Notes on meta-golden Riemannian structures

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Abstract

In this paper, we introduce a meta-golden Riemannian structure on a manifold and investigate the locally decomposability of the manifold with respect to this structure. We also give some examples for meta-golden structures on tangent, cotangent and tensor bundles.

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1 Introduction

The golden ratio $\varphi = \frac{1+\sqrt{5}}{2} = 1.618...$ was first employed by Phidas (490-430 BC) and was first defined by Euclid. It has many applications since antiquity in geometry, art, architecture and music. The logarithmic spiral was also believed to provide the golden ratio. But, in his very recent paper [4], Bartlett showed that it is not true. Instead of this, he showed that logarithmic spiral provides "meta-golden chi ratio" which is denoted by χ. It is known that rectangle of width 1 and length φ can be subdivided into a square and another rectangle proportional to the original rectangle. That smaller proportional rectangle will have width $1/\varphi$ and length 1, and the square a side 1 because $\varphi = 1/\varphi + 1$. Bartlett wondered what kind of rectangle would be constructed in a similar way, if the starting point was a golden rectangle, instead of a square. This precedure gives the relation $\chi = \frac{1}{\varphi} + \frac{1}{\chi}$. It transforms in the quadratic equation $\chi^2 - (1/\varphi)\chi - 1 = 0$. So, the roots become $\frac{(1/\varphi)\pm\sqrt{(1/\varphi)^2+4}}{2} = \frac{\sqrt{5}-1\pm\sqrt{22-2\sqrt{5}}}{4}$. The positive root is $1.355\ldots$ and this is called "the meta-golden chi ratio", i.e., $\chi = 1.355...$[9].

Consider the general quadratic equation $x^2 - px - q = 0$, where p and q are positive integers. The set of positive solutions of this equation $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ are referred to as the Metallic Means Family. These numbers were introduced by Spinadel in [19] and can be seen as generalizations of the golden ratio φ. Inspiring these numbers, Hretcanu and Crasmareanu introduced metallic structures on Riemannian manifolds in [7]. Investigating metallic structures and their subclasses (such as golden, silver, bronze structures etc.) on Riemannian manifolds has occupied an important place in differential geometry (see for example [1, 2, 3, 6, 8, 10, 11, 12, 13]).

Motivated by the above studies, in the present paper, we construct a new structure which is called meta-golden structure on a Riemannian manifold and investigate its integrability. We find an almost product structure which corresponds to the meta-golden structure. We define the twin meta-golden Riemannian metric. We give examples for locally decomposable meta-golden Riemannian manifolds.

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2 Preliminaries

Definition 2.1. A meta-golden Riemannian structure on a Riemannian manifold (M, g) is a $(1,1)-$tensor field J which satisfies the following relations:

$$J^2 = \frac{1}{\varphi}J + I,$$

$$g(JU, V) = g(U, JV) \tag{2.1}$$

or equivalently

$$g(JU, JV) = g(J^2U, V) = g\left((\frac{1}{\varphi}J + I)U, V\right) = \frac{1}{\varphi}g(JU, V) + g(U, V)$$

for every vector fields U, V on M, where φ is the golden ratio which is equal to $\frac{1+\sqrt{5}}{2}$ and I is the $(1,1)-$type identity tensor field.

The Riemannian metric satisfying (2.1) is called $J-$compatible or pure metric, and the triple (M, g, J) is referred to as a meta-golden Riemannian manifold.

Remark 2.2. In [14], the authors introduced a meta-golden structure on a golden manifold, i.e., they considered that φ is a Golden structure ($(1,1)-$tensor field). In this paper, we assume that φ is golden ratio (real number). On the other hand, since φ is an irrational number, our meta-golden structure does not belong to any subclass of metallic structures.

Generally, a $(0,s)-$tensor field t is pure with respect to a $(1,1)-$tensor field ψ if and only if

$$t(\psi U_1, U_2, ..., U_s) = t(U_1, \psi U_2, ..., U_s) = ... = t(U_1, U_2, ..., \psi U_s)$$

for every vector fields $U_i,\ i = 1, ..., s$ on M. The Tachibana operator Φ_ψ applied to the tensor field t is given by

$$\begin{aligned}(\Phi_\psi t)(V, U_1, U_2, ..., U_s) &= (\psi V)t(U_1, U_2, ..., U_s) - Vt(\psi U_1, ..., U_s) \\ &\quad + \sum_{\lambda=1}^{s} t(U_1, ..., (L_{U_\lambda}\psi)V, ..., U_s),\end{aligned}$$

where L_U is the Lie derivative with respect to U [15]. If the pure tensor t fulfills $\Phi_\psi t = 0$, then it is said to be a $\Phi-$tensor. When ψ is a product structure it is a decomposable tensor.

Let (M, g, P) be a locally decomposable Riemannian manifold. Then, we have

$$P^2 = I,\ g(PU, V) = g(U, PV),\ \nabla P = 0,$$

for every vector fields U, V on M, where ∇ is the Levi-Civita connection of g. In [17], it was proved that the condition $\nabla P = 0$ is equivalent to $\Phi_P g = 0$, Φ_P is the Tachibana operator:

$$\begin{aligned}(\Phi_P g)(U, V, W) &= (PU)(g(V, W)) - U(g(PV, W)) + g((L_V P)U, W) \\ &\quad + g(V, (L_W P)U).\end{aligned} \tag{2.2}$$

3 Meta-golden Riemannian structures

In this section, we study some geometric properties of meta-golden structures on Riemannian manifolds.

Theorem 3.1. Let (M, g, J) be a meta-golden Riemannian manifold, where J is a meta-golden structure and g is a Riemannian metric. Then

(i) J is integrable if $\Phi_J g = 0$,

(ii) The condition $\Phi_J g = 0$ is equivalent to $\nabla J = 0$, where ∇ is the Levi-Civita connection of g.

Proof. From the purity of g and taking into account $\nabla g = 0$, we get

$$g(V_1, (\nabla_U J)V_2) = g((\nabla_U J)V_1, V_2), \tag{3.1}$$

for every vector fields U, V_1, V_2.

Using equation (3.1) and $[U, V] = \nabla_U V - \nabla_V U$, we can rewrite equation (2.2) as

$$(\Phi_J g)(U, V_1, V_2) = -g((\nabla_U J)V_1, V_2) + g((\nabla_{V_1} J)U, V_2) + g(V_1, (\nabla_{V_2} J)U). \tag{3.2}$$

Interchanging U and V_2, we obtain

$$(\Phi_J g)(V_2, V_1, U) = -g((\nabla_{V_2} J)V_1, U) + g((\nabla_{V_1} J)V_2, U) + g(V_1, (\nabla_U J)V_2). \tag{3.3}$$

Adding (3.2) and (3.3), we occur

$$(\Phi_J g)(U, V_1, V_2) + (\Phi_J g)(V_2, V_1, U) = 2g((\nabla_{V_1} J)V_2, U). \tag{3.4}$$

Equation (3.4) proves the assertations (i) and (ii). Q.E.D.

The following proposition gives the relationship between the meta-golden structures and the almost product structure.

Proposition 3.2. If J is a meta-golden structure on M, then

$$P = \frac{1}{\sqrt{4 + \frac{1}{\varphi^2}}} \left(2J - \frac{1}{\varphi} I\right) \tag{3.5}$$

is an almost product structure on M. Contrarily, every almost product structure P on M induces two meta-golden structures on M, given as

$$J_1 = \frac{1}{2}\left(\frac{1}{\varphi} I + \sqrt{4 + \frac{1}{\varphi^2}} P\right), \; J_2 = \frac{1}{2}\left(\frac{1}{\varphi} I - \sqrt{4 + \frac{1}{\varphi^2}} P\right). \tag{3.6}$$

Proof. Let J be a meta-golden structure on M. In this case, $P = \frac{1}{\sqrt{4+\frac{1}{\varphi^2}}}\left(2J - \frac{1}{\varphi} I\right)$ is a is an almost product structure on M. In fact,

$$P^2 = \frac{1}{4 + \frac{1}{\varphi^2}}\left(4J^2 - \frac{4}{\varphi} J + \frac{1}{\varphi^2} I\right) = \frac{1}{4 + \frac{1}{\varphi^2}}\left(4(J^2 - \frac{1}{\varphi} J) + \frac{1}{\varphi^2} I\right) = \frac{1}{4 + \frac{1}{\varphi^2}}\left(4 + \frac{1}{\varphi^2}\right) I = I.$$

Contrarily, let P be an almost product structure on M, i.e., $P^2 = I$. Then J_1 and J_2 in (3.6) are meta-golden structures on M. In fact,

$$\begin{aligned} J_1^2 &= \frac{1}{4}\left(\frac{1}{\varphi^2}I + (4+\frac{1}{\varphi^2})I + \frac{2}{\varphi}\sqrt{4+\frac{1}{\varphi^2}}\right)P \\ &= \frac{1}{2}\left(\frac{1}{\varphi^2}I + \frac{1}{\varphi}\sqrt{4+\frac{1}{\varphi^2}}P\right) + I \\ &= \frac{1}{\varphi}\frac{1}{2}\left(\frac{1}{\varphi}I + \sqrt{4+\frac{1}{\varphi^2}}P\right) + I \\ &= \frac{1}{\varphi}J_1 + I. \end{aligned}$$

One can prove that J_2 is another meta-golden structure on M similarly. Q.E.D.

In order that a Riemannian metric g is pure with respect to a meta-golden structure J, it is necessary and sufficient that g is pure with respect to the corresponding almost product structure P. The relation between $\Phi_P g$ and $\Phi_J g$ is

$$\Phi_P g = \frac{2}{\sqrt{4+\frac{1}{\varphi^2}}}\Phi_J g. \tag{3.7}$$

Remark that if $\Phi_P g = 0$, then the Riemannian metric g is decomposable. By virtue of Theorem 3.1, we can say that the meta-golden structure J is integrable if the Riemannian metric g is decomposable. If (M, g, J) is a meta-golden Riemannian manifold with a decomposable metric, then (M, g, J) is said to be a locally decomposable meta-golden Riemannian manifold. Hence, we can express the following proposition.

Proposition 3.3. Let (M, g, J) be a meta-golden Riemannian manifold, where J is a meta-golden structure and g is a Riemannian metric. The manifold M is a locally decomposable meta-golden Riemannian manifold if and only if $\Phi_P g = 0$, where P is the almost product structure associated with J.

The twin meta-golden metric G is defined by

$$G(U, V) = g(JU, V),$$

for every vector fields U, V on M. One can easily show that the metric G is pure with respect to J. Applying the operator Φ_J to G gives us by direct calculation

$$\Phi_J G(U, V, W) = (\Phi_J g)(U, JV, W) + g(N_J(U, V), W),$$

where N_J is the Nijenhuis tensor associated with J. So, we have

Proposition 3.4. Let (M, g, J) be a meta-golden Riemannian manifold, where J is a meta-golden structure and g is a Riemannian metric. In this case, the twin meta-golden metric G is a $\Phi-$tensor field if g is a $\Phi-$tensor field and $N_J = 0$, where N_J is the Nijenhuis tensor associated with J.

Theorem 3.5. Let (M, g, J) be a meta-golden Riemannian manifold, where J is a meta-golden structure and g is a Riemannian metric. In this case, the Riemannian curvature tensor field is a $\Phi-$tensor field.

Proof. The Riemannian curvature tensor field R of the meta-golden Riemannian metric g is pure according to the meta-golden structure J, i.e.

$$R(JU_1, U_2, U_3, U_4) = R(U_1, JU_2, U_3, U_4) = R(U_1, U_2, JU_3, U_4) = R(U_1, U_2, U_3, JU_4),$$

for every vector fields $U_i,\ i = 1, ..., 4$. If we apply the Tachibana operator to R, we occur

$$(\Phi_J R)(V, U_1, U_2, U_3, U_4) = (\nabla_{JV} R)(U_1, U_2, U_3, U_4) - (\nabla_V R)(JU_1, U_2, U_3, U_4), \tag{3.8}$$

where V is a vector field on M. Since R is pure and applying the second Bianchi identity to (3.8), we deduce

$$\begin{aligned}(\Phi_J R)(V, U_1, U_2, U_3, U_4) &= g((\nabla_{JV} R)(U_1, U_2, U_3) - (\nabla_V R)(JU_1, U_2, U_3), U_4)\\ &= g((\nabla_{JV} R)(U_1, U_2, U_3) - J((\nabla_V R)(U_1, U_2, U_3), U_4)\\ &= g(-(\nabla_{U_1} R)(U_2, JV, U_3) - (\nabla_{U_2} R)(JV, U_1, U_3)\\ &\quad -J((\nabla_V R)(U_1, U_2, U_3), U_4).\end{aligned} \tag{3.9}$$

Having in mind $\nabla J = 0$, we have

$$\begin{aligned}(\nabla_{U_2} R)(JV, U_1, U_3) &= \nabla_{U_2}(R(JV, U_1, U_3)) - R(\nabla_{U_2}(JA), U_1, U_3)\\ &\quad -R(JV, \nabla_{U_2} U_1, U_3) - R(JV, U_1, \nabla_{U_2} U_3)\\ &= (\nabla_{U_2} J)(R(V, U_1, U_3)) + J(\nabla_{U_2} R(V, U_1, U_3))\\ &\quad -R((\nabla_{U_2} J)V + J(\nabla_{U_2} V), U_1, U_3)\\ &\quad -R(JV, \nabla_{U_2} U_1, U_3) - R(JV, U_1, \nabla_{U_2} U_3)\\ &= J(\nabla_{U_2} R(V, U_1, U_3)) - J(R(\nabla_{U_2} V, U_1, U_3)\\ &\quad -J(R(V, \nabla_{U_2} U_1, U_3) - J(R(V, U_1, \nabla_{U_2} U_3)\\ &= J((\nabla_{U_2} R)(V, U_1, U_3).\end{aligned} \tag{3.10}$$

By the similar way, we obtain

$$(\nabla_{U_1} R)(U_2, JV, U_3) = J((\nabla_{U_1} R)(U_2, V, U_3). \tag{3.11}$$

Putting (3.10) and (3.11) in (3.9) and by virtue of second Bianchi identity, we get

$$\begin{aligned}(\Phi_J R)(V, U_1, U_2, U_3, U_4) &= g(-J((\nabla_{U_1} R)(U_2, V, U_3)) - J((\nabla_{U_2} R)(V, U_1, U_3))\\ &\quad -J((\nabla_V R)(U_1, U_2, U_3)), U_4)\\ &= 0.\end{aligned}$$

Q.E.D.

By (3.10) and (3.11), we can find a similar equation to (3.7) as follows:

$$\Phi_P R = \frac{2}{\sqrt{4+\frac{1}{\varphi^2}}}\Phi_J R. \tag{3.12}$$

In the light of Theorem 3.5 and equation (3.12), we give the following result.

Proposition 3.6. Let (M, g, J) be a meta-golden Riemannian manifold, where J is a meta-golden structure and g is a Riemannian metric. In this case, the Riemannian curvature tensor field is decomposable.

4 Examples

Example 4.1. Let (M, g) be a Riemannian manifold. We can define another Riemannian metric $\bar{g}$ on M using g as follows:

$$\bar{g}(U, V) = e^{2f} g(U, V),$$

where f is a smooth function and U, V are vector fields on M. The metric $\bar{g}$ is called a conformal metric to g.

Let (M, g, J) be a meta-golden Riemannian manifold. It is clear that $(M, \bar{g}, J)$ is another meta-golden Riemannian manifold. Applying the Tachibana operator Φ_J to $\bar{g}$, we find

$$\begin{aligned}\Phi_J\bar{g}(U,V,W) &= (JU)(e^{2f}g(V,W)) - U(e^{2f}g(JV,W)) + e^{2f}g((L_V J)U, W)\\ &\quad + e^{2f}g(V,(L_W J)Z)\\ &= (JU)(e^{2f})g(V,W) - U(e^{2f})g(JV,W) + e^{2f}(\Phi_J g)(U,V,W).\end{aligned}$$

Using (3.5) and (3.7), we get

$$\begin{aligned}\Phi_J\bar{g}(U,V,W) &= \frac{\sqrt{4+\frac{1}{\varphi^2}}}{2}\{(PU)(e^{2f})g(V,W) - U(e^{2f})g(PV,W)\\ &\quad + e^{2f}(\Phi_P g)(U,V,W)\},\end{aligned}$$

where P is the almost product structure associated with J. So, we have

Theorem 4.2. Let (M, g, J) be a meta-golden Riemannian manifold, where J is a meta-golden structure and g is a Riemannian metric. In this case, $(M, \bar{g}, J)$ is a locally decomposable metallic Riemannian manifold if and only if the function f is a constant function.

Example 4.3. The tangent bundle TM of a Riemannian manifold M is a $2n-$dimensional smooth manifold. A differentiable structure on TM is constructed by the natural projection $\pi : TM \to M$. If (N, x^i) is a local coordinate system on M, then $(\pi^{-1}(N), x^i, x^{\bar{\imath}} = y^i,\ i = 1, ..., n,\ \bar{\imath} = n+1, ..., 2n)$ is a local coordinate system on TM, where (y^i) are the cartesian coordinates in each tangent space T_pM.

Given a vector field $U = U^i \frac{\partial}{\partial x^i}$ on M. The vertical lift U^V and the horizontal lift U^H are defined by, respectively

$$U^V = U^i \frac{\partial}{\partial x^{\bar{\imath}}},\ U^H = U^i \frac{\partial}{\partial x^i} - y^a U^b \Gamma^i_{ab} \frac{\partial}{\partial x^{\bar{\imath}}},$$

where Γ_{ab}^{i} are the coefficients of the Levi-Civita connection of g (Einstein summation convention is used).

The Sasaki metric g^S on TM is defined by

$$g^S(U^H,V^H)=g(U,V),\ g^S(U^H,V^V)=0,\ g^S(U^V,V^V)=g(U,V), \tag{4.1}$$

for every vector fields U,V on M, [20].

We define a meta-golden structure $\bar{J}$ on TM as follows:

$$\begin{cases} \bar{J}(U^H)=\frac{1}{2\varphi}U^H+\frac{1}{2}\sqrt{4+\frac{1}{\varphi^2}}U^V, \\ \bar{J}(U^V)=\frac{1}{2\varphi}U^V+\frac{1}{2}\sqrt{4+\frac{1}{\varphi^2}}U^H, \end{cases} \tag{4.2}$$

which satisfies $\bar{J}^2-\frac{1}{\varphi}\bar{J}-I=0$. One can easily show that $g^S(\bar{J}\tilde{U},\tilde{V})=g^S(\tilde{U},\bar{J}\tilde{V})$ for all vector fields $\tilde{U}=U^H,U^V$ and $\tilde{V}=V^H,V^V$. Therefore, the metric g^S is pure with respect to the meta-golden structure $\bar{J}$. So, we have

Theorem 4.4. Let (M,g) be a Riemannian manifold and (TM,g^S) be its tangent bundle. The structure $\bar{J}$ defined by (4.2) is a meta-golden Riemannian structure, and so the triple $(TM,g^S,\bar{J})$ is a meta-golden Riemannian manifold.

Taking into account (4.2), using (2.2) and the fact that

$$U^H(g(V,W))^V=(Ug(V,W))^V,\ U^V(g(V,W))^V=0,$$

for all vector fields U,V,W on M, we have

$$\begin{aligned}(\Phi_{\bar{J}}g^S)(\tilde{U},\tilde{V}_1,\tilde{V}_2) &= (\bar{J}\tilde{U})(g^S(\tilde{V}_1,\tilde{V}_2))-\tilde{U}(g^S(\bar{J}\tilde{V}_1,\tilde{V}_2))+g^S((L_{\tilde{V}_1}\bar{J})\widetilde{U},\tilde{V}_2)\\ &\quad +g^S(\tilde{V}_1,(L_{\tilde{V}_2}\bar{J})\widetilde{U}),\end{aligned}$$

for all vector fields $\tilde{U},\tilde{V}_1,\tilde{V}_2$. It deduces that

$$\begin{aligned}(\Phi_{\bar{J}}g^S)(U^H,V_1^H,V_2^H) &= \frac{1}{2}\sqrt{4+\frac{1}{\varphi^2}}\,[g^S((R(V_1,U)y)^H,V_2^H)\\ &\quad +g^S(V_1^H,\ (R(V_2,U)y)^H)],\\ (\Phi_{\bar{J}}g^S)(U^V,V_1^V,V_2^H) &= -\frac{1}{2}\sqrt{4+\frac{1}{\varphi^2}}g^S(V_1^V,\ (R(V_2,U)y)^V),\\ (\Phi_{\bar{J}}g^S)(U^V,V_1^H,V_2^V) &= -\frac{1}{2}\sqrt{4+\frac{1}{\varphi^2}}g^S((R(V_1,U)y)^V,V_2^V),\\ (\Phi_{\bar{J}}g^S)(U^H,V_1^H,V_2^V) &= (\Phi_{\bar{J}}g^S)(U^H,V_1^V,V_2^V)=(\Phi_{\bar{J}}g^S)(U^H,V_1^V,V_2^H)=0,\\ (\Phi_{\bar{J}}g^S)(U^V,V_1^V,V_2^V) &= (\Phi_{\bar{J}}g^S)(U^V,V_1^H,V_2^H)=0,\end{aligned}$$

where $y=y^i\frac{\partial}{\partial y^i}$. From Proposition 3.3 and equation (3.7), we express the following theorem.

Theorem 4.5. Let (M,g) be a Riemannian manifold and TM be its tangent bundle with the Sasaki metric g^S and the meta-golden structure $\bar{J}$ defined by (4.2). The triple $(TM,g^S,\bar{J})$ is a locally decomposable meta-golden Riemannian manifold if and only if M is locally flat.

Example 4.6. The cotangent bundle T^*M of a Riemannian manifold M is a $2n$-dimensional smooth manifold. A differentiable structure on T^*M is constructed by the natural projection $\pi : T^*M \to M$. If (N, x^i) is a local coordinate system on M, then $(\pi^{-1}(N), x^i, x^{\bar{\imath}} = p_i,\ i = 1, \cdots, n,\ \bar{\imath} = n+1, \cdots, 2n)$ is a local coordinate system on TM, where (p_i) are the cartesian coordinates in each cotangent space T_p^*M.

Given a 1-form $\alpha = \alpha_j dx^j$ and a vector field $U = U^i \frac{\partial}{\partial x^i}$ on M. The vertical lift α^V of the 1-form α and the horizontal lift U^H of a vector field U are defined by, respectively

$$\alpha^V = \alpha_i \frac{\partial}{\partial x^{\bar{\imath}}},\ U^H = U^i \frac{\partial}{\partial x^i} - p_a U^b \Gamma^a_{ib} \frac{\partial}{\partial x^{\bar{\imath}}},$$

where Γ^a_{ib} are the coefficients of the Levi-Civita connection of g (Einstein summation convention is used).

The Sasaki metric g^S on T^*M is defined by

$$g^S(U^H, V^H) = g(U, V),\ g^S(U^H, \alpha^V) = 0,\ g^S(\beta^V, \alpha^V) = g(\beta, \alpha),$$

for every vector fields U, V and every 1-forms α, β on M, [20].

The Levi-Civita connection ∇^S of the the metric g^S is given by

$$\begin{aligned}
\nabla^S_{\alpha^V} \beta^V &= 0, \\
\nabla^S_{\alpha^V} V^H &= \frac{1}{2}(p(g^{-1} \circ R(\ , V)\tilde{\alpha}))^H, \\
\nabla^S_{U^H} \beta^V &= (\nabla_U \beta)^V + \frac{1}{2}(p(g^{-1} \circ R(\ , U)\tilde{\beta})^H, \\
\nabla^S_{U^H} V^H &= (\nabla_U V)^H + \frac{1}{2}(pR(U, V))^V,
\end{aligned} \tag{4.3}$$

for every vector fields U, V and every 1-forms α, β on M, where $\tilde{\alpha} = g \circ \alpha$, $\tilde{\beta} = g \circ \beta$ are vector fields, $p = p_i dx^i$, $R(\ , A)\tilde{\beta}$ is a $(1,1)-$tensor field and $g^{-1} \circ R(\ , U)\tilde{\beta}$ is a $(2,0)-$tensor field on M, [16].

We define a meta-golden structure $\tilde{J}$ on TM as follows:

$$\begin{cases} \tilde{J}(U^H) = \frac{1}{2\varphi} U^H + \frac{1}{2}\sqrt{4 + \frac{1}{\varphi^2}}\bar{U}^V, \\ \tilde{J}(\alpha^V) = \frac{1}{2\varphi}\alpha^V + \frac{1}{2}\sqrt{4 + \frac{1}{\varphi^2}}\bar{\alpha}^H, \end{cases} \tag{4.4}$$

which satisfies $\tilde{J}^2 - \frac{1}{\varphi}\tilde{J} - I = 0$. Here, $\bar{U} = g \circ U$ is a 1-form and $\bar{\alpha} = g^{-1} \circ \alpha$ is a vector field on M. One can show that the metric g^S is pure with respect to the meta-golden structure $\tilde{J}$. So, we have

Theorem 4.7. Let (M, g) be a Riemannian manifold and (T^*M, g^S) be its cotangent bundle. The structure $\tilde{J}$ defined by (4.4) is a meta-golden Riemannian structure, and so the triple $(T^*M, g^S, \tilde{J})$ is a meta-golden Riemannian manifold.

Now, we compute the covariant derivative of $\tilde{J}$. Using (4.3) and (4.4), we get

$$\begin{aligned}
(\nabla^S_{U^H}\tilde{J})(V^H) &= \frac{1}{4}\sqrt{4+\frac{1}{\varphi^2}}\left(p\left(g^{-1}\circ(R(\ ,U)V-R(U,V))\right)\right)^H,\\
(\nabla^S_{U^H}\tilde{J})(\beta^V) &= \frac{1}{4}\sqrt{4+\frac{1}{\varphi^2}}(pR(U,\tilde{\beta})-pR(\ ,U)\tilde{\beta})^V,\\
(\nabla^S_{\alpha^V}\tilde{J})(V^H) &= -\frac{1}{4}\sqrt{4+\frac{1}{\varphi^2}}(p(R(\ ,V)\tilde{\alpha})^V,\\
(\nabla^S_{\alpha^V}\tilde{J})(\beta^V) &= \frac{1}{4}\sqrt{4+\frac{1}{\varphi^2}}(p(g^{-1}\circ R(\ ,\tilde{\beta})\tilde{\alpha}))^H,
\end{aligned} \tag{4.5}$$

for every for every vector fields U, V and every 1-forms α, β on M. Equations (4.5) give the following theorem.

Theorem 4.8. Let (M, g) be a Riemannian manifold and T^*M be its cotangent bundle with the Sasaki metric g^S and the meta-golden structure $\tilde{J}$ defined by (4.4). The triple $(T^*M, g^S, \tilde{J})$ is a locally decomposable meta-golden Riemannian manifold if and only if M is locally flat.

Example 4.9. The $(1,1)-$tensor bundle T^1_1M of a Riemannian manifold M is a $n+n^2-$dimensional smooth manifold. A differentiable structure on T^1_1M is constructed by the natural projection $\pi : T^1_1M \to M$. If (N, x^i) is a local coordinate system on M, then $(\pi^{-1}(N), x^i, x^{\bar{\imath}} = t^j_i,\ i = 1, ..., n,\ \bar{\imath} = n+1, ..., n+n^2)$ is a local coordinate system on TM, where (t^j_i) are the cartesian coordinates in each $(1,1)-$tensor space $T^1_{1(p)}M$.

Given a $(1,1)-$tensor field $A = A^i_j \frac{\partial}{\partial x^i}\otimes dx^j$ and a vector field $U = U^i\frac{\partial}{\partial x^i}$ on M. The vertical lift A^V $(1,1)-$tensor field and the horizontal lift U^H of a vector field U are defined by, respectively

$$A^V = A^j_i\frac{\partial}{\partial x^{\bar{\imath}}},\ U^H = U^i\frac{\partial}{\partial x^i} - U^b(\Gamma^a_{ib}t^j_a - \Gamma^j_{ba}t^a_i)\frac{\partial}{\partial x^{\bar{\imath}}},$$

where Γ^a_{ib} are the coefficients of the Levi-Civita connection of g (Einstein summation convention is used). The vector fields γA and $\tilde{\gamma}A$ on T^1_1M are respectively given by, [5]

$$\gamma A = t^m_i A^j_m\frac{\partial}{\partial x^{\bar{\imath}}},\ \tilde{\gamma}A = t^j_m A^m_i\frac{\partial}{\partial x^{\bar{\imath}}}.$$

The Sasaki metric g^S on T^1_1M is defined by

$$g^S(U^H, V^H) = g(U, V),\ g^S(U^H, A^V) = 0,\ g^S(A^V, B^V) = G(A, B), \tag{4.6}$$

for every vector fields U, V and every $(1,1)-$tensor fields A, B on M, where $G(A, B) = g_{it}g^{jl}A^i_jB^t_l$, [18].

Equations (4.6) show that the horizontal distribution H is orthogonal to the tensor spaces (fibers) of T^1_1M. Let E be a nowhere vanishing vector field on M. For any vector field U and 1-form $\tilde{E} = g\circ E$ on M, the vertical lift of U is defined as $(U\otimes\tilde{E})^V$ with respect to E. The map $U \to (U\otimes\tilde{E})^V$ is a monomorphism. Therefore, an $n-$dimensional smooth vertical distribution V^E is defined on T^1_1M. If $V^\perp$ is the distribution on $T^1_1(M)$, which is orthogonal to H and V^E, then

$V^{\perp}, V^{E}$ and H are mutually orthogonal distributions with respect to the metric g^S. We construct a $(1,1)-$tensor field J' on $T_1^1(M)$ by

$$\begin{cases} J'(U^H) = \frac{1}{2\varphi}U^H + \frac{1}{2}\sqrt{4+\frac{1}{\varphi^2}}(U\otimes\tilde{E})^V, \\ J'((U\otimes\tilde{E})^V) = \frac{1}{2\varphi}(U\otimes\tilde{E})^V + \frac{1}{2}\sqrt{4+\frac{1}{\varphi^2}}U^H, \\ J'(A^V) = \frac{1}{2\varphi}(1+\sqrt{1+4\varphi^2})A^V, \end{cases} \tag{4.7}$$

for every vector field U and $(1,1)-$tensor field A on M, where $\tilde{E} = g\circ E$ is a 1-form on M. It can be easily shown that $J'^2 - \frac{1}{\varphi}J' - I = 0$, i.e., J' is a meta-golden structure.

Theorem 4.10. Let (M,g) be a Riemannian manifold and T_1^1M be its $(1,1)-$tensor bundle with the Sasaki metric g^S and the meta-golden structure J' defined by (4.7). The triple (T_1^1M, g^S, J') is a meta-golden Riemannian manifold if and only if $g(E,E) = 1$.

Proof. We compute $K(\tilde{U},\tilde{V}) = g^S(J'\tilde{U},\tilde{V}) - g^S(\tilde{U},J'\tilde{V})$ for every vector fields $\tilde{U},\tilde{V}$ on $T_1^1(M)$. The only non-zero component is obtained

$$K((U\otimes\tilde{E})^V, V^H) = \sqrt{4+\frac{1}{\varphi^2}}\{g(U,V) - g(U,V)g(E,E)\},$$

which gives the assertation. Q.E.D.

The Levi-Civita connection of the metric g^S on T_1^1M is given by

$$\begin{aligned} \nabla^S_{A^V}B^V &= 0, \\ \nabla^S_{A^V}V^H &= \frac{1}{2}(g^{bl}R(t_b,A_l)V + g_{at}(t^a(g^{-1}\circ R(\ ,V)\tilde{A}^t)^H, \\ \nabla^S_{U^H}B^V &= (\nabla_U B)^V + \frac{1}{2}(g^{bj}R(t_b,B_j)U + g_{ai}(t^a(g^{-1}\circ R(\ ,U)\tilde{B}^i)))^H, \\ \nabla^S_{U^H}V^H &= (\nabla_U V)^H + \frac{1}{2}(\tilde{\gamma}-\gamma)R(U,V), \end{aligned} \tag{4.8}$$

for every vector fields U,V and $(1,1)-$tensor fields A,B on M, where $A_l = (A^i_l)$, $\tilde{A}^t = (g^{bl}A_l{}^t) = (A^{bt}_{\cdot})$, $t_l = (t^a_l)$, $t^a = (t^a_b)$, $R(\ ,U)V$ is a $(1,1)-$ tensor field and $g^{-1}\circ R(\ ,U)V$ is a vector field.

Using (4.8), we investigate the covariant derivative of J', i.e.

$$(\nabla^S_{\tilde{U}}J')\tilde{V} = \nabla^S_{\tilde{U}}(J'\tilde{V}) - J'(\nabla^S_{\tilde{U}}\tilde{V})$$

for every vector fields $\tilde{U},\tilde{V}$ on T_1^1M. By direct calculations, we obtain

$$\begin{aligned} (\nabla^S_{U^H}J')V^H &= \frac{1}{4}\sqrt{4+\frac{1}{\varphi^2}}(V\otimes(g\circ\nabla_U E))^V - \frac{1}{4}\sqrt{4+\frac{1}{\varphi^2}}(\tilde{\gamma}-\gamma)R(U,V) \\ &\quad + \frac{1}{4}\sqrt{4+\frac{1}{\varphi^2}}\{g^{bj}R(t_b,(V\otimes\tilde{E})_j)U + g_{ai}(t^a(g^{-1}\circ R(\ ,U)(V\otimes\tilde{E})^i\}^H, \end{aligned}$$

$$(\nabla^S_{U^H}J')B^V = \frac{1}{4}\sqrt{4+\frac{1}{\varphi^2}}\{g^{bj}R(t_b,B_j)U+g_{ai}(t^a(g^{-1}\circ R(\ ,U)\widetilde{B}^i)\}^H$$
$$-\frac{1}{4}\sqrt{4+\frac{1}{\varphi^2}}\{g^{bj}R(t_b,B_j)U+g_{ai}(t^a(g^{-1}\circ R(\ ,U)(\widetilde{B}^i)]\otimes\tilde{E}\}^V,$$

$$(\nabla^S_{U^H}J')(V\otimes\tilde{E})^V = \frac{1}{2}(V\otimes(g\circ\nabla_U E))^V+\frac{1}{4}\sqrt{4+\frac{1}{\varphi^2}}(\tilde{\gamma}-\gamma)R(U,V)$$
$$-\frac{1}{4}\sqrt{4+\frac{1}{\varphi^2}}\{[g^{bj}R(t_b,(V\otimes\tilde{E})_j)U+g_{ai}(t^a(g^{-1}\circ R(\ ,U)(V\otimes\tilde{E})^i]\otimes\tilde{E}\}^V,$$

$$(\nabla^S_{A^V}J')V^H = \frac{1}{4}\sqrt{4+\frac{1}{\varphi^2}}\{[g^{bj}R(t_b,A_j)V+g_{ai}(t^a(g^{-1}\circ R(\ ,V)\widetilde{A}^i)]\otimes\tilde{E}\}^V,$$

$$(\nabla^S_{A^V}J')(V\otimes\tilde{E})^V = \frac{1}{4}\sqrt{4+\frac{1}{\varphi^2}}\{g^{bj}R(t_b,A_j)V+g_{ai}(t^a(g^{-1}\circ R(\ ,V)(\widetilde{A}^i))\}^H,$$

$$(\nabla^S_{(U\otimes\tilde{E})^V}J')(V\otimes\tilde{E})^V = \frac{1}{4}\sqrt{4+\frac{1}{\varphi^2}}\{g^{bj}R(t_b,(U\otimes\tilde{E})_j)V+g_{ai}(t^a(g^{-1}\circ R(\ ,V)(U\otimes\tilde{E})^i))\}^H,$$

$$(\nabla^S_{(U\otimes\tilde{E})^V}J')V^H = -\frac{1}{4}\sqrt{4+\frac{1}{\varphi^2}}\{[g^{bj}R(t_b,(U\otimes\tilde{E})_j)V+g_{ai}(t^a(g^{-1}\circ R(\ ,V)(U\otimes\tilde{E})^i))]\otimes\tilde{E}\}^V,$$

$$(\nabla^S_{A^V}J')B^V=0,\ (\nabla^S_{(U\otimes\tilde{E})^V}J')(V\otimes\tilde{E})^V=0.$$

Thus, we can express the final theorem of the paper.

Theorem 4.11. Let (M,g) be a Riemannian manifold and T^1_1M be its $(1,1)-$ tensor bundle with the Sasaki metric g^S and the meta-golden structure J' defined by (4.7). The triple (T^1_1M,g^S,J') is a locally decomposable metallic Riemannian manifold if and only if M is flat, $g(E,E)=1,\ \nabla E=0$, where ∇ is the Levi-Civita connection of g.

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