Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n, w)$ for which

$$\|f\|_{WM_{p,\varphi}(\mathbb{R}^n,w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x, r)^{-1} w(B(x, r))^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r),w)},$$

where $WL_p(B(x, r), w)$ denotes the weighted weak $L_p$ space of measurable functions $f$ for which

$$\|f\|_{WL_p(B(x,r),w)} = \sup_{t>0} \left(\int_{\{y\in B(x,r):|f(y)|>t\}} w(y)dy\right)^{\frac{1}{p}}.$$

**Definition 2.3.** ([12]). Let $1 \leq p < \infty$, $w$ be a weight function on $\mathbb{R}^n$ and $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $M_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n, w)$ the generalized weighted local Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n, w)$ with finite norm

$$\|f\|_{M_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n,w)} = \|f(x_0 + \cdot)\|_{M_{p,\varphi}(\mathbb{R}^n,w)}.$$

Also by $WM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n, w) \equiv WM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n, w)$ we denote the weak generalized weighted local Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n, w)$ for which

$$\|f\|_{WM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n,w)} = \|f(x_0 + \cdot)\|_{WM_{p,\varphi}(\mathbb{R}^n,w)} < \infty.$$

If $\lambda < 0$ or $\lambda > n$, then $M_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n) = \Theta$, where $\Theta$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Moreover, we have the following embeddings between generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ and generalized local Morrey spaces $M_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$:

$$M_{p,\varphi}(\mathbb{R}^n) \subset M_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n), \ \ \|f\|_{M_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)} \leq \|f\|_{M_{p,\varphi}(\mathbb{R}^n)},$$

$$WM_{p,\varphi}(\mathbb{R}^n) \subset WM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n), \ \ \|f\|_{WM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)} \leq \|f\|_{WM_{p,\varphi}(\mathbb{R}^n)}.$$

**Definition 2.4.** ([4]). The vanishing generalized weighted Morrey space $\mathcal{VM}_{p,\varphi}(\mathbb{R}^n, w)$ is defined as the space of functions $f \in M_{p,\varphi}(\mathbb{R}^n, w)$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{w(B(x,r))^{-\frac{1}{p}}}{\varphi(x,r)} \|f\|_{L_p(B(x,r),w)} = 0.$$

The vanishing generalized weighted weak Morrey space $\mathcal{VWM}_{p,\varphi}(\mathbb{R}^n, w)$ is defined as the space of functions $f \in WM_{p,\varphi}(\mathbb{R}^n, w)$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{w(B(x,r))^{-\frac{1}{p}}}{\varphi(x,r)} \|f\|_{WL_p(B(x,r),w)} = 0$$

Everywhere in the sequel we assume that

$$\lim_{r\to 0} \frac{1}{\inf_{x\in\mathbb{R}^n} \varphi(x,r)} = 0 \quad \text{and} \quad \sup_{0<r<\infty} \frac{1}{\inf_{x\in\mathbb{R}^n} \varphi(x,r)} < \infty, \tag{2.1}$$

which makes the spaces $\mathcal{VM}_{p,\varphi}(\mathbb{R}^n, w)$ and $VWM_{p,\varphi}(\mathbb{R}^n, w)$ non-trivial, because bounded functions with compact support belong then to this space.

Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$. We say that $\varphi$ belongs to the class $\mathfrak{M}_{\text{glob}}$, if it satisfies the assumptions in (2.1).

The spaces $\mathcal{VM}_{p,\varphi}(\mathbb{R}^n, w)$ and $\mathcal{VWM}_{p,\varphi}(\mathbb{R}^n, w)$ are Banach spaces with respect to the norm

$$\|f\|_{\mathcal{VM}_{p,\varphi}(\mathbb{R}^n,w)} \equiv \|f\|_{M_{p,\varphi}(\mathbb{R}^n,w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{L_p(B(x,r),w)},$$

$$\|f\|_{\mathcal{VWM}_{p,\varphi}(\mathbb{R}^n,w)} \equiv \|f\|_{WM_{p,\varphi}(\mathbb{R}^n,w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r),w)},$$

respectively.

Extending the definition of vanishing generalized weighted Morrey spaces to the case of weighted local Morrey spaces, we introduce the following definition.

**Definition 2.5.** ([14]). The vanishing generalized weighted local Morrey space $\mathcal{VM}_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n, w)$ and its weak version $\mathcal{VWM}_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n, w)$ are defined as the spaces of functions $f \in M_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n, w)$ and $f \in WM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n, w)$ such that

$$\lim_{r\to 0} \frac{w(B(x_0,r))^{-\frac{1}{p}}}{\varphi(x_0,r)} \|f\|_{L_p(B(x_0,r),w)} = 0 \quad \text{and} \quad \lim_{r\to 0} \frac{w(B(x_0,r))^{-\frac{1}{p}}}{\varphi(x_0,r)} \|f\|_{WL_p(B(x_0,r),w)} = 0,$$

respectively.

The followings are sufficient conditions for the non-triviality of the spaces $\mathcal{VM}^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w)$ and $\mathcal{VWM}^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w)$:

$$\lim_{r\to 0} \frac{1}{\varphi(x_0, r)} = 0 \quad \text{and} \quad \sup_{r>0} \frac{1}{\varphi(x_0, r)} < \infty, \tag{2.2}$$

since bounded functions with compact support belong then to these spaces, see [32].

Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$. We say that $\varphi$ belongs to the class $\mathfrak{M}_{\rm loc}$, if it satisfies the assumptions in (2.2).

The spaces $\mathcal{VM}^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w)$ and $VWM^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w)$ are closed subspaces of the Banach spaces $M^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w)$ and $WM^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w)$, respectively, which may be shown by standard means.

We will also use the following notation

$$\mathfrak{A}_{p,\varphi,w}(f; x_0, r) := \varphi(x_0, r)^{-1} w(B(x_0, r))^{-\frac{1}{p}} \|f\|_{L_p(B(x_0,r),w)}$$

and

$$\mathfrak{A}^W_{p,\varphi,w}(f; x_0, r) := \varphi(x_0, r)^{-1} w(B(x_0, r))^{-\frac{1}{p}} \|f\|_{WL_p(B(x_0,r),w)}$$

for brevity, so that

$$\mathcal{VM}^{\{x_0\}}_{p,\varphi,w}(\mathbb{R}^n) = \Big\{ f \in M^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w) : \lim_{r\to 0} \mathfrak{A}_{p,\varphi,w}(f; x_0, r) = 0 \Big\}$$

and similarly for $\mathcal{VWM}^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w)$.

## 3 The Spanne-type estimate for the operator $M_\rho$ in the spaces $\mathcal{VM}^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w^p)$

In this section, we obtain the Spanne-type boundedness of the generalized fractional maximal operators $M_\rho$ in the vanishing generalized weighted local Morrey spaces $\mathcal{VM}^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w^p)$ and in the vanishing generalized Morrey spaces $\mathcal{VM}_{p,\varphi}(\mathbb{R}^n)$.

In the following theorem Spanne studied boundedness of the Riesz potential in Morrey spaces.

**Theorem A** (Spanne, but published by Peetre, [27]). *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$. Moreover, let $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $\frac{\lambda}{p} = \frac{\mu}{q}$. Then for $p > 1$, the Riesz potential operator $I_\alpha$ is bounded from $L_{p,\lambda}(\mathbb{R}^n)$ to $L_{q,\mu}(\mathbb{R}^n)$ and for $p = 1$, $I_\alpha$ is bounded from $L_{1,\lambda}(\mathbb{R}^n)$ to $WL_{q,\mu}(\mathbb{R}^n)$.*

In particular, the following statement containing Theorem A.

**Theorem B** ([2, 3]). *Let $1 \le p < q < \infty$, $0 < \lambda, \mu < n$ and $0 < \alpha = \frac{n-\lambda}{p} - \frac{n-\mu}{q} < \frac{n}{p}$. Then, for $p > 1$, the Riesz potential operator $I_\alpha$ is bounded from $L_{p,\lambda}(\mathbb{R}^n)$ to $L_{q,\mu}(\mathbb{R}^n)$, and, for $p = 1$, $I_\alpha$ is bounded from $L_{1,\lambda}(\mathbb{R}^n)$ to $WL_{q,\mu}(\mathbb{R}^n)$.*

The following theorem which is the Spanne-type results for the boundedness of the operator $M_\rho$ on the generalized weighted local Morrey spaces $M^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n, w)$.

**Theorem C** (Spanne-type result, [17]). *Let $x_0 \in \mathbb{R}^n$, $1 \le p < q < \infty$, $w^q \in A_{1+\frac{q}{p'}}$, and let the function $\rho$ satisfy the conditions* (3.3)-(3) *and* (3.5). *Let also $(\varphi_1, \varphi_2)$ satisfy the conditions*

$$\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0, s) s^{\frac{n}{p}} \le C\, \varphi_2\big(x_0, \frac{t}{2}\big)\, t^{\frac{n}{q}}, \tag{3.1}$$

$$\sup_{t>r}\frac{\left(\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)(w^p(B(x_0,s)))^{\frac{1}{p}}s^{\frac{n}{p}}\right)\rho(t)}{(w^q(B(x_0,t)))^{\frac{1}{q}}t^{\frac{n}{p}}}\leq C\,\varphi_2(x_0,r), \tag{3.2}$$

*where $C$ does not depend on $x_0$ and $r$. Then the operator $M_\rho$ is bounded from $M^{\{x_0\}}_{p,\varphi_1}(w^p)$ to $WM^{\{x_0\}}_{q,\varphi_2}(w^q)$ and for $p>1$ from $M^{\{x_0\}}_{p,\varphi_1}(w^p)$ to $M^{\{x_0\}}_{q,\varphi_2}(w^q)$. Moreover, for $1\leq p<q<\infty$*

$$\|M_\rho f\|_{WM^{\{x_0\}}_{q,\varphi_2}(w^q)}\lesssim\|f\|_{M^{\{x_0\}}_{p,\varphi_1}(w^p)},$$

*and for $p>1$*

$$\|M_\rho f\|_{M^{\{x_0\}}_{q,\varphi_2}(w^q)}\lesssim\|f\|_{M^{\{x_0\}}_{p,\varphi_1}(w^p)}.$$

In order to achieve our purpose, we assume that

$$\sup_{1\leq t<\infty}\frac{\rho(t)}{t^n}<\infty, \tag{3.3}$$

so that the fractional maximal function $M_\rho f$ is well defined, at least for characteristic functions $1/|x|^{2n}$ of complementary balls:

$$f(x)=\frac{\chi_{\mathbb{R}^n\setminus B(0,1)}(x)}{|x|^{2n}}.$$

In addition, we shall also assume that $\rho$ satisfies the growth condition: there exist constants $C>0$ and $0<2k_1<k_2<\infty$ such that

$$\sup_{r<s\leq 2r}\frac{\rho(s)}{s^n}\leq C\sup_{k_1r<t<k_2r}\frac{\rho(t)}{t^n},\quad r>0. \tag{3.4}$$

This condition is weaker than the usual doubling condition for the function $\frac{\rho(t)}{t^n}$ : there exists a constant $C>0$ such that

$$\frac{1}{C}\frac{\rho(t)}{t^n}\leq\frac{\rho(r)}{r^n}\leq C\frac{\rho(t)}{t^n},$$

whenever $r$ and $t$ satisfy $r,\ t>0$ and $\frac{1}{2}\leq\frac{r}{t}\leq 2$. In the sequel for the generalized fractional maximal operator $M_\rho$ we always assume that $\rho$ satisfies the condition (3.4).

The boundedness of the operator $I_\rho$ in the spaces $L_p(\mathbb{R}^n)$ can be found in [7]. Let $\frac{\rho(t)}{t^n}$ be almost decreasing, that is, there exists a constant $C$ such that $\frac{\rho(t)}{t^n}\leq C\,\frac{\rho(s)}{s^n}$ for $s<t$. In this case, there is a close and strong relation between the operators $M_\rho$ and $I_\rho$ (see, [15]) such that

$$M_\rho f(x)\lesssim I_\rho(|f|)(x).$$

Hence, the following lemma is valid for the operator $M_\rho$.

**Lemma 3.1.** ([17]). Let $w^q\in A_{1+\frac{q}{p'}}, 1\leq p<q<\infty$, the function $\rho$ satisfies the conditions(3.3)-(3.4), and $f\in L_1^{loc}(\mathbb{R}^n,w)$.

$(i)$ If $1<p<q<\infty$ then there exist $C>0$ for all $r>0$ such that the inequality

$$\rho(r)\leq Cr^{\frac{n}{p}-\frac{n}{q}} \tag{3.5}$$

is sufficient condition for the boundedness of generalized fractional maximal operator $M_\rho$ from $L_p(\mathbb{R}^n, w^p)$ to $L_q(\mathbb{R}^n, w^q)$.

$(ii)$ If $p = 1, 1 < q < \infty$ then there exist $C > 0$ for all $r > 0$ such that the inequality

$$\rho(r) \leq C r^{n-\frac{n}{q}} \tag{3.6}$$

is sufficient condition for the boundedness of generalized fractional maximal operator $M_\rho$ from $L_1(\mathbb{R}^n, w)$ to $WL_q(\mathbb{R}^n, w^q)$, where the constant $C$ does not depend on $f$.

The following lemma is weighted local strong and weak $L_p$-estimates for the operator $M_\rho$ which is our main tool to prove our main results.

**Lemma 3.2.** ([17]). Let fixed $x_0 \in \mathbb{R}^n$, $w^q \in A_{1+\frac{q}{p'}}, 1 \leq p < q < \infty$ and $\rho(t)$ satisfy the conditions (3.3)-(3.4).

$(i)$ If $1 < p < q < \infty$ and the condition (3.5) is fulfill, then the inequality

$$\begin{aligned}\|M_\rho f\chi_{B(x_0,r)}\|_{L_q(\mathbb{R}^n,w^q)} &\lesssim \|f\chi_{B(x_0,2r)}\|_{L_p(\mathbb{R}^n,w^p)} \\ &+ (w^q(B(x_0,r)))^{\frac{1}{q}} \left(\sup_{t>r} \frac{\rho(t)}{t^{\frac{n}{p}}} \|f\chi_{B(x_0,t)}\|_{L_p(\mathbb{R}^n,w^p)}\right)\end{aligned} \tag{3.7}$$

holds for the ball $B(x_0, r)$ and for all $f \in L_p^{loc}(\mathbb{R}^n, w^p)$ and,

$(ii)$ if $p = 1, 1 < q < \infty$ and the condition (3.6) is fulfill, then the inequality

$$\begin{aligned}\|M_\rho f\chi_{B(x_0,r)}\|_{WL_q(\mathbb{R}^n,w^q)} &\lesssim \|f\chi_{B(x_0,2r)}\|_{L_1(\mathbb{R}^n,w)} \\ &+ (w^q(B(x_0,r)))^{\frac{1}{q}} \left(\sup_{t>r} \frac{\rho(t)}{t^n} \|f\chi_{B(x_0,t)}\|_{L_1(\mathbb{R}^n,w)}\right)\end{aligned} \tag{3.8}$$

holds for any ball $B(x_0, r)$ and for all $f \in L_1^{loc}(\mathbb{R}^n, w)$.

An extension theorem of Theorem A, Theorem B and Theorem C which is the following theorem is one of our main results in which we generalize the Spanne-type boundedness of the operator $M_\rho$ in vanishing generalized weighted local Morrey spaces $\mathcal{VM}_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n, w^p)$.

**Theorem 3.3.** Let $x_0 \in \mathbb{R}^n$, $1 \leq p < q < \infty$, $w^q \in A_{1+\frac{q}{p'}}$, $\varphi_1, \varphi_2, \in \mathfrak{M}_{\text{loc}}$ and the function $\rho$ satisfy the conditions (3.3)-(3.4) and (3.5)-(3.6). Let also $\varphi_1, \varphi_2$ satisfy the conditions

$$\operatorname*{ess\,inf}_{t<r<\infty} \varphi_1(x_0, r) \left(w^p(B(x_0, r))\right)^{\frac{1}{p}} \leq C\, \varphi_2\big(x_0, \frac{t}{2}\big) \left(w^q(B(x_0, t))\right)^{\frac{1}{q}},$$

$$\sup_{t>r} \left(\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0, s) \left(w^p(B(x_0, s))\right)^{\frac{1}{p}}\right) \frac{\rho(t)}{t^{\frac{n}{p}}} \leq C\, \varphi_2(x_0, r),$$

where $C$ does not depend on $x_0$ and $r$. Then the operator $M_\rho$ is bounded from vanishing generalized weighted local Morrey spaces $\mathcal{VM}_{p,\varphi_1}^{\{x_0\}}(\mathbb{R}^n, w^p)$ to $\mathcal{VM}_{q,\varphi_2}^{\{x_0\}}(\mathbb{R}^n, w^q)$ for $p > 1$ and from the space $\mathcal{VM}_{1,\varphi_1}^{\{x_0\}}(\mathbb{R}^n, w)$ to the weak space $\mathcal{VWM}_{q,\varphi_2}^{\{x_0\}}(\mathbb{R}^n, w^q)$ for $p = 1$.

*Proof.* Since the norm inequality is already provided by Theorem C, so we only have to prove that

$$\lim_{r\to 0}\mathfrak{A}_{p,\varphi_1,w^p}(f;x_0,r)=0 \quad\Longrightarrow\quad \lim_{r\to 0}\mathfrak{A}_{q,\varphi_2,w^q}(M_\rho f;x_0,r)=0, \tag{3.9}$$

and

$$\lim_{r\to 0}\mathfrak{A}^W_{1,\varphi_1,w}(f;x_0,r)=0 \quad\Longrightarrow\quad \lim_{r\to 0}\mathfrak{A}^W_{q,\varphi_2,w^q}(M_\rho f;x_0,r)=0, \tag{3.10}$$

To check (3.9), i.e., to show that

$$\frac{\left(w^q(B(x_0,r))\right)^{-\frac{1}{q}}\|M_\rho f\|_{L^q(B(x_0,r),w^q)}}{\varphi_2(x_0,r)}<\varepsilon \quad \text{for small } \ r,$$

we use the estimate (3.7) where we split the right-hand side:

$$\frac{\left(w^q(B(x_0,r))\right)^{-\frac{1}{q}}\|M_\rho f\|_{L^q(B(x_0,r),w^q)}}{\varphi_2(x_0,r)}\lesssim I(x_0,r)+J_{\delta_0}(x_0,r)+K_{\delta_0}(x_0,r), \tag{3.11}$$

with $\delta_0>0$ and $r<\delta_0$, where

$$I(x_0,r):=\frac{\left(w^q(B(x_0,r))\right)^{-\frac{1}{q}}\|f\|_{L^p(B(x_0,2r),w^p)}}{\varphi_2(x_0,r)},$$

$$J_{\delta_0}(x_0,r):=\frac{1}{\varphi_2(x_0,r)}\left(\sup_{r<t<\delta_0}\|f\|_{L_p(B(x_0,t),w^p)}\frac{\rho(t)}{\left(w^p(B(x_0,t))\right)^{\frac{1}{p}}}\right)$$

and

$$K_{\delta_0}(x_0,r):=\frac{1}{\varphi_2(x,r)}\left(\sup_{t>\delta_0}\|f\|_{L_p(B(x_0,t),w^p)}\frac{\rho(t)}{\left(w^p(B(x_0,t))\right)^{\frac{1}{p}}}\right).$$

For the first term from (3.12) we have

$$I(x_0,r)\lesssim\frac{r^{-\frac{n}{p}}\|f\|_{L^p(B(x_0,r))}}{\varphi_1(x_0,r)}.$$

By hypothesis we get $H(x_0,r)<\frac{\varepsilon}{3}$ for infinitesimals $r$.

We use the fact that $f\in\mathcal{VM}^{\{x_0\}}_{p,\varphi_1}(\mathbb{R}^n,w^p)$ and choose any fixed $\delta_0>0$ such that

$$\frac{t^{-\frac{n}{p}}\|f\|_{L^p(B(x_0,t))}}{\varphi_1(x_0,t)}<\frac{\varepsilon}{3C},\quad t\le\delta_0,$$

where $C$ is constant from (3.11) and (3.13), which yields the estimate of the second term uniform in $r\in(0,\delta_0)$ :

$$J_{\delta_0}(x_0,r)<\frac{\varepsilon}{3C},\quad 0<r<\delta_0.$$

For the third term, we have

$$K_{\delta_0}(x_0,r) \le C_{\delta_0} \frac{\|f\|_{M^{\{x_0\}}_{p,\varphi_1}(\mathbb{R}^n,w^p)}}{\varphi_2(x_0,r)},$$

where

$$C_{\delta_0} = \sup_{t>\delta_0} \varphi_1(x_0,t)\rho(t).$$

Note that $C_{\delta_0} < \infty$ follows from (3.13). Then, by (2.2) we choose small $r$ such that

$$\frac{1}{\varphi_2(x_0,r)} \le \frac{\varepsilon}{3C_{\delta_0}\|f\|_{M^{\{x_0\}}_{p,\varphi_1}(\mathbb{R}^n,w^p)}},$$

which completes the estimation of the third term and the proof. The proof of (3.10) is, line by line, similar to the proof of (3.9) by using (3.8). Q.E.D.

The following new result is valid for the generalized fractional maximal operator $M_\rho$.

**Theorem 3.4.** Let $1 \le p < \infty$, $\varphi \in \mathfrak{M}_{\rm glob}$ and the function $\rho$ satisfy the conditions (3.3)-(3.4) and (3.5)-(3.6). Let also $(\varphi_1,\varphi_2)$ satisfy the conditions for every $\delta > 0$

$$c_\delta = \sup_{t>\delta, x\in\mathbb{R}^n} \varphi_1(x,t)\rho(t) < \infty,$$

and

$$\varphi_1(x,r) r^{\frac{n}{p}} \lesssim \varphi_2\big(x,\frac{r}{2}\big)\, r^{\frac{n}{q}}, \tag{3.12}$$

$$\sup_{r<t<\infty} \varphi_1(x,t)\rho(t) \lesssim \varphi_2(x,r). \tag{3.13}$$

Then the operator $M_\rho$ is bounded from vanishing generalized Morrey spaces $\mathcal{VM}_{p,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{VM}_{q,\varphi_2}(\mathbb{R}^n)$ for $p > 1$ and from the vanishing space $\mathcal{VM}_{1,\varphi_1}(\mathbb{R}^n)$ to the vanishing weak space $\mathcal{VWM}_{q,\varphi_2}(\mathbb{R}^n)$ for $p = 1$.

*Proof.* Taking $w \equiv 1$ and $x_0 = x$, the proof is step by step the same as in the proof of Theorem 3.3. Q.E.D.

## Conflicts of interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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