Algebra of elliptical quaternions

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Abstract

Quaternions are defined by W.R. Hamilton in 1843. By defining a new product on the set of quaternions, Hamilton facilitated the study of motions in Euclidean space. In this study; algebraic structure of elliptical quaternions has been constructed on the field of real numbers and some properties of this kind of quaternions have been studied with the help of Hamilton matrices.

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1 Introduction

Quaternions were introduced in 1843 by W.R. Hamilton. In 1829, complex numbers, which are one real and the other imaginary, figuring that it could be represented in the plane with the aid of two axes, Hamilton set out to find an algebraic notation that would allow performing all operations corresponding to complex numbers. From his former work on this subject published in 1833, he showed that complex numbers can be expressed with binary of numbers instead of a unary number. Then he studied to find algebraic representations of points in three-dimensional space that did not depend on randomly chosen artificial coordinates. However ten years later he was able to achieve the desired result when the product gave up its commutative property. He found that it is possible to represent points in three dimensional space not with triplets, but with quads. By describing quaternions, Hamilton introduced a new product operation for two vectors into vector algebra, where division is also possible [1]. Thus, by defining a new product on the Hamilton quaternions set, it facilitated the study of motions in three-dimensional Euclidean space [2]. Today, quaternions are used in optimization problems, especially in kinematics, computer graphics, animation, and rigid motions transformations [3,4]. There are many studies on quaternions and their matrix representations [5-11]. With the help of real quaternions, the movements of spherical rotations in three-dimensional space can be studied. with the help of a little generalization in real quaternions, it is possible to examine the rotational motions on the ellipsoid instead of the sphere. For this, the inner product and the vector product must be defined in accordance with the metric that accepts the given ellipsoid as a sphere. For each ellipsoid, a corresponding quaternion can be defined. This quaternion is called the elliptical quaternion. Real quaternions are a special case of ellipsoidal quaternions [12,13].

In this study; algebraic structure of elliptical quaternions has been constructed on the field of real numbers and some properties of this kind of quaternions have been studied with the help of Hamilton matrices.

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2 Preliminary

In this section, some basic definitions and theorems used in the study are given.

Definition 2.1. The set of

$$q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3$$

is called real quaternions. Such that ordered q_0, q_1, q_2, q_3 four real numbers accompany to $e_0 = 1, e_1, e_2, e_3$ units which enable

$$\begin{aligned} e_1^2 = e_2^2 = e_3^2 = -1, \\ e_1 \times e_2 = e_3,\ e_2 \times e_3 = e_1,\ e_3 \times e_1 = e_2, \\ e_2 \times e_1 = -e_3,\ e_3 \times e_2 = -e_1,\ e_1 \times e_3 = -e_2 \end{aligned} \tag{2.1}$$

properties. Here, q_0, q_1, q_2, q_3 real numbers are components of q quaternion and it is written as $\{\mathbb{H}, \oplus, \mathbb{R}, +, ., \odot, \times\}$ an associative algebra where quaternions set is $\mathbb{H}$.

Definition 2.2. This algebra is called quaternion algebra and shortly denoted by $\mathbb{H}$. One basis of this algebra is $\{1, e_1, e_2, e_3\}$ and the dimension is four [2].

Definition 2.3. In 3-dimensional Euclidean space; the set defined as

$$\begin{aligned} \mathbb{H}_{a_1,a_2,a_3}(\varepsilon) = \big\{ & q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3 \mid q_i \in \mathbb{R}, 0 \le i \le 3, e_1^2 = -a_1,\ e_2^2 = -a_2,\ e_3^2 = -a_3, \\ & e_1e_2e_3 = -\Delta \big\} \end{aligned}$$

where

$$e_1e_2 = \frac{\Delta}{a_3}e_3 = -e_2e_1,\ e_2e_3 = \frac{\Delta}{a_1}e_1 = -e_3e_2,\ e_3e_1 = \frac{\Delta}{a_2}e_2 = -e_1e_3$$

for the ellipsoid

$$\varepsilon : a_1x^2 + a_2y^2 + a_3z^2 = 1$$

such that $a_1, a_2, a_3 \in \mathbb{R}^+$ and $\sqrt{a_1a_2a_3} = \Delta$. This set is called the elliptical quaternion corresponding to the ε ellipsoid and is denoted by $\mathbb{H}_{a_1,a_2,a_3}(\varepsilon)$ for short. An elliptical quaternion $q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3$ can be written as the sum of the scalar part S_q and the vector part V_q. The set of elliptical quaternions is a set that is not commutative but has associative property. If $a_1 = a_2 = a_3 = 1$ taken here, the set of real quaternions is obtained [12].

Definition 2.4. Let $p = p_0e_0 + p_1e_1 + p_2e_2 + p_3e_3, q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3 \in \mathbb{H}_{a_1,a_2,a_3}(\varepsilon)$ be elliptical quaternions, their sum and their product by a scalar; defined as

$$\begin{aligned} p \oplus q &= (p_0 + q_0)e_0 + (p_1 + q_1)e_1 + (p_2 + q_2)e_2 + (p_3 + q_3)e_3, \\ \text{For } \ \lambda \in \mathbb{R}, \lambda \odot p &= (\lambda p_0)e_0 + (\lambda p_1)e_1 + (\lambda p_2)e_2 + (\lambda p_3)e_3 \end{aligned}$$

respectively [13].

Definition 2.5. Let $p = p_1e_1 + p_2e_2 + p_3e_3,\ q = q_1e_1 + q_2e_2 + q_3e_3 \in \mathbb{H}_{a_1,a_2,a_3}(\varepsilon)$ be elliptic quaternions. Their elliptical inner product and their elliptical vector product for the set of elliptical

quaternions that accepts the ε ellipsoid as a sphere, defined as

$$\langle p, q\rangle_{\varepsilon} = a_1 p_1 q_1 + a_2 p_2 q_2 + a_3 p_3 q_3,$$

$$(p \wedge q)_{\varepsilon} = \Delta \begin{vmatrix} \frac{e_1}{a_1} & \frac{e_2}{a_2} & \frac{e_3}{a_3} \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix}$$

respectively [12].

If $a_1 = a_2 = a_3 = 1$ is taken specifically in the definitions of elliptical inner product and elliptical vector product, standard Euclidean inner product and vector product are obtained, respectively [12].

Definition 2.6. Let $p = p_1 e_1 + p_2 e_2 + p_3 e_3$ and $q = q_1 e_1 + q_2 e_2 + q_3 e_3 \in \mathbb{H}_{a_1, a_2, a_3}(\varepsilon)$ be elliptic quaternions. The elliptical quaternion product of elliptic quaternions is defined by

$$p \times q = (p_0 e_0 + p_1 e_1 + p_2 e_2 + p_3 e_3) \times (q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3)$$

$$\begin{aligned} &= (p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3)\, e_0 \\ &+ \left(p_0 q_1 + p_1 q_0 + p_2 q_3 \frac{\Delta}{a_1} - p_3 q_2 \frac{\Delta}{a_1} \right) e_1 \\ &+ \left(p_0 q_2 - p_1 q_3 \frac{\Delta}{a_2} + p_2 q_0 + p_3 q_1 \frac{\Delta}{a_2} \right) e_2 \\ &+ \left(p_0 q_3 + p_1 q_2 \frac{\Delta}{a_3} - p_2 q_1 \frac{\Delta}{a_3} + p_3 q_0 \right) e_3 \end{aligned}$$

[12].

Theorem 2.7. Expression of the elliptical quaternion product of elliptical quaternions with the help of matrices is

$$p \times q = \begin{bmatrix} p_0 & - & a_1 p_1 & - & a_2 p_2 & - & a_3 p_3 \\ p_1 & & p_0 & - & p_3 \frac{\Delta}{a_1} & & p_2 \frac{\Delta}{a_1} \\ p_2 & & p_3 \frac{\Delta}{a_2} & & p_0 & - & p_1 \frac{\Delta}{a_2} \\ p_3 & - & p_2 \frac{\Delta}{a_3} & & p_1 \frac{\Delta}{a_3} & & p_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

[12].

Conclusion 2.8. The elliptical quaternion product of elliptical quaternions has no commutative property. That is, for $p, q \in \mathbb{H}_{a_1, a_2, a_3}(\varepsilon)$,

$$p \times q \neq q \times p.$$

Specially, for each $p = p_0 e_0, q = q_0 e_0 \in \mathbb{H}_{a_1, a_2, a_3}(\varepsilon)$, there exists commutative property [12].

Theorem 2.9. Let $p = p_1e_1 + p_2e_2 + p_3e_3$ and $q = q_1e_1 + q_2e_2 + q_3e_3 \in \mathbb{H}_{a_1,a_2,a_3}(\varepsilon)$ be elliptical quaternions. Expression of the elliptical quaternion products of $p = p_0e_0 + V_p$ and $q = q_0e_0 + V_q$ elliptical quaternions; with the help of elliptical inner product and elliptical vector product is

$$p \times q = p_0q_0 - \langle V_p, V_q \rangle_\varepsilon + p_0V_q + q_0V_p + (V_p \wedge V_q)_\varepsilon$$

[12].

Definition 2.10. The elliptical quaternion, which is $p_0 = 0$ (the real part of the elliptical quaternion), is called the pure elliptical quaternion [12].

Conclusion 2.11. The elliptical quaternion product of elliptical quaternions is

$$p \times q = -\langle V_p, V_q \rangle_\varepsilon + (V_p \wedge V_q)_\varepsilon$$

where p and q are the pure elliptical quaternions [12].

Definition 2.12. Let $p = p_0e_0 + p_1e_1 + p_2e_2 + p_3e_3 \in \mathbb{H}_{a_1,a_2,a_3}(\varepsilon)$ be elliptical quaternion. The conjugate, the norm and the inverse of the elliptical quaternion defined as

$$\begin{aligned} \bar{p} &= p_0e_0 - p_1e_1 - p_2e_2 - p_3e_3, \\ N_p &= \sqrt{p\bar{p}} = \sqrt{p_0^2 + p_1^2 + p_2^2 + p_3^2}, \\ p^{-1} &= \frac{\bar{p}}{(N_p)^2},\ (N_p)^2 \neq 0 \end{aligned}$$

respectively [12].

3 $\mathbb{H}_{a_1,a_2,a_3}(\varepsilon)$ algebra of elliptical quaternions

Theorem 3.1. In 3-dimensional Euclidean space; the set defined as

$$\begin{aligned} \mathbb{H}_{a_1,a_2,a_3}(\varepsilon) = \{ & q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3 \mid q_i \in \mathbb{R}, 0 \leq i \leq 3, e_1^2 = -a_1,\ e_2^2 = -a_2,\ e_3^2 = -a_3, \\ & e_1e_2e_3 = -\Delta\} \end{aligned}$$

where

$$e_1e_2 = \frac{\Delta}{a_3}e_3 = -e_2e_1,\ e_2e_3 = \frac{\Delta}{a_1}e_1 = -e_3e_2,\ e_3e_1 = \frac{\Delta}{a_2}e_2 = -e_1e_3$$

for the ellipsoid

$$\varepsilon : a_1x^2 + a_2y^2 + a_3z^2 = 1$$

such that $a_1, a_2, a_3 \in \mathbb{R}^+$ and $\sqrt{a_1a_2a_3} = \Delta$.

This set is a vector space on the field $(\mathbb{R}, +, .)$ with the addition and product by a scalar (Definition 2.3). This vector space will be denoted by $\mathbb{H}_{a_1,a_2,a_3}(\varepsilon)$ shortly.

Theorem 3.2. Let $\{\mathbb{H}_{a_1,a_2,a_3}(\varepsilon), \oplus, \mathbb{R}, +, ., \odot\}$ be the vector space of the elliptical quaternions. The quaternion product of elliptical quaternions defined on $\mathbb{H}_{a_1,a_2,a_3}(\varepsilon)$ provides the following properties.

Q1) For $\forall p,q \in \mathbb{H}_{a_1,a_2,a_3}(\varepsilon)$,

$$p \times q \in \mathbb{H}_{a_1,a_2,a_3}(\varepsilon),$$

Q2) For $\forall \lambda \in \mathbb{R}$, $\forall p,q \in \mathbb{H}_{a_1,a_2,a_3}(\varepsilon)$,

$$\lambda \odot (p \times q) = (a \odot p) \times q = p \times (\lambda \odot q),$$

Q3) For $\forall p,q,r \in \mathbb{H}_{a_1,a_2,a_3}(\varepsilon)$,

$$(p \oplus q) \times r = (p \times r) \oplus (q \times r),$$

$$p \times (q \oplus r) = (p \times q) \oplus (p \times r),$$

Q4) For $\forall p,q,r \in \mathbb{H}_{a_1,a_2,a_3}(\varepsilon)$,

$$(p \times q) \times r = p \times (q \times r).$$

Thus, $\{\mathbb{H}_{a_1,a_2,a_3}(\varepsilon), \oplus, \mathbb{R}, +, ., \odot, \times\}$ is an algebra.

Definition 3.3. This algebra over $(\mathbb{R}, +, .)$ field is called an elliptical quaternion algebra and it is denoted by $\mathbb{H}_{a_1,a_2,a_3}(\varepsilon)$.

One of the basis elliptical quaternion algebra is $\{1, e_1, e_2, e_3\}$.

4 Matrix representation of elliptical quaternion algebra elements

Let $p = p_0e_0 + p_1e_1 + p_2e_2 + p_3e_3$ and $q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3 \in \mathbb{H}_{a_1,a_2,a_3}(\varepsilon)$ be elliptic quaternions. The quaternion product of elliptical quaternions is defined as

$$\begin{aligned}
p \times q &= (p_0e_0 + p_1e_1 + p_2e_2 + p_3e_3) \times (q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3) \\
&= (p_0q_0 - a_1p_1q_1 - a_2p_2q_2 - a_3p_3q_3)\, e_0 \\
&+ \left(p_0q_1 + p_1q_0 + p_2q_3\frac{\Delta}{a_1} - p_3q_2\frac{\Delta}{a_1}\right) e_1 \\
&+ \left(p_0q_2 - p_1q_3\frac{\Delta}{a_2} + p_2q_0 + p_3q_1\frac{\Delta}{a_2}\right) e_2 \\
&+ \left(p_0q_3 + p_1q_2\frac{\Delta}{a_3} - p_2q_1\frac{\Delta}{a_3} + p_3q_0\right) e_3
\end{aligned}$$

and can be expressed with the help of a linear operator.

Let p be an elliptical quaternions $p = p_0e_0 + p_1e_1 + p_2e_2 + p_3e_3$,

$$\begin{array}{ccccccc}
L_p & : & \mathbb{H}_{a_1,a_2,a_3}(\varepsilon) & \overset{linear}{\rightarrow} & \mathbb{H}_{a_1,a_2,a_3}(\varepsilon) & & \\
 & & q & \rightarrow & L_p(q) & = & p \times q
\end{array}$$

in linear transformation corresponding to the base of $\{e_0 = 1, e_1, e_2, e_3\}$ blowe

$$\begin{aligned}
L_p(e_0) &= p \times e_0 \\
&= p_0 e_0 + p_1 e_1 + p_2 e_2 + p_3 e_3, \\
L_p(e_1) &= p \times e_1 \\
&= -a_1 p_1 e_0 + p_0 e_1 + p_3 \frac{\Delta}{a_2} e_2 - p_2 \frac{\Delta}{a_3} e_3, \\
L_p(e_2) &= p \times e_2 \\
&= -a_2 p_2 e_0 - p_3 \frac{\Delta}{a_1} e_1 + p_0 e_2 + p_1 \frac{\Delta}{a_3} e_3, \\
L_p(e_3) &= p \times e_3 \\
&= -a_3 p_3 e_0 + p_2 \frac{\Delta}{a_1} e_1 - p_1 \frac{\Delta}{a_2} e_2 + p_0 e_3
\end{aligned}$$

and the matrix representation of corresponding to the linear operatör L_p is denoted by $H^+(p)$,

$$H^+(p) = \begin{bmatrix} p_0 & -a_1 p_1 & -a_2 p_2 & -a_3 p_3 \\ p_1 & p_0 & -p_3 \frac{\Delta}{a_1} & p_2 \frac{\Delta}{a_1} \\ p_2 & p_3 \frac{\Delta}{a_2} & p_0 & -p_1 \frac{\Delta}{a_2} \\ p_3 & -p_2 \frac{\Delta}{a_3} & p_1 \frac{\Delta}{a_3} & p_0 \end{bmatrix}.$$

Thus, $p \times q$ elliptical quaternion product can be expressed as the form $H^+(p)\, q$ matrix product.

Actually,

$$\begin{aligned}
H^+(p)\, q &= \begin{bmatrix} p_0 & -a_1 p_1 & -a_2 p_2 & -a_3 p_3 \\ p_1 & p_0 & -p_3 \frac{\Delta}{a_1} & p_2 \frac{\Delta}{a_1} \\ p_2 & p_3 \frac{\Delta}{a_2} & p_0 & -p_1 \frac{\Delta}{a_2} \\ p_3 & -p_2 \frac{\Delta}{a_3} & p_1 \frac{\Delta}{a_3} & p_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \\
&= \begin{bmatrix} p_0 q_0 - a_1 p_1 q_1 - a_2 p_2 q_2 - a_3 p_3 q_3 \\ p_1 q_0 + p_0 q_1 - p_3 q_2 \frac{\Delta}{a_1} + p_2 q_3 \frac{\Delta}{a_1} \\ p_2 q_0 + p_3 q_1 \frac{\Delta}{a_2} + p_0 q_2 - p_1 q_3 \frac{\Delta}{a_2} \\ p_3 q_0 - p_2 q_1 \frac{\Delta}{a_3} + p_1 q_2 \frac{\Delta}{a_3} + p_0 q_3 \end{bmatrix} \\
&= p \times q.
\end{aligned}$$

Therefore,

$$H^+(p) = p_0 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + p_1 \begin{bmatrix} 0 & -a_1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\Delta}{a_2} \\ 0 & 0 & \frac{\Delta}{a_3} & 0 \end{bmatrix} + p_2 \begin{bmatrix} 0 & 0 & -a_2 & 0 \\ 0 & 0 & 0 & \frac{\Delta}{a_1} \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{\Delta}{a_3} & 0 & 0 \end{bmatrix} + p_3 \begin{bmatrix} 0 & 0 & 0 & -a_3 \\ 0 & 0 & -\frac{\Delta}{a_1} & 1 \\ 0 & \frac{\Delta}{a_2} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

where

$$E_0=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\quad E_1=\begin{bmatrix}0&-a_1&0&0\\1&0&0&0\\0&0&0&-\frac{\Delta}{a_2}\\0&0&\frac{\Delta}{a_3}&0\end{bmatrix},$$

$$E_2=\begin{bmatrix}0&0&-a_2&0\\0&0&0&\frac{\Delta}{a_1}\\1&0&0&0\\0&-\frac{\Delta}{a_3}&0&0\end{bmatrix},\quad E_3=\begin{bmatrix}0&0&0&-a_3\\0&0&-\frac{\Delta}{a_1}&1\\0&\frac{\Delta}{a_2}&0&0\\1&0&0&0\end{bmatrix}.$$

Matrix can be written by

$$H^{+}(p)=p_0E_0+p_1E_1+p_2E_2+p_3E_3$$

is obtain. Here, $E_0=I_4,E_1,E_2,E_3$ matrices in order corresponds to $e_0=1,e_1,e_2,e_3$ units. The properties shown below exist

$$E_1^2=-a_1I_4\ ,\ E_2^2=-a_2I_4,\ E_3^2=-a_3I_4,$$

$$E_1E_2=\frac{\Delta}{a_3}E_3=-E_2E_1,\ E_2E_3=\frac{\Delta}{a_1}E_1=-E_3E_2,\ E_3E_1=\frac{\Delta}{a_2}E_2=-E_1E_3.$$

Let p be an elliptical quaternions $p=p_0e_0+p_1e_1+p_2e_2+p_3e_3$,

$$\begin{array}{ccccccc}R_p&:&\mathbb{H}_{a_1,a_2,a_3}(\varepsilon)&\overset{linear}{\rightarrow}&\mathbb{H}_{a_1,a_2,a_3}(\varepsilon)&&\\q&&\rightarrow&&R_p(q)&=&q\times p\end{array}$$

in linear transformation corresponding to the base of $\{e_0=1,e_1,e_2,e_3\}$ blowe

$$\begin{aligned}R_p(e_0)&=e_0\times p\\&=p_0e_0+p_1e_1+p_2e_2+p_3e_3,\\R_p(e_1)&=e_1\times p\\&=-a_1p_1e_0+p_0e_1-p_3\frac{\Delta}{a_2}e_2+p_2\frac{\Delta}{a_3}e_3,\\R_p(e_2)&=e_2\times p\\&=-a_2p_2e_0+p_3\frac{\Delta}{a_1}e_1+p_0e_2-p_1\frac{\Delta}{a_3}e_3,\\R_p(e_3)&=e_3\times p\\&=-a_3p_3e_0-p_3\frac{\Delta}{a_1}e_1+p_1\frac{\Delta}{a_2}e_2+p_0e_3\end{aligned}$$

and the matrix representation of corresponding to the linear operatör R_p is denoted by $H^{-}(p)$,

$$H^{-}(p)=\begin{bmatrix}p_0&-a_1p_1&-a_2p_2&-a_3p_3\\p_1&p_0&p_3\frac{\Delta}{a_1}&-p_2\frac{\Delta}{a_1}\\p_2&-p_3\frac{\Delta}{a_2}&p_0&p_1\frac{\Delta}{a_2}\\p_3&p_2\frac{\Delta}{a_3}&-p_1\frac{\Delta}{a_3}&p_0\end{bmatrix}.$$

Thus, $q \times p$ elliptical quaternion product can be expressed as the form $H^{-}(p)\, q$ matrix product. Actually,

$$H^{-}(p)\, q = \begin{bmatrix} p_0 & -a_1 p_1 & -a_2 p_2 & -a_3 p_3 \\ p_1 & p_0 & p_3 \frac{\Delta}{a_1} & -p_2 \frac{\Delta}{a_1} \\ p_2 & -p_3 \frac{\Delta}{a_2} & p_0 & p_1 \frac{\Delta}{a_2} \\ p_3 & p_2 \frac{\Delta}{a_3} & -p_1 \frac{\Delta}{a_3} & p_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$= \begin{bmatrix} p_0 q_0 - a_1 p_1 q_1 - a_2 p_2 q_2 - a_3 p_3 q_3 \\ p_1 q_0 + p_0 q_1 + p_3 q_2 \frac{\Delta}{a_1} - p_2 q_3 \frac{\Delta}{a_1} \\ p_2 q_0 - p_3 q_1 \frac{\Delta}{a_2} + p_0 q_2 + p_1 q_3 \frac{\Delta}{a_2} \\ p_3 q_0 + p_2 q_1 \frac{\Delta}{a_3} - p_1 q_2 \frac{\Delta}{a_3} + p_0 q_3 \end{bmatrix}$$

$$= q \times p.$$

Therefore,

$$H^{-}(p) = p_0 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + p_1 \begin{bmatrix} 0 & -a_1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\Delta}{a_2} \\ 0 & 0 & -\frac{\Delta}{a_3} & 0 \end{bmatrix} + p_2 \begin{bmatrix} 0 & 0 & -a_2 & 0 \\ 0 & 0 & 0 & -\frac{\Delta}{a_1} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{\Delta}{a_3} & 0 & 0 \end{bmatrix} + p_3 \begin{bmatrix} 0 & 0 & 0 & -a_3 \\ 0 & 0 & \frac{\Delta}{a_1} & 1 \\ 0 & -\frac{\Delta}{a_2} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

where

$$F_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 & -a_1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\Delta}{a_2} \\ 0 & 0 & -\frac{\Delta}{a_3} & 0 \end{bmatrix},$$

$$F_2 = \begin{bmatrix} 0 & 0 & -a_2 & 0 \\ 0 & 0 & 0 & -\frac{\Delta}{a_1} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{\Delta}{a_3} & 0 & 0 \end{bmatrix}, \quad F_3 = \begin{bmatrix} 0 & 0 & 0 & -a_3 \\ 0 & 0 & \frac{\Delta}{a_1} & 1 \\ 0 & -\frac{\Delta}{a_2} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Matrix can be written by

$$H^{-}(p) = p_0 F_0 + p_1 F_1 + p_2 F_2 + p_3 F_3$$

where

$$F_1^2 = -a_1 I_4 \ , \ F_2^2 = -a_2 I_4 \ , \ F_3^2 = -a_3 I_4,$$

$$F_1 F_2 = \frac{\Delta}{a_3} F_3 = -F_2 F_1, \ F_2 F_3 = \frac{\Delta}{a_1} F_1 = -F_3 F_2, \ F_3 F_1 = \frac{\Delta}{a_2} F_2 = -F_1 F_3$$

is obtain. Here, $F_0 = I_4, F_1, F_2, F_3$ in order corresponds to $e_0 = 1, e_1, e_2, e_3$ units. The properties shown below exist.

Hence, H^{+} is a homomorphism but H^{-} is not a homomorphism. Actually,

i) $H^{+}(p \oplus q) = H^{+}(p) + H^{+}(q)$,
ii) $H^{+}(p \times q) = H^{+}(p)\ H^{+}(q)$,
iii) $H^{-}(p \oplus q) = H^{-}(p) + H^{-}(q)$,
iv) $H^{-}(p \times q) = H^{-}(q)\ H^{-}(p) \neq H^{-}(p)\ H^{-}(q)$.

Here, H^+ and H^-operators are similar to Hamilton operators [14].

The $H^+(p)$ and $H^-(p)$ matrices are called the Hamilton matrices corresponding to the elliptical quaternion p.

Let

$$M=\left\{\begin{bmatrix} p_0 & -a_1p_1 & -a_2p_2 & -a_3p_3 \\ p_1 & p_0 & -p_3\frac{\Delta}{a_1} & p_2\frac{\Delta}{a_1} \\ p_2 & p_3\frac{\Delta}{a_2} & p_0 & -p_1\frac{\Delta}{a_2} \\ p_3 & -p_2\frac{\Delta}{a_3} & p_1\frac{\Delta}{a_3} & p_0 \end{bmatrix} \mid p_i\in\mathbb{R}, a_i\in\mathbb{R}^+, 1\leq i\leq 3, \sqrt{a_1a_2a_3}=\Delta, e_1^2=-a_1,\ e_2^2=-a_2,\right.$$
$$\left. e_3^2=-a_3,\ e_1e_2e_3=-\Delta\right\}$$

be set of an elliptical quaternions. The transform

$$\begin{array}{rcccccc} H^+ & : & \mathbb{H}_{a_1,a_2,a_3}(\varepsilon) & \longrightarrow & M & & \\ & & p & \longrightarrow & H^+(p) & = & p\times q \end{array}$$

is a linear isomorphism.

Let

$$N=\left\{\begin{bmatrix} p_0 & -a_1p_1 & -a_2p_2 & -a_3p_3 \\ p_1 & p_0 & p_3\frac{\Delta}{a_1} & -p_2\frac{\Delta}{a_1} \\ p_2 & -p_3\frac{\Delta}{a_2} & p_0 & p_1\frac{\Delta}{a_2} \\ p_3 & p_2\frac{\Delta}{a_3} & -p_1\frac{\Delta}{a_3} & p_0 \end{bmatrix} \mid p_i\in\mathbb{R}, a_i\in\mathbb{R}^+, 1\leq i\leq 3, \sqrt{a_1a_2a_3}=\Delta, e_1^2=-a_1,\ e_2^2=-a_2,\right.$$
$$\left. e_3^2=-a_3,\ e_1e_2e_3=-\Delta\right\}$$

be set of an elliptical quaternions. The transform

$$\begin{array}{rcccccc} H^- & : & \mathbb{H}_{a_1,a_2,a_3}(\varepsilon) & \longrightarrow & N & & \\ & & p & \longrightarrow & H^-(p) & = & q\times p \end{array}$$

is a linear isomorphism.

The following equations are obtained from the operations of $H^+$and H^-

$$H^+(e_0)=H^-(e_0)=I_4$$
$$H^+(e_1)=E_1, H^+(e_2)=E_2, H^+(e_3)=E_3,$$
$$H^-(e_1)=F_1, H^-(e_2)=F_2, H^-(e_3)=F_3$$

where

$$H^{+}(e_0)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\quad H^{+}(e_1)=\begin{bmatrix}0&-a_1&0&0\\1&0&0&0\\0&0&0&-\frac{\Delta}{a_2}\\0&0&\frac{\Delta}{a_3}&0\end{bmatrix},$$

$$H^{+}(e_2)=\begin{bmatrix}0&0&-a_2&0\\0&0&0&\frac{\Delta}{a_1}\\1&0&0&0\\0&-\frac{\Delta}{a_3}&0&0\end{bmatrix},\quad H^{+}(e_3)=\begin{bmatrix}0&0&0&-a_3\\0&0&-\frac{\Delta}{a_1}&1\\0&\frac{\Delta}{a_2}&0&0\\1&0&0&0\end{bmatrix}$$

and

$$H^{-}(e_0)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\quad H^{-}(e_1)=\begin{bmatrix}0&-a_1&0&0\\1&0&0&0\\0&0&0&\frac{\Delta}{a_2}\\0&0&-\frac{\Delta}{a_3}&0\end{bmatrix},$$

$$H^{-}(e_2)=\begin{bmatrix}0&0&-a_2&0\\0&0&0&-\frac{\Delta}{a_1}\\1&0&0&0\\0&\frac{\Delta}{a_3}&0&0\end{bmatrix},\quad H^{-}(e_3)=\begin{bmatrix}0&0&0&-a_3\\0&0&\frac{\Delta}{a_1}&1\\0&-\frac{\Delta}{a_2}&0&0\\1&0&0&0\end{bmatrix}.$$

These matrices satisfy the equation (1). Moreover the properties of the matrices $H^{+}(e_0)$, $H^{+}(e_1)$, $H^{+}(e_2), H^{+}(e_3)$ and $H^{-}(e_0)$, $H^{-}(e_1), H^{-}(e_2), H^{-}(e_3)$ are similar to the $e_0=1, e_1, e_2, e_3$ units.

Since H^{+} and H^{-} are linear, we write

$$\begin{aligned}H^{+}(p)&=p_0H^{+}(e_0)+p_1H^{+}(e_1),+p_2H^{+}(e_2)+p_3H^{+}(e_3),\\&=p_0E_0+p_1E_1+p_2E_2+p_3E_3\end{aligned}$$

and

$$\begin{aligned}H^{-}(p)&=p_0H^{-}(e_0)+p_1H^{-}(e_1),+p_2H^{-}(e_2)+p_3H^{-}(e_3),\\&=p_0F_0+p_1F_1+p_2F_2+p_3F_3.\end{aligned}$$

Let p is unit elliptical quaternion and

$$\eta=\begin{bmatrix}1&0&0&0\\0&a_1&0&0\\0&0&a_2&0\\0&0&0&a_3\end{bmatrix}.$$

Thus, the following properties are satisfied for matrices produced from operators H^{+}, H^{-}

i) $\left(H^{+}(p)\right)^{T}\eta H^{+}(p)=\eta$

ii) $\left(H^{-}(p)\right)^{T}\eta H^{-}(p)=\eta.$

Conclusion 4.1. Matrices produced from operators H^+ and H^-, where p is a unit elliptical quaternion, are orthogonal.

Theorem 4.2. The following properties are provided for the $p,q,r,s \in \mathbb{H}_{a_1,a_2,a_3}(\varepsilon)$ elliptical quaternions.

$i)$ $p \times q = H^+(p)\, q = H^-(q)\, p,$

$ii)$ $H^+(p \times q) = H^+(H^+(p)\, q) = H^+(p)\ H^+(q),$

$iii)$ $H^-(p \times q) = H^-(H^-(p)\, q) = H^-(q)\ H^-(p),$

$iv)$ $H^+(\lambda p) = \lambda H^+(p), \quad \lambda \in \mathbb{R},$

$v)$ $H^-(\lambda p) = \lambda H^-(p), \quad \lambda \in \mathbb{R},$

$vi)$ $H^+(p \times q \oplus r \times s) = H^+(p)\ H^+(q) + H^+(r)\ H^+(s),$

$vii)$ $H^-(p \times q \oplus r \times s) = H^-(p)\ H^-(q) + H^-(r)\ H^-(s),$

$viii)$ $H^+(H^-(p)\, q) = H^+(q)\ H^+(p),$

$ix)$ $H^-(H^+(p)\, q) = H^-(q)\ H^-(p),$

$x)$ $H^+(p)\ H^-(q) = H^-(q)\ H^+(p),$

$xi)$ $H^+\left(p^{-1}\right) = H^+(p)^{-1}, \quad (N_p)^2 \neq 0,$

$xii)$ $H^-\left(p^{-1}\right) = H^-(p)^{-1}, \quad (N_p)^2 \neq 0,$

$xiii)$ $\det\ [H^{-+}(p)] = (N_p)^2 = \langle p,p\rangle^2,$

$xiv)$ $\det\ [H^-(p)] = (N_p)^2 = \langle p,p\rangle^2,$

$xv)$ $Iz\ [H^{-+}(p)] = 4p_0,$

$xvii)$ $Iz\ [H^-(p)] = 4p_0.$

5 Conclusion

Quaternions, which are four-dimensional number systems, have an important study field in the motions in kinematics and differential geometry on real numbers in three-dimensional space. Quaternions are very easy to show rotations in three-dimensional space. In this study, algebraic structure has been constructed on the field of real numbers of elliptical quaternions. Some algebraic properties of elliptical quaternions and Hamilton operators have been mentioned. The relation of elliptical

quaternions with Hamilton operators is given. Some of the application areas of quaternions are computer programming, numbers image processing, signal processing, image enhancement, animation, physics and biology. Therefore, quaternions create new fields of study for researchers.

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