Refinement of Berezin number inequalities with Kantorovich ratio

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Abstract

We construct Berezin number inequalities for Hilbert space operators using the notion of Kantorovich ratio and increasing convex function in this study. Also, we present new results related to Berezin number inequality. We give the Berezin number inequality by using operator version of improvement of arithmetic-geometric mean inequality.

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1 Introduction

Let $\mathbb{B}(\mathbb{H})$ define the C^*-algebra of bounded linear operators on a complex Hilbert space. Let $\mathbb{B}^{-1}(\mathbb{H})$ be the set of all invertible operators. The absolute of T called by $|T|$, that is, $|T| = (TT^*)^{1/2}$, where T^* is the selfadjoint operator of T. The alphabet I stands for the identity operator on $\mathbb{H}$. A reproducing kernel Hilbert space (RKHS in short) $\mathbb{H} = \mathbb{H}(\Omega)$ is a Hilbert space of all complex valued functions on a non-empty set Ω, which has the property that for every $\tau \in \Omega$ the map $\psi_\tau : \mathbb{H} \to \mathbb{C}$ defined by $\psi_\tau(f) = f(\tau)$, is continuous linear functional on $\mathbb{H}$. Throughout the article, a reproducing kernel Hilbert space on the set Ω is denoted by $\mathbb{H}$. Reproducing kernel Hilbert spaces have become an important tool in approximation theory, machine learning theory, theory of integral operators and they play a valuable role in complex analysis. Now, according to the Riesz representation theorem, for each $\tau \in \Omega$ there exits a unique function $k_\tau \in \mathbb{H}$ such that $f(\tau) = \langle f, k_\tau \rangle$ for all $f \in \mathbb{H}$. The collection of functions $\{k_\tau : \tau \in \Omega\}$ is defined the reproducing kernel of $\mathbb{H}$ and the collection of functions $\left\{\widehat{k}_\tau = \frac{k_\tau}{\|k_\tau\|} : \tau \in \Omega\right\}$ is defined the normalized reproducing kernel of $\mathbb{H}$. The function $\widetilde{T}$ defined on Ω by $\widetilde{T}(\tau) = \langle T\widehat{k}_\tau, \widehat{k}_\tau \rangle$, is called the Berezin transform (or symbol) of T. The Berezin range (or set) of T, denoted as $\mathrm{Ber}(T)$, is defined as $\mathrm{Ber}(T) = \mathrm{Range}(\widetilde{T}) = \{\widetilde{T}(\tau) : \tau \in \Omega\}$. The Berezin radius (or number) of T denoted as $\mathrm{ber}(T)$ is defined as

$$\mathrm{ber}(T) = \sup_{\tau \in \Omega} |\widetilde{T}(\tau)|.$$

The Berezin number of an operator T satisfies the following properties:

(i) $\mathrm{ber}(\alpha T) = |\alpha| \, \mathrm{ber}(T)$ for all $\alpha \in \mathbb{C}$

(ii) $\mathrm{ber}(T + R) \leq \mathrm{ber}(T) + \mathrm{ber}(R)$

The Berezin symbol is widely used in various questions involving unique operator and analysis. For further information about Berezin symbol we refer the reader to [6, 7, 16, 17] and references therein.

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For $T \in \mathbb{B}(\mathbb{H})$, the numerical range of T, denoted as $W(T)$, is the collection of complex scalars $\langle Tx, x\rangle$ for $x \in \mathbb{H}$ with $\|x\| = 1$. More precisely,

$$W(T) = \{\langle Tx, x\rangle : x \in \mathbb{H} \text{ and } \|x\| = 1\}.$$

The numerical radius of T is denoted by $w(T)$. Recall that

$$w(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

It is clear from definitions that $\widetilde{T}$ is a bounded function, $\mathrm{Ber}(T)$ lies in the numerical range $W(T)$, and so $\mathrm{ber}(T)$ does not exceed the numerical radius $w(T)$ of operator T.

Also, it is clear that for $T \in \mathbb{B}(\mathbb{H})$,

$$\mathrm{Ber}(T) \subseteq W(T) \text{ and } \mathrm{ber}(T) \leq w(T).$$

For more detailed information on numerical radius, references to [10, 12, 20, 23, 24, 25] are recommended.

In particular, for all $T \in \mathbb{B}(\mathbb{H})$, the following inequalities hold

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\| \tag{1.1}$$

and

$$\mathrm{ber}(T) \leq w(T) \leq \|T\|. \tag{1.2}$$

Huban et al. [13, 14, 15] have proved the following inequalities:

$$\mathrm{ber}(T) \leq \frac{1}{2}\left(\|T\|_{\mathrm{ber}} + \|T^*\|_{\mathrm{ber}}\right) \leq \frac{1}{2}\left(\|T\|_{\mathrm{ber}} + \left\|T^2\right\|_{\mathrm{ber}}^{1/2}\right) \leq \|T\|_{\mathrm{ber}} \tag{1.3}$$

and

$$\mathrm{ber}^2(T) \leq \frac{1}{2}\left\||T|^2 + |T^*|^2\right\|_{\mathrm{ber}} \leq \|T\|_{\mathrm{ber}}^2. \tag{1.4}$$

for $T \in \mathbb{B}(\mathbb{H})$. Also, they showed that if $T, R \in \mathbb{B}(\mathbb{H})$, then

$$\mathrm{ber}(T^*R) \leq \frac{1}{2}\|TT^* + R^*R\|_{\mathrm{ber}}. \tag{1.5}$$

Bakherad and Garayev [2] proved that

$$\mathrm{ber}^r(T^*XR) \leq \frac{1}{2}\|(T^*|X^*|T)^r + (R^*|X|R)^r\|_{\mathrm{ber}}. \tag{1.6}$$

The Young inequality is defined by

$$a^v b^{1-v} \leq va + (1-v)b \tag{1.7}$$

for positive number a, b and $v \in [0, 1]$. In (1.7), the inequality holds iff $a = b$. The inequality (1.7) is defined the v-weighted aritmetic geometric mean inequality. For the special case, when $v = \frac{1}{2}$, we have

$$\sqrt{ab} \leq \frac{a+b}{2},$$

which is called the fundamental arithmetic-geometric mean inequality.

Now, we need the following definition in the third section.

Definition 1.1. Let $\mathbb{B}^{+}(\mathbb{H})$ define the set of all the positive operators in $\mathbb{B}(\mathbb{H})$. Let $T \in \mathbb{B}^{+}(\mathbb{H})$, let R be a self-adjoint operator, and let f be a continuos function on a real interval J containing $sp\left(T^{-\frac{1}{2}} R T^{-\frac{1}{2}}\right)$. Then by using the continuous functional calculus, the f-connection is denoted by σ_f and defined as

$$T\sigma_f R = T^{\frac{1}{2}} f\left(T^{-\frac{1}{2}} R T^{-\frac{1}{2}}\right) T^{\frac{1}{2}}.$$

Note that the above definition for the functions t^{υ} and $(1-\upsilon)+\upsilon t$, where $\upsilon \in [0,1]$, leads to the operator weighted geometric mean and the operator weighted arithmetic mean, respectively.

The Kantorovich's ratio is defined by

$$K(t,2) = \frac{(t+1)^2}{4t}$$

for a positive real number t. The Kantorovich's ratio has the following properties:

(i) $K(1,2) = 1$.

(ii) $K(t,2) = K\left(\frac{1}{t},2\right) > 1$.

(iii) $K(t,2)$ is a monotone function on the interval $[1,\infty)$ and monotone decreasing on the interval $[0,1]$.

Zou et al. [28] demonstrated the following conclusion in 2011, which is a revised version of the traditional Young inequality in terms of the Kantorovich constant.

For $a, b > 0$ and $v \in [0,1]$,

$$K(t,2)\, a^{v} b^{1-v} \leq va + (1-v)\, b,$$

where $r = \min\{v, 1-v\}$, $0 \leq v \leq 1$. Taking $v = \frac{1}{2}$, we find

$$\sqrt{ab} \leq \frac{1}{\sqrt{K(t,2)}} \left(\frac{a+b}{2}\right), \tag{1.8}$$

where $t = \frac{b}{a}$.

It should be noted that the well-known idea of operator mean was established by Kudo and Ando in article [21].

Let $\mathbb{B}(\mathbb{H})^{++}$ denote the set of all strictly positive operators in $\mathbb{B}(\mathbb{H})$. For $T, R \in \mathbb{B}(\mathbb{H})^{++}$ and $0 \leq v \leq 1$, the v-weighted geometric mean $T\sharp_v R$, the v-arithmetic mean $T\nabla_v R$ are denoted, respectively, by

$$T\sharp_v R = T^{\frac{1}{2}} \left(T^{-\frac{1}{2}} R T^{-\frac{1}{2}}\right)^{v} T^{\frac{1}{2}},$$
$$T\nabla_v R = (1-v)\, T + vR.$$

For convenience, we define $T\sharp R$ and $T\nabla R$ for the particular situation, where $v = \frac{1}{2}$.

$$T\sharp R \leq T\nabla R \tag{1.9}$$

was found in an operator version of (1.7).

Linear operators induced by functions are frequently encountered in functional analysis; they include Hankel operators, composition operators, and Toeplitz operators. The inducing function

is sometimes referred to as the symbol of the resultant operator. In many circumstances, a linear operator on a Hilbert space $\mathbb{H}$ also gives rise to a function on a subset of a topological space. Hence, we frequently examine operators induced by functions, and we may similarly research functions induced by operators. The Berezin symbol is an outstanding example of an operator-function link. F. Berezin proposed the Berezin transform in [6] and it has proven to be a fundamental tool in operator theory, since many essential features of significant operators are contained in their Berezin transforms. The Berezin symbol and Berezin number has large application in the study of various questions of operator theory in the functional Hilbert space, quantum physics and non-commutative geometry. These are the important tools to study operators on Hardy and Bergman spaces, especially for Toeplitz and Hankel operators. Recall that the Hardy space $\mathbb{H}^2(\mathbb{D})$ of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is a RKHS of analytic functions on $\mathbb{D}$ with reproducing kernel $k_\tau(z) = 1/(1-\overline{\tau}z)$ (see, Paulsen and Raghupati [26]). Since, the collection of normalized reproducing kernel of $\mathbb{H}$ is a subset of the unit sphere of $\mathbb{H}$, so the numerical radius and the Berezin number of an operator on $\mathbb{H}$ may not be equal. Thus it is natural to ask, when the numerical radius is equal to the Berezin number? Note that if there exists a sequence $\{\tau_n\}$ in Ω such that $\lim_{n\to\infty}\left|\widetilde{T}(\tau_n)\right| = w(T)$, then $\mathrm{ber}(T) = w(T)$. The Berezin number inequalities have been studied by many mathematicians over the years, interested readers can see [3, 4, 5, 8, 9, 11, 27]. In this paper, we obtain Berezin radius inequalities for Hilbert space operators using the notion of Kantorovich ratio and increasing convex function. Also, we present new results related to Berezin radius inequality. We give the Berezin radius inequality by using operator version of improvement of arithmetic-geometric mean inequality.

2 Preliminaries

We start with the following lemmas that will be used to develop new results in this study. Kittaneh [18] proved the first lemma, known as the generalized mixed Schwarz inequality.

Lemma 2.1. (i) Let $T \in \mathbb{B}(\mathbb{H})$ and for any $x, y \in \mathbb{H}$. If f and g is non-negative continuos functions on $[0,\infty)$ satisfying $f(t)\,g(t) = t$ $(t \geq 0)$, then

$$|\langle Tx, y\rangle| \leq \sqrt{\|f(|T|)\,x\|\,\|g(|T^*|)\,y\|}. \tag{2.1}$$

(ii) In particular,

$$|\langle Tx, y\rangle| \leq \left\langle |T|^{2\alpha} x, x\right\rangle^{\frac{1}{2}} \left\langle |T^*|^{2(1-\alpha)} y, y\right\rangle^{\frac{1}{2}}, \ (0 \leq \alpha \leq 1). \tag{2.2}$$

The second lemma is proved by Mond and Pečarić in [22].

Lemma 2.2. Let $T \in \mathbb{B}(\mathbb{H})$ be a self-adjoint operator with spectrum contained in the interval J, and $x \in \mathbb{H}$ be a unit vector. If f is a convex function on J, then

$$f(\langle Tx, x\rangle) \leq \langle f(T)\,x, x\rangle. \tag{2.3}$$

The inequality above is reversed if f is concave.

Lemma 2.3. If $f : [0,d] \to [0,\infty]$, $d > 0$, is an increasing convex function with $f(0) = 0$ and $\alpha \in [0,1]$, then we have

$$f(\alpha x) = \alpha f(x). \tag{2.4}$$

The fourth lemma is showed by Aujla and Silva [1].

Lemma 2.4. Let f be a non-negative increasing convex function on $[0,\infty)$ and let $T,R\in\mathbb{B}(\mathbb{H})$ be positive operators. Then

$$\|f((1-\upsilon)T+\upsilon R)\|\leq\|(1-\upsilon)f(T)+\upsilon f(R)\| \tag{2.5}$$

for every $0\leq\upsilon\leq 1$.

The fifth lemma is presented by Kittaneh in [19].

Lemma 2.5. Let $T,R\in\mathbb{B}(\mathbb{H})$ be positive operators, then

$$\|T+R\|\leq\frac{1}{2}\left(\|T\|+\|R\|+\sqrt{(\|T\|+\|R\|)^2+4\left\|T^{\frac{1}{2}}R^{\frac{1}{2}}\right\|^2}\right).$$

Kittaneh also showed that $\left\|T^{\frac{1}{2}}R^{\frac{1}{2}}\right\|\leq\|TR\|^{\frac{1}{2}}$ for positive operators $T,R\in\mathbb{B}(\mathbb{H})$ (see, [19, Lemma 2]).

The last lemma is operators version for improvement of arithmetic-geometric mean inequalities.

Lemma 2.6 ([23])**.** Let $T,R\in\mathbb{B}(\mathbb{H})^{++}$ be such that $mR\leq T$ with $m>1$. Then

$$K(m,2)\,T\sharp R\leq T\nabla R$$

or

$$K(m,2)\,T^{\frac{1}{2}}\left(T^{-\frac{1}{2}}RT^{-\frac{1}{2}}\right)^{\frac{1}{2}}T^{\frac{1}{2}}\leq\frac{R+T}{2}. \tag{2.6}$$

3 Main results

Let us first demonstrate our conclusions for the convex function, then illustrate how they connect to the current ones.

Theorem 3.1. Let $\mathbb{H}=\mathbb{H}(\Omega)$ be an RKHS. Let $T\in\mathbb{B}^{-1}(\mathbb{H})$ and let f be an increasing convex function with $f(0)=0$. If $m|T|\leq|T^*|$ or $m|T^*|\leq|T|$ with $m>1$, then

$$f(\mathrm{ber}(T))\leq\frac{1}{2\sqrt{K(m,2)}}\|f(|T|)+f(|T^*|)\|_{\mathrm{ber}},$$

where $K(m,2)=\frac{(m+1)^2}{4m}$.

Proof. Let $\widehat{k}_\tau$ be a normalized reproducing kernel in $\mathbb{H}$. Using (ii) of Lemma 2.1 for $\alpha=\frac{1}{2}$ and

monotoncity convexity of f, we have

$$\begin{aligned}
f\left(\left|\left\langle T\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|\right) &\leq f\left(\sqrt{\left\langle |T|\,\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\left\langle |T^*|\,\widehat{k}_\tau,\widehat{k}_\tau\right\rangle}\right)\\
&\leq f\left(\frac{1}{\sqrt{K\left(t,2\right)}}\left(\frac{\left\langle |T|\,\widehat{k}_\tau,\widehat{k}_\tau\right\rangle+\left\langle |T^*|\,\widehat{k}_\tau,\widehat{k}_\tau\right\rangle}{2}\right)\right) \text{ (by (1.8))}\\
&\leq \frac{1}{\sqrt{K\left(t,2\right)}}f\left(\frac{\left\langle |T|\,\widehat{k}_\tau,\widehat{k}_\tau\right\rangle+\left\langle |T^*|\,\widehat{k}_\tau,\widehat{k}_\tau\right\rangle}{2}\right) \text{ (by (2.4))}\\
&\leq \frac{1}{2\sqrt{K\left(t,2\right)}}\left(f\left(\left\langle |T|\,\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right)+f\left(\left\langle |T^*|\,\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right)\right)\\
&\leq \frac{1}{2\sqrt{K\left(t,2\right)}}\left(\left\langle f\left(|T|\right)\widehat{k}_\tau,\widehat{k}_\tau\right\rangle+\left\langle f\left(|T^*|\right)\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right) \text{ (by (2.3))}\\
&\leq \frac{1}{2\sqrt{K\left(m,2\right)}}\left\langle \left(f\left(|T|\right)+f\left(|T^*|\right)\right)\widehat{k}_\tau,\widehat{k}_\tau\right\rangle,
\end{aligned}$$

and

$$f\left(\left|\left\langle T\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|\right)\leq \frac{1}{2\sqrt{K\left(m,2\right)}}\left\langle \left(f\left(|T|\right)+f\left(|T^*|\right)\right)\widehat{k}_\tau,\widehat{k}_\tau\right\rangle,$$

where $t=\frac{\left\langle |T|\widehat{k}_\tau,\widehat{k}_\tau\right\rangle}{\left\langle |T^*|\widehat{k}_\tau,\widehat{k}_\tau\right\rangle}$ and $K\left(m,2\right)=\frac{(m+1)^2}{4m}$. Taking the supremum over $\tau\in\Omega$ in the above inequality, we have

$$\sup_{\tau\in\Omega} f\left(\left|\left\langle T\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|\right)\leq \sup_{\tau\in\Omega}\left(\frac{1}{2\sqrt{K\left(m,2\right)}}\left(\left\langle \left(f\left(|T|\right)+f\left(|T^*|\right)\right)\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right)\right).$$

Thus, we get

$$f\left(\mathrm{ber}(T)\right)\leq \frac{1}{2\sqrt{K\left(m,2\right)}}\left\|f\left(|T|\right)+f\left(|T^*|\right)\right\|_{\mathrm{ber}}.$$

Then the desired result has been obtained. Q.E.D.

Considering $f\left(t\right)=t^r$, $r\geq 1$, in Theorem 3.1 we infer the following corollary.

Corollary 3.2. Let $T\in\mathbb{B}^{-1}\left(\mathbb{H}\right)$ be such that $m\left|T\right|\leq\left|T^*\right|$ or $m\left|T^*\right|\leq\left|T\right|$ with $m>1$. Then

$$\mathrm{ber}^r(T)\leq \frac{1}{2\sqrt{K\left(m,2\right)}}\left\|\left|T\right|^r+\left|T^*\right|^r\right\|_{\mathrm{ber}} \text{ (for } r\geq 1).$$

In particular,

$$\begin{aligned}
\mathrm{ber}(T) &\leq \frac{1}{2\sqrt{K\left(m,2\right)}}\left\|\left|T\right|+\left|T^*\right|\right\|_{\mathrm{ber}}\leq \frac{1}{2\sqrt{K\left(m,2\right)}}\left(\left\|T\right\|_{\mathrm{ber}}+\left\|T^*\right\|_{\mathrm{ber}}\right)\\
&\leq \frac{1}{2\sqrt{K\left(m,2\right)}}\left(\left\|T\right\|_{\mathrm{ber}}+\left\|T^2\right\|_{\mathrm{ber}}^{1/2}\right) \qquad (3.1)
\end{aligned}$$

and

$$\operatorname{ber}^2(T) \leq \frac{1}{2\sqrt{K(m,2)}} \left\| |T|^2 + |T^*|^2 \right\|_{\operatorname{ber}}, \tag{3.2}$$

where $K(m,2) = \frac{(m+1)^2}{4m}$.

Remark 3.3. The inequalities (3.1) and (3.2) are refinement of inequalities (1.3) and (1.4), respectively.

Theorem 3.4. Let $\mathbb{H} = \mathbb{H}(\Omega)$ be a RKHS. Let $T, S \in \mathbb{B}^{-1}(\mathbb{H})$ and let f be an increasing convex function with $f(0) = 0$. If $m^2 T^*T \leq S^*S$ or $m^2 S^*S \leq T^*T$ with $m > 1$, then

$$f(\operatorname{ber}(S^*T)) \leq \frac{1}{2\sqrt{K(m,2)}} \left\| f\left(|T|^2\right) + f\left(|S|^2\right) \right\|_{\operatorname{ber}},$$

where $K(m,2) = \frac{(m+1)^2}{4m}$.

Proof. Let $\widehat{k}_\tau$ be a normalized reproducing kernel in $\mathbb{H}$. Then from Cauchy-Schwarz inequality and monotoncity and convexity of f, we have

$$\begin{aligned}
f\left(\left|\left\langle S^*T\widehat{k}_\tau, \widehat{k}_\tau \right\rangle\right|\right) &\leq f\left(\left|\left\langle T\widehat{k}_\tau, S\widehat{k}_\tau \right\rangle\right|\right) \\
&\leq f\left(\left\|T\widehat{k}_\tau\right\|_{\operatorname{ber}} \left\|S\widehat{k}_\tau\right\|_{\operatorname{ber}}\right) \\
&\leq f\left(\left\langle |T|^2 \widehat{k}_\tau, \widehat{k}_\tau \right\rangle^{1/2} \left\langle |S|^2 \widehat{k}_\tau, \widehat{k}_\tau \right\rangle^{1/2}\right) \\
&\leq f\left(\frac{1}{\sqrt{K(t,2)}} \left(\frac{\left\langle |T|^2 \widehat{k}_\tau, \widehat{k}_\tau \right\rangle + \left\langle |S|^2 \widehat{k}_\tau, \widehat{k}_\tau \right\rangle}{2}\right)\right) \text{ (by (1.8))} \\
&\leq \frac{1}{\sqrt{K(t,2)}} f\left(\left(\frac{\left\langle |T|^2 \widehat{k}_\tau, \widehat{k}_\tau \right\rangle + \left\langle |S|^2 \widehat{k}_\tau, \widehat{k}_\tau \right\rangle}{2}\right)\right) \text{ (by (2.4))} \\
&\leq \frac{1}{2\sqrt{K(t,2)}} \left(f\left(\left\langle |T|^2 \widehat{k}_\tau, \widehat{k}_\tau \right\rangle\right) + f\left(\left\langle |S|^2 \widehat{k}_\tau, \widehat{k}_\tau \right\rangle\right)\right) \\
&\leq \frac{1}{2\sqrt{K(t,2)}} \left(\left\langle f\left(|T|^2\right) \widehat{k}_\tau, \widehat{k}_\tau \right\rangle + \left\langle f\left(|S|^2\right) \widehat{k}_\tau, \widehat{k}_\tau \right\rangle\right) \text{ (by (2.3))} \\
&\leq \frac{1}{2\sqrt{K(m,2)}} \left\langle \left(f\left(|T|^2\right) + f\left(|S|^2\right)\right) \widehat{k}_\tau, \widehat{k}_\tau \right\rangle,
\end{aligned}$$

and

$$f\left(\left|\left\langle S^*T\widehat{k}_\tau, \widehat{k}_\tau \right\rangle\right|\right) \leq \frac{1}{2\sqrt{K(m,2)}} \left\langle \left(f\left(|T|^2\right) + f\left(|S|^2\right)\right) \widehat{k}_\tau, \widehat{k}_\tau \right\rangle,$$

where $t = \frac{\left\langle |T|^2 \widehat{k}_\tau, \widehat{k}_\tau \right\rangle}{\left\langle |S|^2 \widehat{k}_\tau, \widehat{k}_\tau \right\rangle}$ and $K(m,2) = \frac{(m+1)^2}{4m}$. Taking the supremum over $\tau \in \Omega$ in the above inequality, we have

$$\sup_{\tau\in\Omega} f\left(\left|\left\langle S^*T\widehat{k}_\tau, \widehat{k}_\tau \right\rangle\right|\right) \leq \sup_{\tau\in\Omega} \left(\frac{1}{2\sqrt{K(m,2)}} \left\langle \left(f\left(|T|^2\right) + f\left(|S|^2\right)\right) \widehat{k}_\tau, \widehat{k}_\tau \right\rangle\right).$$

Hence, we have

$$f\left(\mathrm{ber}(S^*T)\right) \leq \frac{1}{2\sqrt{K\left(m,2\right)}} \left\| f\left(|T|^2\right) + f\left(|S|^2\right) \right\|_{\mathrm{ber}},$$

as desired. Q.E.D.

Considering $f\left(t\right) = t^r$, $r \geq 1$, on $[0,\infty)$ in Theorem 3.4 we infer the following corollary.

Corollary 3.5. Let $T, S \in \mathbb{B}^{-1}\left(\mathbb{H}\right)$. If $m^2T^*T \leq S^*S$ or $m^2S^*S \leq T^*T$ with $m > 1$, then

$$\mathrm{ber}^r(S^*T) \leq \frac{1}{2\sqrt{K\left(m,2\right)}} \left\| |T|^{2r} + |S|^{2r} \right\|_{\mathrm{ber}}, \tag{3.3}$$

where $K\left(m,2\right) = \frac{(m+1)^2}{4m}$. In particular,

$$\mathrm{ber}(S^*T) \leq \frac{1}{2\sqrt{K\left(m,2\right)}} \left\| TT^* + S^*S \right\|_{\mathrm{ber}}, \tag{3.4}$$

where $K\left(m,2\right) = \frac{(m+1)^2}{4m}$.

Remark 3.6. Inequality (3.4) is an refinement of inequality (1.5).

We use the Lemma 2.3 to the following theorem.

Theorem 3.7. Let $\mathbb{H} = \mathbb{H}\left(\Omega\right)$ be a RKHS. Let $T, R, X \in \mathbb{B}^{-1}\left(\mathbb{H}\right)$ and let f and g be non-negative continuous on $[0,\infty)$ satisfying $f\left(t\right)g\left(t\right) = t$ for all $t \in [0,\infty)$, and $h : [0,\infty) \to [0,\infty)$ be a non-negative increasing convex function such that $h\left(0\right) = 0$. If $m^4T^*g^2\left(|X^*|\right)T \leq R^*f^2\left(|X|\right)R$ or $m^4R^*f^2\left(|X|\right)R \leq T^*g^2\left(|X^*|\right)T$ with $m > 1$, then

$$h\left(\mathrm{ber}(T^*XR)\right) \leq \frac{1}{2\sqrt{K\left(m,2\right)}} \left\| h\left(R^*f^2\left(|X|\right)R\right) + h\left(T^*g^2\left(|X^*|\right)T\right) \right\|_{\mathrm{ber}},$$

where $K\left(m,2\right) = \frac{(m+1)^2}{4m}$.

Proof. Let $\tau \in \Omega$ be an arbitrary number. Using the (i) of Lemma 2.1, we get

$$\left|\left\langle T^*XR\widehat{k}_\tau, \widehat{k}_\tau \right\rangle\right| \leq \left|\left\langle XR\widehat{k}_\tau, T\widehat{k}_\tau \right\rangle\right| \leq \sqrt{\left\langle R^*f^2\left(|X|\right)R\widehat{k}_\tau, \widehat{k}_\tau \right\rangle \left\langle T^*g^2\left(|X^*|\right)T\widehat{k}_\tau, \widehat{k}_\tau \right\rangle}. \tag{3.5}$$

From inequality (1.8), we have

$$\begin{aligned}
&\sqrt{\left\langle R^*f^2\left(|X|\right)R\widehat{k}_\tau, \widehat{k}_\tau \right\rangle \left\langle T^*g^2\left(|X^*|\right)T\widehat{k}_\tau, \widehat{k}_\tau \right\rangle} \\
&\leq \frac{1}{2\sqrt{K\left(t,2\right)}} \left(\left\langle R^*f^2\left(|X|\right)R\widehat{k}_\tau, \widehat{k}_\tau \right\rangle + \left\langle T^*g^2\left(|X^*|\right)T\widehat{k}_\tau, \widehat{k}_\tau \right\rangle \right) \\
&\leq \frac{1}{2\sqrt{K\left(m,2\right)}} \left(\left\langle R^*f^2\left(|X|\right)R\widehat{k}_\tau, \widehat{k}_\tau \right\rangle + \left\langle T^*g^2\left(|X^*|\right)T\widehat{k}_\tau, \widehat{k}_\tau \right\rangle \right),
\end{aligned}$$

where $t=\frac{\left\langle R^{*}f^{2}(|X|)R\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle}{\left\langle T^{*}g^{2}(|X^{*}|)T\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle}$ and $K\left(m,2\right)=\frac{(m+1)^{2}}{4m}$.

It follows from last inequality and (3.5) that

$$\left|\left\langle T^{*}XR\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle\right|\leq\frac{1}{2\sqrt{K\left(m,2\right)}}\left(\left\langle R^{*}f^{2}\left(|X|\right)R\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle+\left\langle T^{*}g^{2}\left(|X^{*}|\right)T\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle\right),$$

where $K\left(m,2\right)=\frac{(m+1)^{2}}{4m}$. Taking the supremum over $\tau\in\Omega$ in the above inequality, we have

$$\mathrm{ber}(T^{*}XR)\leq\frac{1}{2\sqrt{K\left(m,2\right)}}\left\|R^{*}f^{2}\left(|X|\right)R+T^{*}g^{2}\left(|X^{*}|\right)T\right\|_{\mathrm{ber}},$$

where $K\left(m,2\right)=\frac{(m+1)^{2}}{4m}$. Also

$$\begin{aligned}h\left(\mathrm{ber}(T^{*}XR)\right)&\leq h\left(\frac{2}{2\sqrt{K\left(m,2\right)}}\left\|\frac{R^{*}f^{2}\left(|X|\right)R+T^{*}g^{2}\left(|X^{*}|\right)T}{2}\right\|_{\mathrm{ber}}\right)\\&\leq\frac{1}{\sqrt{K\left(m,2\right)}}h\left(\left\|\frac{R^{*}f^{2}\left(|X|\right)R+T^{*}g^{2}\left(|X^{*}|\right)T}{2}\right\|_{\mathrm{ber}}\right)\ \text{(by (2.4))}\\&\leq\frac{1}{\sqrt{K\left(m,2\right)}}\left\|h\left(\frac{R^{*}f^{2}\left(|X|\right)R+T^{*}g^{2}\left(|X^{*}|\right)T}{2}\right)\right\|_{\mathrm{ber}}\\&\text{(by non-negative increasing convex function)}\\&\leq\frac{1}{2\sqrt{K\left(m,2\right)}}\left\|h\left(R^{*}f^{2}\left(|X|\right)R\right)+h\left(T^{*}g^{2}\left(|X^{*}|\right)T\right)\right\|_{\mathrm{ber}}\ \text{(by (2.5))},\end{aligned}$$

where $\frac{1}{2\sqrt{K(m,2)}}\leq1$ and $K\left(m,2\right)=\frac{(m+1)^{2}}{4m}$. From the Jensen inequality, we get

$$h\left(\left\|K\right\|_{\mathrm{ber}}\right)=h\left(\sup_{\tau\in\Omega}\left\langle K\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle\right)\leq\sup_{\tau\in\Omega}\left\langle h\left(K\right)\widehat{k}_{\tau},\widehat{k}_{\tau}\right\rangle=\left\|h\left(K\right)\right\|_{\mathrm{ber}}$$

for each positive operator $K\in\mathbb{B}^{-1}\left(\mathbb{H}\right)$. This completes the proof. Q.E.D.

Corollary 3.8. We know that $f\left(t\right)=t^{r}$, $r\geq1$ is an increasing convex function on $[0,\infty)$. Let $T,R,X\in\mathbb{B}^{-1}\left(\mathbb{H}\right)$. If $m^{4}T^{*}g^{2}\left(|X^{*}|\right)T\leq R^{*}f^{2}\left(|X|\right)R$ or $m^{4}R^{*}f^{2}\left(|X|\right)R\leq T^{*}g^{2}\left(|X^{*}|\right)T$ with $m>1$, then

$$\mathrm{ber}^{r}(T^{*}XR)\leq\frac{1}{2\sqrt{K\left(m,2\right)}}\left\|\left(R^{*}\left|X\right|R\right)^{r}+\left(T^{*}\left|X^{*}\right|T\right)^{r}\right\|_{\mathrm{ber}},\tag{3.6}$$

where $K\left(m,2\right)=\frac{(m+1)^{2}}{4m}$. In particular

$$\mathrm{ber}(T^{*}XR)\leq\frac{1}{2\sqrt{K\left(m,2\right)}}\left\|R^{*}\left|X\right|R+T^{*}\left|X^{*}\right|T\right\|_{\mathrm{ber}},$$

where $K\left(m,2\right)=\frac{(m+1)^{2}}{4m}$.

The inequality (3.6) given above is the refinement of the inequality (1.6).

Remark 3.9. In Corollary 3.8,

(i) If $X = I$, then the inequality (3.6) is equivalent to inequality (3.3), i.e.,

$$\operatorname{ber}^r(T^*R) \leq \frac{1}{2\sqrt{K(m,2)}} \left\| |R|^{2r} + |T|^{2r} \right\|_{\operatorname{ber}},$$

where $K(m,2) = \frac{(m+1)^2}{4m}$.

(ii) If $T = R = I$, then we obtain the Corollary 3.2, i.e.,

$$\operatorname{ber}^r(X) \leq \frac{1}{2\sqrt{K(m,2)}} \left\| |X|^{2r} + |X^*|^{2r} \right\|_{\operatorname{ber}},$$

where $K(m,2) = \frac{(m+1)^2}{4m}$.

Theorem 3.10. Let $\mathbb{H} = \mathbb{H}(\Omega)$ be a RKHS. Let $T, R, X \in \mathbb{B}^{-1}(\mathbb{H})$ such that R is a selfadjoint and $f : [0,\infty) \to [0,\infty)$ be an increasing convex function such that $f(0) = 0$. If

$$m^5 X^* T^{1/2} f^2\left(T^{-1/2} R T^{-1/2}\right) T^{1/2} X \leq T$$

or $m^5 T \leq X^* T^{1/2} f^2\left(T^{-1/2} R T^{-1/2}\right) T^{1/2} X$ with $m > 1$, then

$$\operatorname{ber}((T\sigma_f R) X) \leq \frac{1}{2\sqrt{K(m,2)}} \left\| X^* T^{1/2} f^2\left(T^{-1/2} R T^{-1/2}\right) T^{1/2} X + T \right\|_{\operatorname{ber}},$$

where $K(m,2) = \frac{(m+1)^2}{4m}$.

Proof. For $\tau \in \Omega$, we have

$$\begin{aligned}
&\left|\left\langle ((T\sigma_f R) X)\widehat{k}_\tau, \widehat{k}_\tau \right\rangle\right| \\
&= \left|\left\langle T^{1/2} f\left(T^{-1/2} R T^{-1/2}\right) T^{1/2} X\widehat{k}_\tau, \widehat{k}_\tau \right\rangle\right| \\
&\leq \left|\left\langle f\left(T^{-1/2} R T^{-1/2}\right) T^{1/2} X\widehat{k}_\tau, T^{1/2}\widehat{k}_\tau \right\rangle\right| \\
&\leq \left\| f\left(T^{-1/2} R T^{-1/2}\right) T^{1/2} X\widehat{k}_\tau \right\|_{\operatorname{ber}} \left\| T^{1/2}\widehat{k}_\tau \right\|_{\operatorname{ber}} \\
&\leq \sqrt{\left\langle f\left(T^{-1/2} R T^{-1/2}\right) T^{1/2} X\widehat{k}_\tau, f\left(T^{-1/2} R T^{-1/2}\right) T^{1/2} X\widehat{k}_\tau \right\rangle \left\langle (T^{1/2}\widehat{k}_\tau, T^{1/2}\widehat{k}_\tau \right\rangle} \\
&= \sqrt{\left\langle X^* T^{1/2} f^2\left(T^{-1/2} R T^{-1/2}\right) T^{1/2} X\widehat{k}_\tau, \widehat{k}_\tau \right\rangle \left\langle T\widehat{k}_\tau, \widehat{k}_\tau \right\rangle} \\
&\leq \frac{1}{2\sqrt{K(t,2)}} \left(\left\langle X^* T^{1/2} f^2\left(T^{-1/2} R T^{-1/2}\right) T^{1/2} X\widehat{k}_\tau, \widehat{k}_\tau \right\rangle + \left\langle T\widehat{k}_\tau, \widehat{k}_\tau \right\rangle \right) \text{ (by (1.8))} \\
&\leq \frac{1}{2\sqrt{K(m,2)}} \left(\left\langle \left(X^* T^{1/2} f^2\left(T^{-1/2} R T^{-1/2}\right) T^{1/2} X + T \right) \widehat{k}_\tau, \widehat{k}_\tau \right\rangle \right)
\end{aligned}$$

and

$$\left|\left\langle((T\sigma_f R)\,X)\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|\leq\frac{1}{2\sqrt{K\,(m,2)}}\left(\left\langle\left(X^*T^{1/2}f^2\left(T^{-1/2}RT^{-1/2}\right)T^{1/2}X+T\right)\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right),$$

where $t=\frac{\left\langle X^*T^{1/2}f^2\left(T^{-1/2}RT^{-1/2}\right)T^{1/2}X\widehat{k}_\tau,\widehat{k}_\tau\right\rangle}{\left\langle T\widehat{k}_\tau,\widehat{k}_\tau\right\rangle}$ and $K\,(m,2)=\frac{(m+1)^2}{4m}$. Taking the supremum over $\tau\in\Omega$ in the above inequality, we get

$$\begin{aligned}&\sup_{\tau\in\Omega}\left|\left\langle((T\sigma_f R)\,X)\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|\\&\leq\sup_{\tau\in\Omega}\left(\frac{1}{2\sqrt{K\,(m,2)}}\left(\left\langle\left(X^*T^{1/2}f^2\left(T^{-1/2}RT^{-1/2}\right)T^{1/2}X+T\right)\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right)\right),\end{aligned}$$

where $K\,(m,2)=\frac{(m+1)^2}{4m}$. Thus, we have

$$\mathrm{ber}((T\sigma_f R)\,X)\leq\frac{1}{2\sqrt{K\,(m,2)}}\left\|X^*T^{1/2}f^2\left(T^{-1/2}RT^{-1/2}\right)T^{1/2}X+T\right\|_{\mathrm{ber}}.$$

This completes the proof. Q.E.D.

Putting $f\,(t)=\sqrt{t}$ in Theorem 3.10, we get the following result.

Corollary 3.11. Let $T,R,X\in\mathbb{B}^{-1}\,(\mathbb{H})$. If one of $m^5X^*T^{1/2}f^2\left(T^{-1/2}RT^{-1/2}\right)T^{1/2}X\leq T$ or $m^5T\leq X^*T^{1/2}f^2\left(T^{-1/2}RT^{-1/2}\right)T^{1/2}X$ with $m>1$ in Theorem 3.10 is satisfied, then

$$\mathrm{ber}((T\sharp R)\,X)\leq\frac{1}{2\sqrt{K\,(m,2)}}\left\|X^*RX+T\right\|_{\mathrm{ber}},$$

where $K\,(m,2)=\frac{(m+1)^2}{4m}$.

If $f\,(t)=\sqrt[4]{t}$ is taken in Theorem 3.10 and Lemma 2.6 is applied, the following result is obtained.

Corollary 3.12. Let $T,R,X\in\mathbb{B}^{-1}\,(\mathbb{H})$. If one of $m^5X^*T^{1/2}f^2\left(T^{-1/2}RT^{-1/2}\right)T^{1/2}X\leq T$ or $m^5T\leq X^*T^{1/2}f^2\left(T^{-1/2}RT^{-1/2}\right)T^{1/2}X$ with $m>1$ in Theorem 3.10 is satisfied, then

$$\mathrm{ber}((T\sharp R)\,X)\leq\frac{1}{2\sqrt{K\,(m,2)}}\left\|\frac{X^*\,(T+R)\,X}{2K\,(m,2)}+T\right\|_{\mathrm{ber}},$$

where $K\,(m,2)=\frac{(m+1)^2}{4m}$.

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