New coding/decoding algorithm on k-order Fibonacci numbers

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Abstract

In this paper, we define a new encryption algorithm by generalizing the 2×2 encryption algorithm and dividing it into $k \times k$ type block matrices. This coding algorithm bound to the Q_k matrices. Therefore, this algorithm is different from the classical algebraic algorithm. Since matrix operations are at the forefront in this algorithm, it is easily and quickly performed by today's modern computers.

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1 Introduction

A recurrence relation is an equation that defines a sequence based on a rule that gives the next term as a function of the previous terms. The simplest form of a recurrence relation is the case where the next term depends only on the immediately previous term. If we denote the *nth* term in the sequence by x_n, such a recurrence relation is of the form

$$x_{n+1} = f(x_n)$$

for some function f. One such example is $x_{n+1} = 2 - \frac{x_n}{2}$.

A recurrence relation can also be higher order, where the term x_{n+1} could depend not only on the previous term x_n but also on earlier terms such as x_{n-1}, x_{n-2}, etc. A second order recurrence relation depends just on x_n and x_{n-1} and is of the form

$$x_{n+1} = f(x_n, x_{n-1})$$

for some function f with two inputs. For example, the recurrence relation $x_{n+1} = x_n + x_{n-1}$ can generate the Fibonacci numbers.

We can see more details about recurrence relation in [1].

Fibonacci numbers are defined by the recurrence relation of $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ such that $F_0 = 0$ and $F_1 = 1$. There are a lot of generalizations of Fibonacci numbers satisfied and studied by some authors. For more information we can see in [1, 2]. The Fibonacci Q-matrix is defined in [3, 4] as follows:

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

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and nth power of the Fibonacci Q-matrix is shown in [5, 6] by

$$Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.$$

Er defined an interesting generalization of Fibonacci numbers in [7]. This generalization is order$-k$ Fibonacci numbers by the following recurrence relation

$$g_n^i = \sum_{j=1}^{k} g_{n-j}^i, \quad \text{for} \quad n > 0 \quad \text{and } 1 \leq i \leq k$$

with boundary conditions for $1 - k \leq n \leq 0$,

$$g_n^i = \begin{cases} 1 & \text{if } i = 1 - n \\ 0 & otherwise \end{cases}$$

Also, Er defined the $k \times k$ dimensional Q_k and Q_k^n matrices, which play the role of $Q-$matrix as follows:

$$Q_k = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{k \times k}$$

and

$$Q_k^n = \begin{bmatrix} g_{n+1}^{(k)} & \cdots & g_n^{(k)} + g_{n-1}^{(k)} & g_n^{(k)} \\ g_n^{(k)} & \cdots & g_{n+1}^{(k)} + g_n^{(k)} & g_{n-1}^{(k)} \\ \vdots & \ddots & \vdots & \vdots \\ g_{n-k+3}^{(k)} & \cdots & g_{n-k+2}^{(k)} + g_{n-k+1}^{(k)} & g_{n-k+2}^{(k)} \\ g_{n-k+2}^{(k)} & \cdots & g_{n-k+3}^{(k)} + g_{n-k+2}^{(k)} & g_{n-k+1}^{(k)} \end{bmatrix}.$$

A lot of studies has been done and continuous to be done on this subject in order to ensure the security of information about data transfer over communication channels. Therefore, encoding/decoding algorithms play an important role in ensuring information security. In particular, the Fibonacci coding theory is one of the most preferred in this field. We can see examples of these in many studies. For example, A.P. Stakhov in [8] introduced a new approach to a coding theory of 2006 that uses the generalization of the Cassini formula for Fibonacci numbers and Q_p matrices. In 2009, M. Basu and B. Prasad presented generalized relations between code elements for Fibonacci coding theory in [9]. Also, M. Basu and M. Das introduced a new coding theory for Tribonacci matrices in [10] and in 2014 they defined the coding theory on Fibonacci n-step numbers by generalizing the coding theory for Tribonacci numbers in [11]. Asci and Aydinyuz defined $k-$order Gaussian Fibonacci polynomials with boundary conditions and introduced a new coding theory called $k-$order Gaussian Fibonacci polynomials coding theory in [12]. They established the code elements for values of k. In [13], Kizilates defined new families of Horadam numbers associated

with finite operators and their applications. Asci and Aydinyuz introduced a method for encrypting $k-$order Fibonacci polynomials using an AES-like cryptographic algorithm in [14].

In [15], Tas, Ucar, Ozgur and Kaymak presented a new method of coding/decoding algorithms using Fibonacci Q -matrices. The method is based on the blocked message matrices. Main idea of their method depends on dividing the message matrix into the block matrices of size 2×2.

In this paper, we define a new $k-$order Fibonacci numbers coding/decoding theory using the matrices of the $k-$order Fibonacci numbers and in this method we can divide the message into $k \times k$ block matrices instead of 2×2. Therefore, with the generalization we have made, we will able to divide the message texts of any length into arbitrary blocks. In the last part, we give illustrative examples of this coding method.

2 A new coding/decoding method for $k-$order Fibonacci numbers

In this section, we describe an encoding algorithm for $k-$order Fibonacci numbers. When using this coding method, we use the $k-$order Fibonacci matrix. In this method we can divide the message into $k \times k$ block matrices.

The number of the block matrices B_i is denoted by b. According to b, we choose the number of n as follows:

$$n = \begin{cases} b & , \quad b \leq 3 \\ \left[\left|\frac{b}{2}\right|\right] & , \quad b > 3 \end{cases} .$$

Using the chosen n, we write the following character table according to $mod27$ (We can optionally increase this table according to the characters used in our message text.). We begin to choose the first character as "n".

A	B	C	D	E	F	G	H	I	J
n	n+1	n+2	n+3	n+4	n+5	n+6	n+7	n+8	n+9
K	L	M	N	O	P	Q	R	S	T
n+10	n+11	n+12	n+13	n+14	n+15	n+16	n+17	n+18	n+19
U	V	W	X	Y	Z	0			
n+20	n+21	n+22	n+23	n+24	n+25	n+26			

Now we give the new encoding and decoding algorithm that we define.

2.1 Coding algorithm

- **Step 1:** Divide the matrix $M_{k\times k}$ into block matrices to B_i. If $n \in \mathbb{Z}^+$ and $k = 2n + 1$, we get B_i for $i = 1$. If $n \in \mathbb{Z}^+$ and $k = 2n$, then we get B_i for i is the positive divisor of k^2 of p^2 where p is prime.
- **Step 2:** Choose n.
- **Step 3:** Determine $b_j^i \quad (1 \leq j \leq T^2)$.
- **Step 4:** Compute $\det(B_i) \to d_i$.
- **Step 5:** Construct $F = \left[d_i,\ b_k^i\right]_{k\in 1,2,3,\ldots,b_j^i,\ldots,T^2}$.
- **Step 6:** End of algorithm.

2.2 Decoding algorithm

- **Step 1:** Compute Q^n.
- **Step 2:** Determine q_j $(1 \leq j \leq m^2)$.
- **Step 3:** $(B_i)_{T\times T} .(Q^n)_{T\times T} = (K_i)_{T\times T}$.
- **Step 4:** Compute determinant of above equation

$$\det\left((Q^n)_{T\times T}.(B_i)_{T\times T}\right) = \det\left((K_i)_{T\times T}\right).$$

- **Step 5:** $x_i = b_j^i$.
- **Step 6:** Determine B_i.
- **Step 7:** Generate M.
- **Step 8:** End of algorithm.

3 Illustrative examples for coding/decoding algorithm

Example 3.1. Let us consider the message matrix for the following message text:

"GOOD MORNING"

and create the code matrix using the message text (We can fill in the message matrix by placing "0" in the empty places.)

$$M = \begin{bmatrix} \text{G} & \text{O} & \text{O} & \text{D} \\ 0 & \text{M} & \text{O} & \text{R} \\ \text{N} & \text{I} & \text{N} & \text{G} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- **Step 1:** Divide the message matrix M of the size 4×4 named B_i $(1 \leq i \leq 4)$ from right to left each size 2×2.

$$B_1 = \begin{bmatrix} \text{O} & \text{D} \\ \text{O} & \text{R} \end{bmatrix}$$
$$B_2 = \begin{bmatrix} \text{G} & \text{O} \\ 0 & \text{M} \end{bmatrix}$$
$$B_3 = \begin{bmatrix} \text{N} & \text{G} \\ 0 & 0 \end{bmatrix}$$
$$B_4 = \begin{bmatrix} \text{N} & \text{I} \\ 0 & 0 \end{bmatrix}.$$

- **Step 2:** Due to $b = 4 > 3$, we calculate $n = \left[\left|\frac{b}{2}\right|\right] = 2$. For $n = 2$, we can get this table according to the alphabet table as follows:

G	O	O	D	0	M	O	R	N	I	N	G
8	16	16	5	28	14	16	19	15	10	15	8

.

- **Step 3:** We have the elements of the blocks B_i $(1 \leq i \leq 4)$ as follows:

$$\begin{array}{lcllll} b_1^1 & = & 16 & b_2^1 = 5 & b_3^1 = 16 & b_4^1 = 19 \\ b_1^2 & = & 8 & b_2^2 = 16 & b_3^2 = 28 & b_4^2 = 14 \\ b_1^3 & = & 15 & b_2^3 = 8 & b_3^3 = 28 & b_4^3 = 28 \\ b_1^4 & = & 15 & b_2^4 = 10 & b_3^4 = 28 & b_4^4 = 28 \end{array}$$

- **Step 4:** Now, we calculate the determinants of the blocks B_i as follows:

$$\begin{aligned} d_1 &= \det B_1 = 224 \\ d_2 &= \det B_2 = -336 \\ d_3 &= \det B_3 = 196 \\ d_4 &= \det B_4 = 140. \end{aligned}$$

- **Step 5:** Construct F as follows:

$$F = \begin{bmatrix} 224 & 16 & 5 & 16 \\ -336 & 8 & 16 & 28 \\ 196 & 15 & 8 & 28 \\ 140 & 15 & 10 & 28 \end{bmatrix}.$$

Decoding algorithm

- **Step 1:** Calculate Q^2 matrices as follows

$$Q^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

- **Step 2:** Multiply each block by Q^2

$$\begin{aligned} B_1.Q^2 &= \begin{bmatrix} 16 & 5 \\ 16 & x_1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 37 & 21 \\ x_1+32 & x_1+16 \end{bmatrix} \\ B_2.Q^2 &= \begin{bmatrix} 8 & 16 \\ 28 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 32 & 24 \\ x_2+56 & x_2+28 \end{bmatrix} \\ B_3.Q^2 &= \begin{bmatrix} 15 & 8 \\ 28 & x_3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 38 & 23 \\ x_3+56 & x_3+28 \end{bmatrix} \\ B_4.Q^2 &= \begin{bmatrix} 15 & 10 \\ 28 & x_4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 40 & 25 \\ x_4+56 & x_4+28 \end{bmatrix} \end{aligned}$$

- **Step 3:** From the determinants of both sides of the equations, we get by

$$\begin{aligned} x_1 &= b_4^1 = 19 \\ x_2 &= b_4^2 = 14 \\ x_3 &= b_4^3 = 28 \\ x_4 &= b_4^4 = 28 \end{aligned}$$

- **Step 4:** Rename x_i as follows:

$$\begin{aligned} x_1 &= b_4^1 = 19 \\ x_2 &= b_4^2 = 14 \\ x_3 &= b_4^3 = 28 \\ x_4 &= b_4^4 = 28 \end{aligned}$$

- **Step 5:** Generate B_i by

$$\begin{aligned} B_1 &= \begin{bmatrix} 16 & 5 \\ 16 & 19 \end{bmatrix} \\ B_2 &= \begin{bmatrix} 8 & 16 \\ 28 & 14 \end{bmatrix} \\ B_3 &= \begin{bmatrix} 15 & 8 \\ 28 & 28 \end{bmatrix} \\ B_4 &= \begin{bmatrix} 15 & 10 \\ 28 & 28 \end{bmatrix} \end{aligned}$$

- **Step 6:** We obtain the message matrix M as

$$M = \begin{bmatrix} 8 & 16 & 16 & 5 \\ 28 & 14 & 16 & 19 \\ 15 & 10 & 15 & 8 \\ 28 & 28 & 28 & 28 \end{bmatrix}$$

and we fill in the matrix according to the alphabet table as follows

$$M = \begin{bmatrix} \mathrm{G} & \mathrm{O} & \mathrm{O} & \mathrm{D} \\ 0 & \mathrm{M} & \mathrm{O} & \mathrm{R} \\ \mathrm{N} & \mathrm{I} & \mathrm{N} & \mathrm{G} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example 3.2. Let us consider the message matrix for the following message text:

"THE PEN IS MIGHTIER THAN THE SWORDS"

and create the code matrix using the message text (We can fill in the message matrix by placing "0" in the empty places.)

$$M = \begin{bmatrix} T & H & E & 0 & P & E \\ N & 0 & I & S & 0 & M \\ I & G & H & T & I & E \\ R & 0 & T & H & A & N \\ 0 & T & H & E & 0 & S \\ W & O & R & D & 0 & 0 \end{bmatrix}.$$

- **Step 1:** Divide the message matrix M of the size 6×6 named B_i $(1 \leq i \leq 4)$ from right to left each size 3×3.

$$B_1 = \begin{bmatrix} 0 & P & E \\ S & 0 & M \\ T & I & E \end{bmatrix}$$
$$B_2 = \begin{bmatrix} T & H & E \\ N & 0 & I \\ I & G & H \end{bmatrix}$$
$$B_3 = \begin{bmatrix} H & A & N \\ E & 0 & S \\ D & 0 & 0 \end{bmatrix}$$
$$B_4 = \begin{bmatrix} R & 0 & T \\ 0 & T & H \\ W & O & R \end{bmatrix}$$

- **Step 2:** Due to $b = 4 > 3$, we calculate $n = \left[\left|\frac{b}{2}\right|\right] = 2$. For $n = 2$, we can get according to the alphabet table as follows:

T	H	E	0	P	E	N	0	I	S	0	M	I	G	H	T	I	E	R
21	9	6	28	17	6	15	28	10	20	28	14	10	8	9	21	10	6	19

0	T	H	A	N	0	T	H	E	0	S	W	O	R	D
28	21	9	2	15	28	21	9	6	28	20	24	16	19	5

- **Step 3:** We have the elements of the blocks B_i $(1 \leq i \leq 4)$ as follows

$$\begin{aligned} b_1^1 &= 28 \quad b_2^1 = 17 \quad b_3^1 = 6 \quad b_4^1 = 20 \\ b_5^1 &= 28 \quad b_6^1 = 14 \quad b_7^1 = 21 \quad b_8^1 = 10 \quad b_9^1 = 6 \end{aligned}$$

$$\begin{aligned} b_1^2 &= 21 \quad b_2^2 = 9 \quad b_3^2 = 6 \quad b_4^2 = 15 \\ b_5^2 &= 28 \quad b_6^2 = 10 \quad b_7^2 = 10 \quad b_8^2 = 8 \quad b_9^2 = 9 \end{aligned}$$

$$\begin{aligned} b_1^3 &= 9 \quad b_2^3 = 2 \quad b_3^3 = 15 \quad b_4^3 = 6 \\ b_5^3 &= 28 \quad b_6^3 = 20 \quad b_7^3 = 5 \quad b_8^3 = 28 \quad b_9^3 = 28 \end{aligned}$$

$$\begin{aligned} b_1^4 &= 19 \quad b_2^4 = 28 \quad b_3^4 = 21 \quad b_4^4 = 28 \\ b_5^4 &= 21 \quad b_6^4 = 9 \quad b_7^4 = 24 \quad b_8^4 = 16 \quad b_9^4 = 19 \end{aligned}$$

- **Step 4:** Now, we calculate the determinants of the blocks B_i as follows:

$$\begin{aligned} d_1 &= \det B_1 = 1414 \\ d_2 &= \det B_1 = 2337 \\ d_3 &= \det B_3 = 2300 \\ d_4 &= \det B_4 = -5179 \end{aligned}$$

- **Step 5:** Construct F as

$$F = \begin{bmatrix} 1414 & 28 & 17 & 20 & 28 & 14 & 21 & 10 & 6 \\ 2337 & 21 & 9 & 15 & 28 & 10 & 10 & 8 & 9 \\ 2300 & 9 & 2 & 6 & 28 & 20 & 5 & 28 & 28 \\ -5179 & 19 & 28 & 28 & 21 & 9 & 24 & 16 & 19 \end{bmatrix}.$$

Decoding algorithm

- **Step 1:** Calculate Q^2 as follows:

$$Q^2 = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

- **Step 2:** Multiply each block by Q^2 as

$$\begin{aligned} B_1.Q^2 &= \begin{bmatrix} 28 & 17 & x_1 \\ 20 & 28 & 14 \\ 21 & 10 & 6 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x_1+73 & 73 & 45 \\ 82 & 68 & 48 \\ 58 & 52 & 31 \end{bmatrix} \\ B_2.Q^2 &= \begin{bmatrix} 21 & 9 & x_2 \\ 15 & 28 & 10 \\ 10 & 8 & 9 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x_2+51 & 51 & 30 \\ 68 & 58 & 43 \\ 37 & 28 & 18 \end{bmatrix} \\ B_3.Q^2 &= \begin{bmatrix} 9 & 2 & x_3 \\ 6 & 28 & 20 \\ 5 & 28 & 28 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x_3+20 & 20 & 11 \\ 60 & 40 & 34 \\ 66 & 38 & 33 \end{bmatrix} \\ B_4.Q^2 &= \begin{bmatrix} 19 & 28 & x_4 \\ 28 & 21 & 9 \\ 24 & 16 & 19 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x_4+66 & 66 & 47 \\ 86 & 77 & 49 \\ 83 & 64 & 40 \end{bmatrix} \end{aligned}$$

- **Step 3:** From the determinants of both sides of the equations, we get as follows:

$$\begin{aligned} x_1 &= b_3^1 = 6 \\ x_2 &= b_3^2 = 6 \\ x_3 &= b_3^3 = 15 \\ x_4 &= b_3^4 = 21 \end{aligned}$$

- **Step 4:** Rename x_i as follows

$$\begin{aligned} x_1 &= b_3^1 = 6 \\ x_2 &= b_3^2 = 6 \\ x_3 &= b_3^3 = 15 \\ x_4 &= b_3^4 = 21 \end{aligned}$$

- **Step 5:** Generate B_i as

$$B_1 = \begin{bmatrix} 28 & 17 & 6 \\ 20 & 28 & 14 \\ 21 & 10 & 6 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 21 & 9 & 6 \\ 15 & 28 & 10 \\ 10 & 8 & 9 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 9 & 2 & 15 \\ 6 & 28 & 20 \\ 5 & 28 & 28 \end{bmatrix}$$

$$B_4 = \begin{bmatrix} 19 & 28 & 21 \\ 28 & 21 & 9 \\ 24 & 16 & 19 \end{bmatrix}$$

- **Step 6:** We obtain the message matrix M as follows:

$$M = \begin{bmatrix} 21 & 9 & 6 & 28 & 17 & 6 \\ 15 & 28 & 10 & 20 & 28 & 14 \\ 10 & 8 & 9 & 21 & 10 & 6 \\ 19 & 28 & 21 & 9 & 2 & 15 \\ 28 & 21 & 9 & 6 & 28 & 20 \\ 24 & 16 & 19 & 5 & 28 & 28 \end{bmatrix}$$

and we fill in the matrix according to the alphabet table as follows

$$M = \begin{bmatrix} T & H & E & 0 & P & E \\ N & 0 & I & S & 0 & M \\ I & G & H & T & I & E \\ R & 0 & T & H & A & N \\ 0 & T & H & E & 0 & S \\ W & O & R & D & 0 & 0 \end{bmatrix}.$$

Example 3.3. Let us consider the message matrix for the following message text:

"I LIKE TURKISH VAN CATS"

and create the code matrix using the message text (We can fill in the message matrix by placing "0" in the empty places.)

$$M = \begin{bmatrix} I & 0 & L & I & K \\ E & 0 & T & U & R \\ K & I & S & H & 0 \\ V & A & N & 0 & C \\ A & T & S & 0 & 0 \end{bmatrix}.$$

- **Step 1:** Divide the message matrix M of the size 5×5 named B_i $(1 \leq i \leq 4)$ from left to right each size 5×5.

$$B_1 = \begin{bmatrix} I & 0 & L & I & K \\ E & 0 & T & U & R \\ K & I & S & H & 0 \\ V & A & N & 0 & C \\ A & T & S & 0 & 0 \end{bmatrix}$$

- **Step 2:** Due to $b = 1 < 3$, we calculate $n = b = 1$. For $n = 1$, we can get this table according to the alphabet table as follows:

I	0	L	I	K	E	0	T	U	R	K	I	S	H	0
9	27	12	9	11	5	27	20	21	18	11	9	19	8	27

V	A	N	0	C	A	T	S	0	0
22	1	14	27	3	1	20	19	27	27

- **Step 3:** We have the elements of the blocks B_i $(1 \leq i \leq 4)$ as follows

$$\begin{array}{rcl} b_1^1 & = & 9 \\ b_5^1 & = & 11 \\ b_9^1 & = & 21 \\ b_{13}^1 & = & 19 \\ b_{17}^1 & = & 1 \\ b_{21}^1 & = & 1 \\ b_{25}^1 & = & 27 \end{array} \qquad \begin{array}{ll} b_2^1 = 27 & b_3^1 = 12 \\ b_6^1 = 5 & b_7^1 = 27 \\ b_{10}^1 = 18 & b_{11}^1 = 11 \\ b_{14}^1 = 8 & b_{15}^1 = 27 \\ b_{18}^1 = 14 & b_{19}^1 = 27 \\ b_{22}^1 = 20 & b_{23}^1 = 19 \end{array} \qquad \begin{array}{l} b_4^1 = 9 \\ b_8^1 = 20 \\ b_{12}^1 = 9 \\ b_{16}^1 = 22 \\ b_{20}^1 = 3 \\ b_{24}^1 = 27 \end{array}$$

- **Step 4:** Now we calculate the determinants of the blocks B_i as follows:

$$\det B_1 = 1759340$$

- **Step 5:** Construct F as follows:

$$F = \begin{bmatrix} 1759340 & 9 & 27 & 12 & 9 & 5 & 27 & 20 & 21 & 18 & 11 & 9 \\ 19 & 8 & 27 & 22 & 1 & 14 & 27 & 3 & 1 & 20 & 19 & 27 & 27 \end{bmatrix}$$

Decoding algorithm

- **Step 1:** Calculate Q^1 as follows:

$$Q^1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

- **Step 2:** Multiply each block by Q^1 as follows:

$$B.Q^1 = \begin{bmatrix} 9 & 27 & 12 & 9 & x \\ 5 & 27 & 20 & 21 & 18 \\ 11 & 9 & 19 & 8 & 27 \\ 22 & 1 & 14 & 27 & 3 \\ 1 & 20 & 19 & 27 & 27 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 36 & 21 & 18 & x+9 & 9 \\ 32 & 25 & 26 & 23 & 5 \\ 20 & 30 & 19 & 38 & 11 \\ 23 & 36 & 49 & 25 & 22 \\ 21 & 20 & 28 & 28 & 1 \end{bmatrix}$$

- **Step 3:** From the determinants of both sides of the equations, we get

$$x_1 = b_4^1 = 11$$

- **Step 4:** Rename x_i as follows

$$x_1 = b_4^1 = 11$$

- **Step 5:** Generate B_i as

$$B = \begin{bmatrix} 9 & 27 & 12 & 9 & 11 \\ 5 & 27 & 20 & 21 & 18 \\ 11 & 9 & 19 & 8 & 27 \\ 22 & 1 & 14 & 27 & 3 \\ 1 & 20 & 19 & 27 & 27 \end{bmatrix}$$

- **Step 6:** We obtain the message matrix M as

$$M = \begin{bmatrix} 9 & 27 & 12 & 9 & 11 \\ 5 & 27 & 20 & 21 & 18 \\ 11 & 9 & 19 & 8 & 27 \\ 22 & 1 & 14 & 27 & 3 \\ 1 & 20 & 19 & 27 & 27 \end{bmatrix}$$

and we fill in the matrix according to the alphabet table as follows

$$M = \begin{bmatrix} I & 0 & L & I & K \\ E & 0 & T & U & R \\ K & I & S & H & 0 \\ V & A & N & 0 & C \\ A & T & S & 0 & 0 \end{bmatrix}.$$

4 Conclusion

In this paper, we defined a new $k-$order Fibonacci numbers coding/decoding theory using the matrices of the $k-$order Fibonacci numbers and in this method we can divide the message into $k \times k$ block matrices instead of 2×2. Therefore, with the generalization we made, we divided the message texts of any length into arbitrary blocks. In the last part, we gave illustrative examples of this coding method. With this generalization, we generalized the 2×2 type block matrices made in [15] and moved them to the $k \times k$ type block matrices. Therefore, we can divide our message text of any length into block matrices depending on the k value we choose arbitrarily. In this method, we can both complete the encryption process faster without further processing and encrypt our message text of the desired length.

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