Singularly perturbed two-point fuzzy boundary value problems

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Abstract

In this work, we firstly introduce singularly perturbed two-point fuzzy boundary value problems (SPTPFBVPs). Then, we provide an algorithm for obtaining their solutions using Zadeh's extension principle. We present some results on the behavior of the α-cuts of the solutions. To demonstrate the robustness of the algorithm, we fuzzify several examples with different boundary layers from the literature. Subsequently, we successfully apply the algorithm to these examples.

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1 Introduction

In differential equations, singular perturbation problems (SPPs) arise when a small parameter multiplies with the highest order derivative of the given differential equation. Boundary layers occur in the solution of a singular perturbation problem (SPP). This means that the solution has layers where the solution changes rapidly. However the solution changes uniformly and slowly away from this transiton layer [1, 2, 3, 4, 5, 6, 7]. These narrow regions are usually found at the domain's edges. Transition points, shock layers, boundary layers, skin layers, edge layers, and other terminology are used to define these changes at solutions in applied problems [8]. These problems can be seen in a variety of real-world situations. In science and engineering, singularly perturbed problems play an important role. These problems are important in many fields of applied mathematics, including quantum mechanics, fluid mechanics, chemical reactor theory, optimal control, aerodynamics, reaction-diffusion processes, geophysics, and many others. Due to the nature of layer behavior, the mathematical treatment of such problems faces major computational challenges [9].

The concept of fuzzy set theory was initially introduced by Zadeh in 1965, while the definition of a fuzzy differential equation was first presented in the literature by Byatt [10]. There are different suggestions for obtaining the solution of a fuzzy differential equation (FDE) in the literature. One of these suggestions is the Zadeh's extension principle approach [11]. For solving a FDE by using the extension principle given by Zadeh, firstly the corresponding crisp initial value problem's solution is obtained, then this solution is fuzzified by the extension principle. FDEs are important in science

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and engineering as they provide a tool to account for vagueness in models. Consequently, numerous studies have been conducted on the applications of fuzzy set theory such as population models [12, 13], civil engineering models [14], computational biology models [15] and etc.

In this paper, we used the approach in [16] to find α-cuts for the solutions to the singularly perturbed two-point fuzzy boundary value problems (SPTPFBVP). With this approach, there is no need to consider the signs of a solution and its derivatives as in [17, 18], That is why, one can easily obtain the solution of a singularly perturbed two-point fuzzy boundary value problem. We also gave some results about the fuzziness of the solutions to SPFBVP.

2 Fuzzy set theory and basic concepts

In this part, we give some definitions in fuzzy set theory which we need in this study.

Definition 2.1. Let X be a universe of discourse and $\mu_{\tilde{A}} : X \to [0,1]$ be a function. The set

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$$

is called a fuzzy set in X. Here $\mu_{\tilde{A}}$ is called membership function [19].

We will denote the set of all fuzzy sets in X by $F(X)$.

Definition 2.2. Let $\tilde{A} \in F(\mathbb{R})$. The definition of α-cut (level sets) of $\tilde{A}$ is

$$A(\alpha) = \{x \in \mathbb{R} : \alpha \in (0,1], \mu_{\tilde{A}}(x) \geq \alpha \},$$

and

$$A(0) = cl(\{x \in \mathbb{R} : \mu_{\tilde{A}}(x) > 0 \}).$$

Here "cl" denotes the closure of a set [20].

Definition 2.3. If $\tilde{A} \in F(\mathbb{R})$ satisfies the following conditions, then it is called as a fuzzy number in $\mathbb{R}$:

1. $\widetilde{A}$ is a normal set, i.e., there exists at least an element x in $\mathbb{R}$ with $\mu_{\widetilde{A}}(x) = 1$.
2. $\mu_{\widetilde{A}}$ is fuzzy convex (a quasi-concave function), i.e., $\mu_{\widetilde{A}}(ka + (1-k)b) \geq \min\{\mu_{\widetilde{A}}(a), \mu_{\widetilde{A}}(b)\}$ for every $a, b \in \mathbb{R}$ and $k \in [0,1]$.
3. $\mu_{\widetilde{A}}$ is upper semicontinuous, i.e., for every $\varepsilon > 0$ and $x_0 \in \mathbb{R}$ there exists a $\delta > 0$ such that $\mu_{\widetilde{A}}(x) - \mu_{\widetilde{A}}(x_0) < \varepsilon$ whenever $|x - x_0| < \delta$.
4. $\widetilde{A}$ is compactly supported, i.e., $A(0)$ is closed and bounded [19].

Definition 2.4. A triangular fuzzy number (TFN) $\tilde{A} \in F_N(\mathbb{R})$ is defined with the following membership function:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2, \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x \leq a_3, \\ 0, & \text{otherwise.} \end{cases}$$

Here $a_1 \leq a_2 \leq a_3$ and it is denoted by $\tilde{A} = (a_1, a_2, a_3)$ [19].

The following definition is known as the Zadeh's extension principle.

Definition 2.5. Let X and Y be crisp sets, $\widetilde{A}$ be a fuzzy set in X, and $f : X \to Y$ be a classical function. Zadeh's extension of the function f is the fuzzy function $\widetilde{f} : F(X) \to F(Y)$ applied to $\widetilde{A}$ that provides the fuzzy subset of Y. The membership function of $\widetilde{f}(\widetilde{A})$ is defined by

$$\mu_{\widetilde{f}(\widetilde{A})}(y) = \begin{cases} \sup\{\mu_{\widetilde{A}}(x) : x \in f^{-1}(y)\} & , \quad f^{-1}(y) \neq \varnothing \\ 0 & , \quad f^{-1}(y) = \varnothing \end{cases}$$

where $f^{-1}(y) = x \in X : f(x) = y$ [19, 21].

Theorem 2.6. If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then its extension $\widetilde{f} : F(\mathbb{R}) \to F(\mathbb{R})$ obtained by Zadeh's extension principle is well-defined, continuous and

$$\widetilde{f}(\widetilde{A})(\alpha) = f(A(\alpha))$$

for all $\alpha \in [0, 1]$ and for all $\widetilde{A} \in F(\mathbb{R})$, where $\widetilde{f}(\widetilde{A}) = \{f(u) : u \in \widetilde{A}(\alpha)\}$ [20].

3 Singularly perturbed two-point fuzzy boundary value problems

In this part, we will give an algorithm for the solution SPTPFBVPs and some properties of solutions of them. First we will give an algorithm to solve the following SPFBVP of the form:

$$Lx(t) = \varepsilon x''(t) + a_1(t)x'(t) + a_2(t)x(t) = \sum_{j=1}^{n} h_j(t), \tag{3.1}$$

$$x(t_0) = \widetilde{\gamma}_0, \tag{3.2}$$

$$x(t_1) = \widetilde{\gamma}_1. \tag{3.3}$$

where $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$are fuzzy numbers in $\mathbb{R}$; $h_j(t)$ $(j : 1, 2, ..., n)$, $a_1(t)$ and $a_2(t)$ are continuous functions on $\Omega = [t_0, t_1]$, and ε is the perturbation parameter $0 < \varepsilon \ll 1$.

Now let us construct an algorithm for finding the solution of (3.1)-(3.3). Consider the following crisp singularly perturbed boundary value problem:

$$Lx(t) = \varepsilon x''(t) + a_1(t)x'(t) + a_2(t)x(t) = \sum_{j=1}^{n} h_j(t), \tag{3.4}$$

$$x(t_0) = \gamma_0, \tag{3.5}$$

$$x(t_1) = \gamma_1. \tag{3.6}$$

Assume that the differential equation in (3.4) has the general solution as

$$x(t; \varepsilon) = c_1 x_1(t; \varepsilon) + c_2 x_2(t; \varepsilon) + x_p(t) \tag{3.7}$$

where c_1 and c_2 are arbitrary constants, x_1 and x_2 are the linearly independent solutions of

$$Lx(t) = \varepsilon x''(t) + a_1(t)x'(t) + a_2(t)x(t) = 0 \tag{3.8}$$

and x_p is the particular solution of (3.4).
Now we will obtain the solution of the crisp initial value problem (3.4)-(3.6). By imposing the initial conditions to the general solution given in (3.7) we obtain the following system:

$$c_1 x_1(t_0;\varepsilon) + c_2 x_2(t_0;\varepsilon) = \gamma_0 - x_p(t_0;\varepsilon) \tag{3.9}$$
$$c_1 x_1(t_1;\varepsilon) + c_2 x_2(t_1;\varepsilon) = \gamma_1 - x_p(t_1;\varepsilon) \tag{3.10}$$

In matrix-vector notations, we can rewrite this system as:

$$\mathbf{Wc} = \mathbf{N}$$

where

$$\mathbf{W}=\begin{bmatrix} x_1(t_0;\varepsilon) & x_2(t_0;\varepsilon) \\ x_1(t_1;\varepsilon) & x_2(t_1;\varepsilon) \end{bmatrix},$$
$$\mathbf{c}=\begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$
$$\mathbf{N}=\begin{bmatrix} \gamma_0 - x_p(t_0;\varepsilon) \\ \gamma_1 - x_p(t_1;\varepsilon) \end{bmatrix},$$

such that $|W| \neq 0$. By using the Cramer's Rule, we obtain

$$c_1 = \frac{|\mathbf{W}_1|}{|\mathbf{W}|},$$
$$c_2 = \frac{|\mathbf{W}_2|}{|\mathbf{W}|}$$

where

$$\mathbf{W}_1=\begin{bmatrix} \gamma_0 - x_p(t_0;\varepsilon) & x_2(t_0;\varepsilon) \\ \gamma_1 - x_p(t_1;\varepsilon) & x_2(t_1;\varepsilon) \end{bmatrix},$$
$$\mathbf{W}_2=\begin{bmatrix} x_1(t_0;\varepsilon) & \gamma_0 - x_p(t_0;\varepsilon) \\ x_1(t_1;\varepsilon) & \gamma_1 - x_p(t_1;\varepsilon) \end{bmatrix}.$$

Thus, c_1 and c_1 are

$$\begin{aligned} c_1 &= \frac{(\gamma_0 - x_p(t_0;\varepsilon))x_2(t_1;\varepsilon) - (\gamma_1 - x_p(t_1;\varepsilon))x_2(t_0;\varepsilon)}{|\mathbf{W}|} \\ &= \frac{\gamma_0 x_2(t_1;\varepsilon) - x_p(t_0;\varepsilon)x_2(t_1;\varepsilon) - \gamma_1 x_2(t_0;\varepsilon) + x_p(t_1;\varepsilon)x_2(t_0;\varepsilon)}{|\mathbf{W}|} \\ &= \frac{\gamma_0 x_2(t_1;\varepsilon) - \gamma_1 x_2(t_0;\varepsilon) + x_p(t_1;\varepsilon)x_2(t_0;\varepsilon) - x_p(t_0;\varepsilon)x_2(t_1;\varepsilon)}{|\mathbf{W}|}, \end{aligned}$$

and

$$\begin{aligned} c_2 &= \frac{x_1(t_0;\varepsilon)(\gamma_1 - x_p(t_1;\varepsilon)) - x_1(t_1;\varepsilon)(\gamma_0 - x_p(t_0;\varepsilon))}{|\mathbf{W}|} \\ &= \frac{x_1(t_0;\varepsilon)\gamma_1 - x_1(t_0;\varepsilon)x_p(t_1;\varepsilon) - x_1(t_1;\varepsilon)\gamma_0 + x_1(t_1;\varepsilon)x_p(t_0;\varepsilon)}{|\mathbf{W}|} \\ &= \frac{\gamma_1 x_1(t_0;\varepsilon) - \gamma_0 x_1(t_1;\varepsilon) + x_p(t_0;\varepsilon)x_1(t_1;\varepsilon) - x_p(t_1;\varepsilon)x_1(t_0;\varepsilon)}{|\mathbf{W}|}. \end{aligned}$$

For simplicity, let

$$\begin{aligned} A_1 &= \frac{x_2(t_1;\varepsilon)}{|\mathbf{W}|}, \\ A_2 &= -\frac{x_2(t_0;\varepsilon)}{|\mathbf{W}|}, \\ A_3 &= \frac{x_p(t_1;\varepsilon)x_2(t_0;\varepsilon) - x_p(t_0;\varepsilon)x_2(t_1;\varepsilon)}{|\mathbf{W}|}, \end{aligned}$$

and

$$\begin{aligned} B_1 &= \frac{x_1(t_0;\varepsilon)}{|\mathbf{W}|}, \\ B_2 &= -\frac{x_1(t_1;\varepsilon)}{|\mathbf{W}|}, \\ B_3 &= \frac{x_p(t_0;\varepsilon)x_1(t_1;\varepsilon) - x_p(t_1;\varepsilon)x_1(t_0;\varepsilon)}{|\mathbf{W}|}. \end{aligned}$$

So we can obtain c_1 and c_2 as

$$\begin{aligned} c_1 &= A_1\gamma_0 + A_2\gamma_1 + A_3, \\ c_2 &= B_1\gamma_1 + B_2\gamma_0 + B_3. \end{aligned}$$

Substituting c_1 and c_2 into (3.7), we obtain

$$\begin{aligned} x(t;\varepsilon) &= c_1x_1(t;\varepsilon) + c_2x_2(t;\varepsilon) + x_p(t;\varepsilon) \\ &= (A_1\gamma_0 + A_2\gamma_1 + A_3)x_1(t;\varepsilon) \\ &+ (B_1\gamma_1 + B_2\gamma_0 + B_3)x_2(t;\varepsilon) + x_p(t;\varepsilon) \\ &= (A_1x_1(t;\varepsilon) + B_2x_2(t;\varepsilon))\gamma_0 + (A_2x_1(t;\varepsilon) + B_1x_2(t;\varepsilon))\gamma_1 \\ &+ A_3x_1(t;\varepsilon) + B_3x_2(t;\varepsilon) + x_p(t;\varepsilon). \end{aligned}$$

Let

$$\begin{aligned} M(t) &= A_1x_1(t;\varepsilon) + B_2x_2(t;\varepsilon), \\ N(t) &= A_2x_1(t;\varepsilon) + B_1x_2(t;\varepsilon), \\ H(t) &= A_3x_1(t;\varepsilon) + B_3x_2(t;\varepsilon) + x_p(t;\varepsilon). \end{aligned}$$

Hence the solution of (3.4)-(3.6) is

$$x(t;\varepsilon) = M(t)\gamma_0 + N(t)\gamma_1 + H(t). \tag{3.11}$$

The application of the Zadeh's Extension Principle to (2.11) leads to the extension:

$$\widetilde{x}(t;\varepsilon) = M(t)\widetilde{\gamma}_0 + N(t)\widetilde{\gamma}_1 + H(t) \tag{3.12}$$

which is the solution of the problem (3.1)-(3.3) with fuzzy boundary conditions $\widetilde{\gamma}_0$, and $\widetilde{\gamma}_1$ in $F_N(\mathbb{R})$. Note that since the function $x(t;\varepsilon)$ is continuous, $\widetilde{x}(t;\varepsilon)$ is well-defined and continuous by Theorem 2.6, hence the existence of the solution $\widetilde{x}(t;\varepsilon)$ to the problem is guaranteed.

Let the α-cuts of $\widetilde{x}(t;\varepsilon)$, $\widetilde{\gamma}_0$, and $\widetilde{\gamma}_1$ be as follows:

$$\begin{aligned} x(t;\varepsilon,\alpha) &= [x_1(t;\varepsilon,\alpha), x_2(t;\varepsilon,\alpha)], \\ \gamma_0(\alpha) &= [\gamma_{01}(\alpha), \gamma_{02}(\alpha)], \\ \gamma_1(\alpha) &= [\gamma_{11}(\alpha), \gamma_{12}(\alpha)]. \end{aligned}$$

Hence, we can write the solution in (3.12) in terms of these α-cuts as follows:

$$\begin{aligned} [x_1(t;\varepsilon,\alpha), x_2(t;\varepsilon,\alpha)] = M(t)[\gamma_{01}(\alpha), \gamma_{02}(\alpha)] + \\ N(t)[\gamma_{11}(\alpha), \gamma_{12}(\alpha)] + H(t). \end{aligned}$$

Here $x_1(t;\varepsilon,\alpha)$ and $x_2(t;\varepsilon,\alpha)$ are defined as

$$\begin{aligned} x_1(t;\varepsilon,\alpha) &= \min\{x(t;\varepsilon,\alpha) : t \in [t_0,t_1]\}, \\ x_2(t;\varepsilon,\alpha) &= \max\{x(t;\varepsilon,\alpha) : t \in [t_0,t_1]\} \end{aligned}$$

for any fixed ε and α. Since (3.12) is linear in $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ the end points of α-cuts of them switch according to the sign of the functions $M(t)$ and $N(t)$. That is why, it is easy to obtain $x_1(t;\varepsilon,\alpha)$ and $x_2(t;\varepsilon,\alpha)$ by using the Heaviside step function instead of the their definition given above. Thus, the desired α-cuts solutions of SPTPFBVP $x_1(t;\varepsilon,\alpha)$ and $x_2(t;\varepsilon,\alpha)$ are obtained as

$$\begin{aligned} x_1(t;\varepsilon,\alpha) = (\gamma_{02}(\alpha) + (\gamma_{01}(\alpha) - \gamma_{02}(\alpha))\theta(M(t)))M(t) \\ + (\gamma_{12}(\alpha) + (\gamma_{11}(\alpha) - \gamma_{12}(\alpha))\theta(N(t)))N(t) + H(t), \end{aligned}$$

and

$$\begin{aligned} x_2(t;\varepsilon,\alpha) = (\gamma_{01}(\alpha) + (\gamma_{02}(\alpha) - \gamma_{01}(\alpha))\theta(M(t)))M(t) \\ + (\gamma_{11}(\alpha) + (\gamma_{12}(\alpha) - \gamma_{11}(\alpha))\theta(N(t)))N(t) + H(t). \end{aligned}$$

Now we will give some results on the behaviour of α-cuts of the solution. Let us take the boundary conditions as triangular fuzzy numbers in $\mathbb{R}$ such that

$$\begin{aligned} \widetilde{\gamma}_0 &= (m_1, m_2, m_3), \\ \widetilde{\gamma}_1 &= (n_1, n_2, n_3). \end{aligned}$$

Let $s(t;\varepsilon)$ denote the difference between the lower and upper endpoints of the fuzzy solution $\widetilde{x}(t;\varepsilon)$ for a fixed ε:

$$\begin{aligned}
s(t;\varepsilon) &:= x_2(t;\varepsilon,0) - x_1(t;\varepsilon,0)\\
&= \frac{1}{|W(\varepsilon)|}[(1-2\theta(\frac{x_1(t_0;\varepsilon)x_2(t;\varepsilon)-x_1(t;\varepsilon)x_2(t_0;\varepsilon)}{|W(\varepsilon)|}))n_1(x_1(t_0;\varepsilon)x_2(t;\varepsilon)\\
&- x_1(t;\varepsilon)x_2(t_0;\varepsilon)) + \theta(\frac{x_1(t_0;\varepsilon)x_2(t;\varepsilon)-x_1(t;\varepsilon)x_2(t_0;\varepsilon)}{|W(\varepsilon)|})]\\
&n_3(x_1(t_0;\varepsilon)x_2(t;\varepsilon) - x_1(t;\varepsilon)x_2(t_0;\varepsilon)) + (-1+2\theta(\frac{x_1(t;\varepsilon)x_2(t_1;\varepsilon)-x_1(t_1;\varepsilon)x_2(t;\varepsilon)}{|W(\varepsilon)|}))\\
&(m_1-m_3)(x_1(t;\varepsilon)x_2(t_1;\varepsilon) - x_1(t_1;\varepsilon)x_2(t;\varepsilon))].
\end{aligned}$$

Let

$$p(t;\varepsilon) = \frac{x_1(t_0;\varepsilon)x_2(t;\varepsilon) - x_1(t;\varepsilon)x_2(t_0;\varepsilon)}{|W(\varepsilon)|},$$

and

$$q(t;\varepsilon) = \frac{x_1(t;\varepsilon)x_2(t_1;\varepsilon) - x_1(t_1;\varepsilon)x_2(t;\varepsilon)}{|W(\varepsilon)|}.$$

So we obtain the following results

$$\begin{aligned}
s(t;\varepsilon) &= x_2(t;\varepsilon,0) - x_1(t;\varepsilon,0)\\
&= [(1-2\theta(p(t)))n_1p(t) + \theta(p(t))n_3p(t)\\
&+ (-1+2\theta(q(t)))(m_1-m_3)q(t)]
\end{aligned}$$

Let us consider the solution of the form $x(t) = e^{rt}$ for the case $a_1(t)$ and $a_2(t)$ are a constant functions, then

$$\begin{aligned}
x_1(t;\varepsilon) &= e^{r_1(\varepsilon)t},\\
x_2(t;\varepsilon) &= e^{r_2(\varepsilon)t}
\end{aligned}$$

where $r_1 := r_1(\varepsilon)$ and $r_2 := r_2(\varepsilon)$ are the roots of the characteristic equation with $r_1(\varepsilon) \le r_2(\varepsilon)$.

Theorem 3.1. s is a nonnegative function on $[t_0, t_1]$.

Proof. Let $x_1(t;\varepsilon) = e^{r_1t}$ and $x_2(t;\varepsilon) = e^{r_2t}$.

$$\begin{aligned}
p(t;\varepsilon) &= \frac{e^{r_1t_0+r_2t} - e^{r_2t_0+r_1t}}{|W(\varepsilon)|},\\
q(t;\varepsilon) &= \frac{e^{r_2t_1+r_1t} - e^{r_1t_1+r_2t}}{|W(\varepsilon)|}.
\end{aligned}$$

Observe that

$$\begin{aligned}|W(\varepsilon)| &= \begin{vmatrix} x_1(t_0;\varepsilon) & x_2(t_0;\varepsilon) \\ x_1(t_1;\varepsilon) & x_2(t_1;\varepsilon) \end{vmatrix} \\ &= \begin{vmatrix} e^{r_1 t_0} & e^{r_2 t_0} \\ e^{r_1 t_1} & e^{r_2 t_1} \end{vmatrix} \\ &= e^{r_1 t_0 + r_2 t_1} - e^{r_1 t_1 + r_2 t_0} > 0.\end{aligned}$$

Since for every $t \in [t_0, t_1]$

$$\begin{aligned}(t - t_0)(r_2 - r_1) &\geq 0 \\ r_2 t - r_1 t - r_2 t_0 - r_1 t_0 &\geq 0 \\ r_2 t + r_1 t_0 &\geq r_1 t + r_2 t_0\end{aligned}$$

we obtain that

$$e^{r_2 t + r_1 t_0} \geq e^{r_1 t + r_2 t_0}.$$

Hence $p(t)$ is a nonnegative function. And since for every $t \in [t_0, t_1]$ we get

$$\begin{aligned}(t_1 - t)(r_2 - r_1) &\geq 0 \\ r_2 t_1 - r_1 t_1 - r_2 t + r_1 t &\geq 0 \\ r_2 t_1 + r_1 t &\geq r_1 t_1 + r_2 t\end{aligned}$$

and consequently

$$e^{r_2 t_1 + r_1 t} \geq e^{r_1 t_1 + r_2 t}.$$

Hence, $q(t)$ is a nonnegative function, too.

Since $\theta(p(t;\varepsilon)) = 1$ and $\theta(q(t;\varepsilon)) = 1$, we obtain that

$$s(t;\varepsilon) = (n_3 - n_1)p(t;\varepsilon) + (m_3 - m_1)q(t;\varepsilon)$$

which is a nonnegative function on $[t_0, t_1]$ for any fixed $\varepsilon \in (0, 1]$, indeed. Q.E.D.

Let us now compute the derivative of s

$$\begin{aligned}\frac{d}{dt}s(t;\varepsilon) &= (n_3 - n_1)\frac{d}{dt}p(t;\varepsilon) + (m_3 - m_1)\frac{d}{dt}q(t;\varepsilon) \\ &= \frac{n_3 - n_1}{|W(\varepsilon)|}\left(r_2 e^{r_1 t_0 + r_2 t} - r_1 e^{r_2 t_0 + r_1 t}\right) \\ &+ \frac{m_3 - m_1}{|W(\varepsilon)|}(r_1 e^{r_2 t_1 + r_1 t} - r_2 e^{r_1 t_1 + r_2 t}) \\ &= A\left(r_2 e^{r_1 t_0 + r_2 t} - r_1 e^{r_2 t_0 + r_1 t}\right) + B(r_1 e^{r_2 t_1 + r_1 t} - r_2 e^{r_1 t_1 + r_2 t}) \\ &= r_1 e^{r_1 t}(-A e^{r_2 t_0} + B e^{r_2 t_1}) + r_2 e^{r_2 t}(A e^{r_1 t_0} - B e^{r_1 t_1})\end{aligned}$$

where

$$A = \frac{n_3 - n_1}{|W(\varepsilon)|},$$
$$B = \frac{m_3 - m_1}{|W(\varepsilon)|}.$$

Since the length of a triangular fuzzy number $\widetilde{A} = (a_1, a_2, a_3)$ is defined by

$$len(\widetilde{A}) = a_3 - a_1,$$

we can also write that

$$A = \frac{len(\widetilde{\gamma}_1)}{|W(\varepsilon)|},$$
$$B = \frac{len(\widetilde{\gamma}_0)}{|W(\varepsilon)|}.$$

Theorem 3.2. i) Let $0 < r_1 < r_2$. If

$$len(\widetilde{\gamma}_0) \geq len(\widetilde{\gamma}_1)$$

and

$$t_0 \leq \frac{1}{r_1 - r_2} \ln(\frac{r_2}{r_1} \frac{(Ae^{r_1 t_0} - Be^{r_1 t_1})}{(Ae^{r_2 t_0} - Be^{r_2 t_1})})$$

then s is nondecreasing on $I = [t_0, \frac{1}{r_1 - r_2} \ln(\frac{r_2}{r_1} \frac{(Ae^{r_1 t_0} - Be^{r_1 t_1})}{(Ae^{r_2 t_0} - Be^{r_2 t_1})})]$.

ii) Let $r_1 < r_2 < 0$. If

$$len(\widetilde{\gamma}_0) \leq len(\widetilde{\gamma}_1)$$

and

$$t_1 \geq \frac{1}{r_1 - r_2} \ln(\frac{r_2}{r_1} \frac{(Ae^{r_1 t_0} - Be^{r_1 t_1})}{(Ae^{r_2 t_0} - Be^{r_2 t_1})}),$$

then s is nondecreasing on $I = [\frac{1}{r_1 - r_2} \ln(\frac{r_2}{r_1} \frac{(Ae^{r_1 t_0} - Be^{r_1 t_1})}{(Ae^{r_2 t_0} - Be^{r_2 t_1})}), t_1]$.

Proof. **i)** Let $0 < r_1 < r_2$. If $len(\ \widetilde{\gamma}_0\ ) = m_3 - m_1 \geq n_3 - n_1 = len(\ \widetilde{\gamma}_1\ )$ then $B \geq A$. For every $t \in I$ we have

$$
\begin{aligned}
t &\leq \frac{1}{r_1 - r_2} \ln[\frac{r_2}{r_1} \frac{(Ae^{r_1 t_0} - Be^{r_1 t_1})}{(Ae^{r_2 t_0} - Be^{r_2 t_1})}] \\
t(r_1 - r_2) &\geq \ln[\frac{r_2}{r_1} \frac{(Ae^{r_1 t_0} - Be^{r_1 t_1})}{(Ae^{r_2 t_0} - Be^{r_2 t_1})}] \\
e^{r_1 t - r_2 t} &\geq \frac{r_2}{r_1} \frac{(Ae^{r_1 t_0} - Be^{r_1 t_1})}{(Ae^{r_2 t_0} - Be^{r_2 t_1})} \\
\frac{r_1}{r_2} e^{r_1 t - r_2 t} &\geq \frac{(Ae^{r_1 t_0} - Be^{r_1 t_1})}{(Ae^{r_2 t_0} - Be^{r_2 t_1})} \\
r_1 e^{r_1 t}(Ae^{r_2 t_0} - Be^{r_2 t_1}) &\leq r_2 e^{r_2 t}(Ae^{r_1 t_0} - Be^{r_1 t_1}) \\
r_1 e^{r_1 t}(-Ae^{r_2 t_0} + Be^{r_2 t_1}) &\geq r_2 e^{r_2 t}(-Ae^{r_1 t_0} + Be^{r_1 t_1}).
\end{aligned}
$$

So, we obtain that

$$
\frac{d}{dt} s(t, \varepsilon) = r_1 e^{r_1 t}(-Ae^{r_2 t_0} + Be^{r_2 t_1}) + r_2 e^{r_2 t}(Ae^{r_1 t_0} - Be^{r_1 t_1}) \geq 0.
$$

Hence, s is nondecreasing on I.

ii) The proof can be easily obtained, similar to the previous one. Q.E.D.

4 Numerical examples

To assess the computational effectiveness of the proposed method, we will apply the algorithm to fuzzified versions of singularly perturbed two-point boundary value problems selected from [9]. These problems include one left end layer, two right end layers, two dual layers, and two internal layers.

Example 4.1. Consider the left end boundary layer SPTPFBVP

$$
\begin{aligned}
\varepsilon x''(t) + x'(t) - x(t) &= 0 \\
x(0) &= (0.8, 1, 1.2) \\
x(1) &= (0.8, 1, 1.2)
\end{aligned}
$$

The exact solution of the corresponding crisp problem is

$$
x(t) = \frac{[(e^{c_2} - 1)e^{c_1 t} + (1 - e^{c_1})e^{c_2 t}]}{e^{c_2} - e^{c_1}}
$$

where $c_1 = \frac{(-1+\sqrt{1+4\varepsilon})}{2\varepsilon}$ and $c_2 = \frac{(-1-\sqrt{1+4\varepsilon})}{2\varepsilon}$. The endpoints of the 0-cut of the fuzzy solutions are

$$x_1(t;\varepsilon,0) = \frac{1}{e^{\frac{\sqrt{4\varepsilon+1}}{\varepsilon}}-1}\left(\begin{array}{c} -0.4e^{-\frac{t\left(\sqrt{4\varepsilon+1}+1\right)}{2\varepsilon}}\left(e^{\frac{\sqrt{4\varepsilon+1}}{\varepsilon}}-e^{\frac{t\sqrt{4\varepsilon+1}}{\varepsilon}}\right) \\ \theta\left(\frac{e^{-\frac{t\left(\sqrt{4\varepsilon+1}+1\right)}{2\varepsilon}}\left(e^{\frac{\sqrt{4\varepsilon+1}}{\varepsilon}}-e^{\frac{t\sqrt{4\varepsilon+1}}{\varepsilon}}\right)}{-1+e^{\frac{\sqrt{4\varepsilon+1}}{\varepsilon}}}\right) \\ +e^{-\frac{(2t-1)\left(\sqrt{4\varepsilon+1}+1\right)}{2\varepsilon}}\left(\begin{array}{c} 1.2e^{\frac{(t-1)\left(\sqrt{4\varepsilon+1}+1\right)}{2\varepsilon}}\left(e^{\frac{\sqrt{4\varepsilon+1}}{\varepsilon}}-e^{\frac{t\sqrt{4\varepsilon+1}}{\varepsilon}}\right) \\ -0.8\left(e^{\frac{t\left(\sqrt{4\varepsilon+1}+1\right)}{2\varepsilon}}-e^{\frac{3t\sqrt{4\varepsilon+1}+t}{2\varepsilon}}\right) \\ \theta\left(\frac{e^{-\frac{(t-1)\left(\sqrt{4\varepsilon+1}+1\right)}{2\varepsilon}}\left(-1+e^{\frac{t\sqrt{4\varepsilon+1}}{\varepsilon}}\right)}{-1+e^{\frac{\sqrt{4\varepsilon+1}}{\varepsilon}}}\right) \end{array}\right) \end{array}\right),$$

$$x_2(t;\varepsilon,0) = \frac{e^{\frac{\sqrt{4\varepsilon+1}+1}{2\varepsilon}}}{1-e^{\frac{\sqrt{4\varepsilon+1}}{\varepsilon}}}\left(\begin{array}{c} e^{-\frac{(t+1)\left(\sqrt{4\varepsilon+1}+1\right)}{2\varepsilon}}\left(e^{\frac{\sqrt{4\varepsilon+1}}{\varepsilon}}-e^{\frac{t\sqrt{4\varepsilon+1}}{\varepsilon}}\right) \\ \left(\begin{array}{c} 0.8\left(\theta\left(\frac{e^{-\frac{t\left(\sqrt{4\varepsilon+1}+1\right)}{2\varepsilon}}\left(e^{\frac{\sqrt{4\varepsilon+1}}{\varepsilon}}-e^{\frac{t\sqrt{4\varepsilon+1}}{\varepsilon}}\right)}{-1+e^{\frac{\sqrt{4\varepsilon+1}}{\varepsilon}}}\right)-1\right) \\ -1.2\theta\left(\frac{e^{-\frac{t\left(\sqrt{4\varepsilon+1}+1\right)}{2\varepsilon}}\left(e^{\frac{\sqrt{4\varepsilon+1}}{\varepsilon}}-e^{\frac{t\sqrt{4\varepsilon+1}}{\varepsilon}}\right)}{-1+e^{\frac{\sqrt{4\varepsilon+1}}{\varepsilon}}}\right) \end{array}\right) \\ -e^{-\frac{t\left(\sqrt{4\varepsilon+1}+1\right)}{2\varepsilon}}\left(e^{\frac{t\sqrt{4\varepsilon+1}}{\varepsilon}}-1\right) \\ \left(0.4\theta\left(\frac{e^{-\frac{(t-1)\left(\sqrt{4\varepsilon+1}+1\right)}{2\varepsilon}}\left(-1+e^{\frac{t\sqrt{4\varepsilon+1}}{\varepsilon}}\right)}{-1+e^{\frac{\sqrt{4\varepsilon+1}}{\varepsilon}}}\right)+0.8\right) \end{array}\right).$$

The graph of the lower part and the upper part of the $\alpha-$cuts of the solution in Example 4.1 at $\alpha = 0$ are shown in Figure 1. The α-cut region of the solution given in Example 4.1 is shown in Figure 2. Note that just as the classical case in [9] on the interval $[t_0, t_1] = [0, 1]$, the fuzzy boundary layer exist in the vicinity of $t = t_0$ since $a_1(t) = 1 > t_0$ all through the domain.

Example 4.2. Consider the right end boundary layer SPTPFBVP

$$\begin{aligned} \varepsilon x''(t) - x'(t) - (1+\varepsilon)x(t) &= 0 \\ x(0) &= (0.9 + e^{(-\frac{1+\varepsilon}{\varepsilon})}, 1 + e^{(-\frac{1+\varepsilon}{\varepsilon})}, 1.2 + e^{(-\frac{1+\varepsilon}{\varepsilon})}) \\ x(1) &= (0.9 + \frac{1}{\varepsilon}, 1 + \frac{1}{\varepsilon}, 1.2 + \frac{1}{\varepsilon}) \end{aligned}$$

The exact solution of the corresponding crisp problem is

$$x(t) = e^{\frac{(1+\varepsilon)(t-1)}{\varepsilon}} + e^{-t}.$$

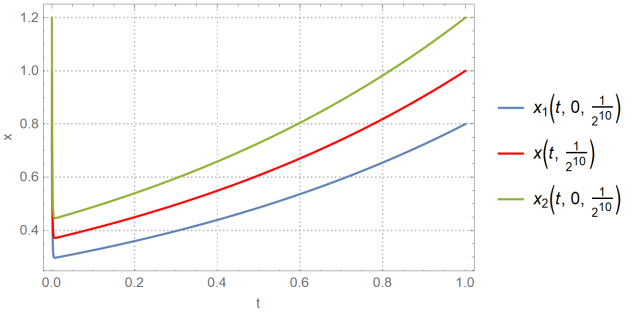


FIGURE 1. The graph of the lower part and the upper part of the $\alpha-$cuts of the solution in Example 4.1 at $\alpha = 0$.

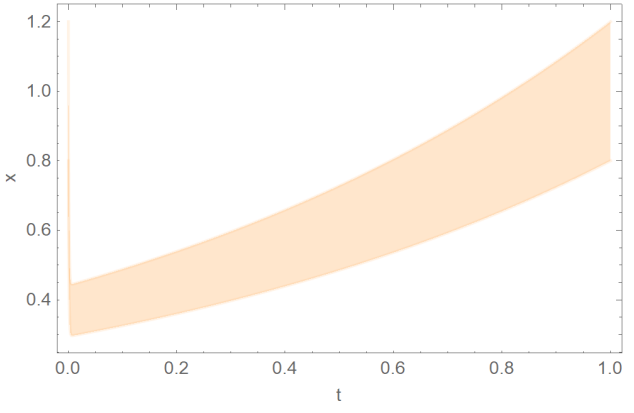


FIGURE 2. The α-cut region of the solution given in Example 4.1

The endpoints of the α-cuts of the fuzzy solutions are

$$x_1(t;\varepsilon,0) = \frac{e^{-t}}{e^{\frac{1}{\varepsilon}+2}-1}\left(\begin{array}{c}\left(0.2e^{t\left(\frac{1}{\varepsilon}+2\right)}-1.5e^{\frac{1}{\varepsilon}}\right)\theta\left(\frac{e^{-t}\left(e^{2+\frac{1}{\varepsilon}}-e^{t\left(2+\frac{1}{\varepsilon}\right)}\right)}{-1+e^{2+\frac{1}{\varepsilon}}}\right)\\ +\left(176e^{t\left(\frac{1}{\varepsilon}+2\right)}-176\right)\theta\left(\frac{e^{1-t}\left(-1+e^{t\left(2+\frac{1}{\varepsilon}\right)}\right)}{-1+e^{2+\frac{1}{\varepsilon}}}\right)\\ -1.1e^{t\left(\frac{1}{\varepsilon}+2\right)}+8.1e^{\frac{1}{\varepsilon}}\end{array}\right),$$

$$x_2(t;\varepsilon,0)=\frac{e^{-t}}{e^{\frac{1}{\varepsilon}+2}-1}\left(\begin{array}{c}\left(1.5e^{\frac{1}{\varepsilon}}-0.2e^{t\left(\frac{1}{\varepsilon}+2\right)}\right)\theta\left(\frac{e^{-t}\left(e^{2+\frac{1}{\varepsilon}}-e^{t\left(2+\frac{1}{\varepsilon}\right)}\right)}{-1+e^{2+\frac{1}{\varepsilon}}}\right)\\+\left(0.5e^{t\left(\frac{1}{\varepsilon}+2\right)}-0.5\right)\theta\left(\frac{e^{1-t}\left(-1+e^{t\left(2+\frac{1}{\varepsilon}\right)}\right)}{-1+e^{2+\frac{1}{\varepsilon}}}\right)\\+175e^{t\left(\frac{1}{\varepsilon}+2\right)}+6.6e^{\frac{1}{\varepsilon}}-176\end{array}\right).$$

The graph of the lower part and the upper part of the $\alpha-$cuts of the solution in Example 4.2 at $\alpha=0$ are shown in Figure 3. The α-cut region of the solution given in Example 4.2 is shown in Figure 4. Note that just as the classical case in [9] on the interval $[t_0,t_1]=[0,1]$, the fuzzy boundary layers exist in the vicinity of $t=t_1$ since $a_1(t)=-1<t_0$ all through the domain $[t_0,t_1]$.

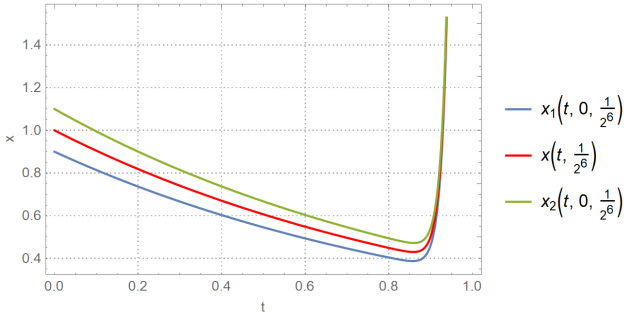


FIGURE 3. The graph of the lower part and the upper part of the $\alpha-$cuts of the solution in Example 4.2 at $\alpha=0$.

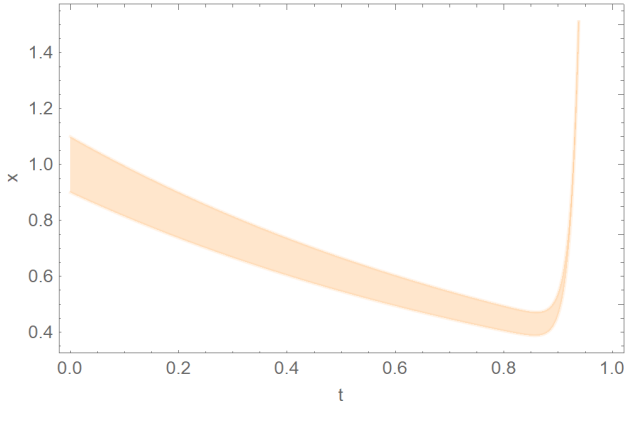


FIGURE 4. The α-cut region of the solution given in Example 4.2

Example 4.3. Consider the right end boundary layer SPTPFBVP

$$\begin{aligned} -\varepsilon x''(t) + x'(t) &= e^t \\ x(0) &= (-0.1, 0, 0.1) \\ x(1) &= (-0.2, 0, 0.2) \end{aligned}$$

The exact solution of the corresponding crisp problem is

$$x(t) = \frac{1}{1-\varepsilon}\left[e^t - \frac{1 - e^{1-\frac{1}{\varepsilon}} + (e^t - 1)e^{\frac{t-1}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}}\right].$$

The endpoints of the α-cuts of the fuzzy solutions are

$$x_1(t;\varepsilon,0) = \frac{1}{\left(e^{\frac{1}{\varepsilon}} - 1\right)(\varepsilon - 1)} \left(\begin{array}{c} \left(e^{\frac{1}{\varepsilon}}(0.2 - 0.2\varepsilon) + (0.2\varepsilon - 0.2)e^{t/\varepsilon}\right)\theta\left(\frac{e^{\frac{1}{\varepsilon}} - e^{t/\varepsilon}}{-1+e^{\frac{1}{\varepsilon}}}\right) \\ + \left((0.2 - 0.2\varepsilon)e^{t/\varepsilon} + 0.2\varepsilon - 0.2\right)\theta\left(\frac{-1+e^{t/\varepsilon}}{-1+e^{\frac{1}{\varepsilon}}}\right) - 0.1\varepsilon e^{t/\varepsilon} - e^{t+\frac{1}{\varepsilon}} \\ +1.8e^{t/\varepsilon} + e^t + 0.1e^{\frac{1}{\varepsilon}}\varepsilon + 0.9e^{\frac{1}{\varepsilon}} - 2.7 \end{array} \right),$$

$$x_2(t;\varepsilon,0) = \frac{1}{\left(e^{\frac{1}{\varepsilon}} - 1\right)(\varepsilon - 1)} \left(\begin{array}{c} \left((0.2 - 0.2\varepsilon)e^{t/\varepsilon} + e^{\frac{1}{\varepsilon}}(0.2\varepsilon - 0.2)\right)\theta\left(\frac{e^{\frac{1}{\varepsilon}} - e^{t/\varepsilon}}{-1+e^{\frac{1}{\varepsilon}}}\right) \\ + \left((0.4\varepsilon - 0.4)e^{t/\varepsilon} - 0.4\varepsilon + 0.4\right)\theta\left(\frac{-1+e^{t/\varepsilon}}{-1+e^{\frac{1}{\varepsilon}}}\right) - 0.1\varepsilon e^{t/\varepsilon} - e^{t+\frac{1}{\varepsilon}} \\ +1.8e^{t/\varepsilon} + e^t - 0.1e^{\frac{1}{\varepsilon}}\varepsilon + 0.2\varepsilon + 1.1e^{\frac{1}{\varepsilon}} - 2.9 \end{array} \right).$$

The graph of the lower part and the upper part of the $\alpha-$cuts of the solution in Example 4.3 at $\alpha = 0$ are shown in Figure 5. The α-cut region of the solution given in Example 4.3 is shown in Figure 6. Note that just as the classical case in [9] on the interval $[t_0, t_1] = [0, 1]$, the fuzzy boundary layers exist in the vicinity of $t = t_1$ since $a_1(t) = -1 < t_0$ all through the domain $[t_0, t_1]$

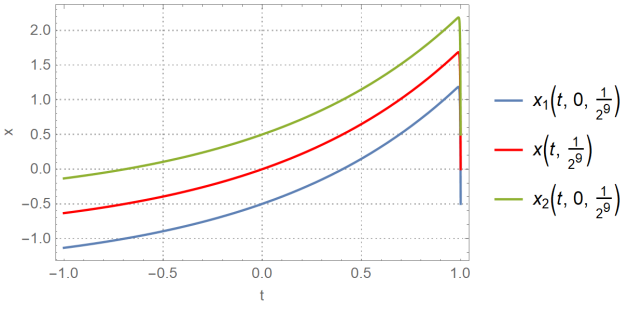


FIGURE 5. The graph of the lower part and the upper part of the $\alpha-$cuts of the solution in Example 4.3 at $\alpha = 0$.

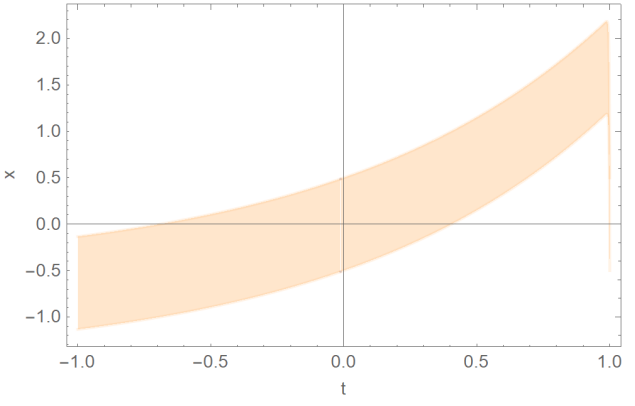


FIGURE 6. The α-cut region of the solution given in Example 4.3

Example 4.4. Consider the dual boundary layer SPTPFBVP

$$\begin{aligned} -\varepsilon x''(t) - 2(2t-1)x'(t) - 4x(t) &= 0 \\ x(0) &= (0.5, 1, 0.1.5) \\ x(1) &= (0.9, 1, 1.9) \end{aligned}$$

The exact solution of the corresponding crisp problem is

$$x(t) = e^{\frac{-2t(1-t)}{\varepsilon}}.$$

The endpoints of the α-cuts of the fuzzy solutions are

$$x_1(t;\varepsilon,0) = \frac{e^{\frac{2(t-1)t}{\varepsilon}}}{\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)} \begin{pmatrix} \left(0.5\operatorname{erf}\left(\frac{2t-1}{\sqrt{2}\sqrt{\varepsilon}}\right) - 0.5\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)\right)\theta\left(\frac{e^{\frac{2(t-1)t}{\varepsilon}}\left(\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right) - \operatorname{erf}\left(\frac{2t-1}{\sqrt{2}\sqrt{\varepsilon}}\right)\right)}{2\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)}\right) \\ + \left(0.45\operatorname{erf}\left(\frac{2t-1}{\sqrt{2}\sqrt{\varepsilon}}\right) + 0.45\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)\right)\theta\left(\frac{e^{\frac{2(t-1)t}{\varepsilon}}\left(\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right) + \operatorname{erf}\left(\frac{2t-1}{\sqrt{2}\sqrt{\varepsilon}}\right)\right)}{2\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)}\right) \\ -0.75\operatorname{erf}\left(\frac{2t-1}{\sqrt{2}\sqrt{\varepsilon}}\right) + 0.75\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right) \end{pmatrix},$$

$$x_2(t;\varepsilon,0) = \frac{e^{\frac{2(t-1)t}{\varepsilon}}}{\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)} \begin{pmatrix} \left(0.5\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right) - 0.5\operatorname{erf}\left(\frac{2t-1}{\sqrt{2}\sqrt{\varepsilon}}\right)\right)\theta\left(\frac{e^{\frac{2(t-1)t}{\varepsilon}}\left(\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right) - \operatorname{erf}\left(\frac{2t-1}{\sqrt{2}\sqrt{\varepsilon}}\right)\right)}{2\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)}\right) \\ + \left(0.5\operatorname{erf}\left(\frac{2t-1}{\sqrt{2}\sqrt{\varepsilon}}\right) + 0.5\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)\right)\theta\left(\frac{e^{\frac{2(t-1)t}{\varepsilon}}\left(\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right) + \operatorname{erf}\left(\frac{2t-1}{\sqrt{2}\sqrt{\varepsilon}}\right)\right)}{2\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)}\right) \\ +0.2\operatorname{erf}\left(\frac{2t-1}{\sqrt{2}\sqrt{\varepsilon}}\right) + 0.7\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right) \end{pmatrix}.$$

The graph of the lower part and the upper part of the $\alpha-$cuts of the solution in Example 4.4 at $\alpha = 0$ are shown in Figure 7. The α-cut region of the solution given in Example 4.4 is shown in Figure 8. Here $a_1(t) = -2(2t-1), a_2(t) = -4$ and $[t_0, t_1] = [0, 1]$. Let $t^* = \frac{t_0+t_1}{2}$. As $a_1(t^*) = 0$,

$\frac{d}{dt}a_1(t^*) < 0$, and for all $t \in [t_0, t_1]$ there is a positive number a_0 such that $|a_1(t)| \geq a_0 > 0$ and $a_2(t) < 0$ one can see that there are dual fuzzy boundary layers in the vicinity of $t = t_0$ and $t = t_1$ just as the classical case in [9].

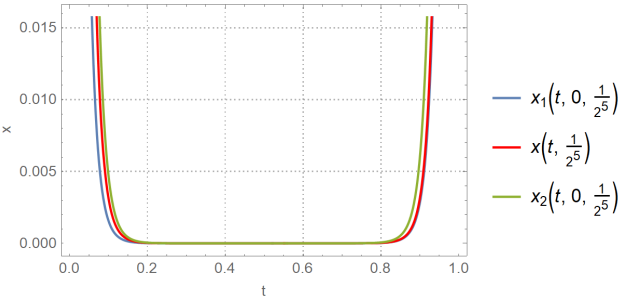


FIGURE 7. The graph of the lower part and the upper part of the $\alpha-$cuts of the solution in Example 4.4 at $\alpha = 0$.

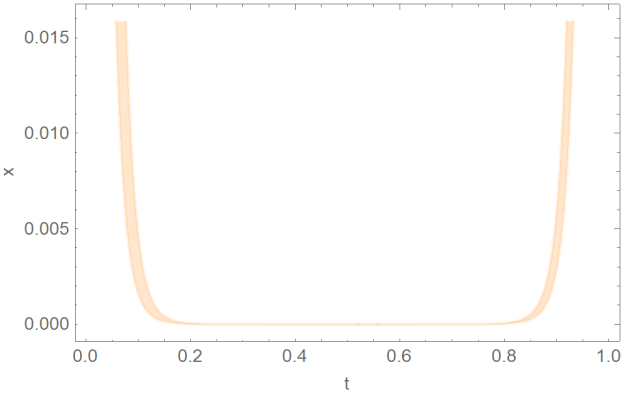


FIGURE 8. The α-cut region of the solution given in Example 4.4

Example 4.5. Consider the dual boundary layer SPTPFBVP

$$\begin{aligned} -\varepsilon x''(t) - tx'(t) - x(t) &= 0 \\ x(-1) &= (0.5, 1, 1.5) \\ x(1) &= (1.8, 2, 2.2) \end{aligned}$$

The exact solution of the corresponding crisp problem is

$$x(t) = \frac{e^{\frac{t^2}{2\varepsilon} - \frac{1}{2\varepsilon}} \left(\operatorname{erf}\left(\frac{t}{\sqrt{2}\sqrt{\varepsilon}} \right) + 3 \operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}} \right) \right)}{2 \operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}} \right)}.$$

The endpoints of the α-cuts of the fuzzy solutions are

$$x_1(t;\varepsilon,0)=\frac{e^{\frac{t^2-1}{2\varepsilon}}}{\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)}\left(\begin{array}{c}\left(0.5\operatorname{erf}\left(\frac{t}{\sqrt{2}\sqrt{\varepsilon}}\right)-0.5\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)\right)\theta\left(\frac{e^{\frac{t^2-1}{2\varepsilon}}\left(\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)-\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)\right)}{2\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)}\right)\\+\left(0.9\operatorname{erf}\left(\frac{t}{\sqrt{2}\sqrt{\varepsilon}}\right)+0.9\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)\right)\theta\left(\frac{e^{\frac{t^2-1}{2\varepsilon}}\left(\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)+\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)\right)}{2\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)}\right)\\-0.75\operatorname{erf}\left(\frac{t}{\sqrt{2}\sqrt{\varepsilon}}\right)+0.75\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)\end{array}\right),$$

$$x_2(t;\varepsilon,0)=\frac{e^{\frac{t^2-1}{2\varepsilon}}}{\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)}\left(\begin{array}{c}\left(0.5\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)-0.5\operatorname{erf}\left(\frac{t}{\sqrt{2}\sqrt{\varepsilon}}\right)\right)\theta\left(\frac{e^{\frac{t^2-1}{2\varepsilon}}\left(\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)-\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)\right)}{2\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)}\right)\\+\left(0.2\operatorname{erf}\left(\frac{t}{\sqrt{2}\sqrt{\varepsilon}}\right)+0.2\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)\right)\theta\left(\frac{e^{\frac{t^2-1}{2\varepsilon}}\left(\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)+\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)\right)}{2\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)}\right)\\+0.65\operatorname{erf}\left(\frac{t}{\sqrt{2}\sqrt{\varepsilon}}\right)+1.15\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)\end{array}\right).$$

The graph of the lower part and the upper part of the $\alpha-$cuts of the solution in Example 4.5 at $\alpha=0$ are shown in Figure 9. The α-cut region of the solution given in Example 4.5 is shown in Figure 10. Here $a_1(t)=-t, a_2(t)=-1$ and $[t_0,t_1]=[-1,1]$. Let $t^*=\frac{t_0+t_1}{2}$. As $a_1(t^*)=0$, $\frac{d}{dt}a_1(t^*)<0$, and for all $t\in[t_0,t_1]$ there is a positive number a_0 such that $|a_1(t)|\geq a_0>0$ and $a_2(t)<0$ one can see that there are dual fuzzy boundary layers in the vicinity of $t=t_0$ and $t=t_1$ just as the classical case in [9].

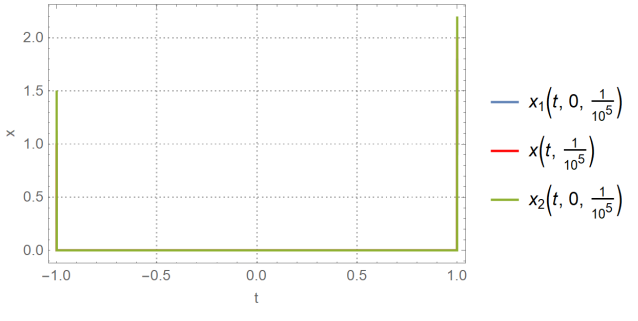


FIGURE 9. The graph of the lower part and the upper part of the $\alpha-$cuts of the solution in Example 4.5 at $\alpha=0$.

Example 4.6. Consider the internal layer SPTPFBVP

$$\begin{aligned}\varepsilon x''(t)+2(2t-1)x'(t)-4x(t)&=0\\x(0)&=(0.5,1,1.5)\\x(1)&=(0.5,1,1.5)\end{aligned}$$

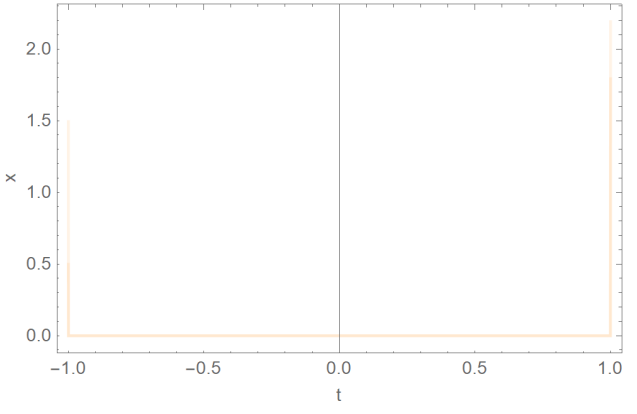


FIGURE 10. The α-cut region of the solution given in Example 4.5

The exact solution of the corresponding crisp problem is

$$x(t)=\frac{-e^{\frac{1}{2\varepsilon}-\frac{(1-2t)^2}{2\varepsilon}}\left(2e^{\frac{(1-2t)^2}{2\varepsilon}}\sqrt{2\pi}t\,\mathrm{erf}(\frac{1-2t}{2\varepsilon})-\mathrm{erf}(\frac{(1-2t)^2}{2\varepsilon})\sqrt{2\pi}\,\mathrm{erf}(\frac{1-2t}{2\varepsilon})-2\varepsilon\right)}{e^{\frac{1}{2\varepsilon}}\sqrt{2\pi}\,\mathrm{erf}(\frac{1}{\sqrt{2}\varepsilon})+2\sqrt{\varepsilon}}.$$

The endpoints of the α-cuts of the fuzzy solutions are

$$x_1(t;\varepsilon,0)=\frac{0.2e^{-\frac{2t^2}{\varepsilon}}}{e^{\frac{1}{2\varepsilon}}\,\mathrm{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)+0.8\sqrt{\varepsilon}}\left(\begin{array}{c}\left(\begin{array}{c}(1.2-2.5t)e^{\frac{4t^2+1}{2\varepsilon}}\,\mathrm{erf}\left(\frac{1-2t}{\sqrt{2}\sqrt{\varepsilon}}\right)\\+(2.5t-1.2)\,\mathrm{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)e^{\frac{4t^2+1}{2\varepsilon}}\\+\sqrt{\varepsilon}\left((2.t-1)e^{\frac{2t^2}{\varepsilon}}+1.e^{\frac{2t}{\varepsilon}}\right)\end{array}\right)\\\theta\left(t+\frac{e^{\frac{1}{2\varepsilon}}\left(\sqrt{2\pi}(1-2t)\,\mathrm{erf}\left(\frac{1-2t}{\sqrt{2}\sqrt{\varepsilon}}\right)+2e^{-\frac{(1-2t)^2}{2\varepsilon}}\sqrt{\varepsilon}\right)}{e^{\frac{1}{2\varepsilon}}\sqrt{2\pi}\,\mathrm{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)+4\sqrt{\varepsilon}}-\frac{1}{2}\right)\end{array}\right)$$

$$-0.4\left(\begin{array}{c}\left(\begin{array}{c}(1.2-2.5t)\,\mathrm{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)e^{\frac{4t^2+1}{2\varepsilon}}\\+(1.2-2.5t)e^{\frac{4t^2+1}{2\varepsilon}}\,\mathrm{erf}\left(\frac{1-2t}{\sqrt{2}\sqrt{\varepsilon}}\right)\\+\sqrt{\varepsilon}\left((1-2.t)e^{\frac{2t^2}{\varepsilon}}+e^{\frac{2t}{\varepsilon}}\right)\end{array}\right)\\\left(\theta\left(\frac{1}{2}\left(-2t+\frac{e^{\frac{1}{2\varepsilon}}\left(\sqrt{2\pi}(1-2t)\,\mathrm{erf}\left(\frac{1-2t}{\sqrt{2}\sqrt{\varepsilon}}\right)+2e^{-\frac{(1-2t)^2}{2\varepsilon}}\sqrt{\varepsilon}\right)}{e^{\frac{1}{2\varepsilon}}\sqrt{2\pi}\,\mathrm{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)+2\sqrt{\varepsilon}}+1\right)\right)-1.5\right)\end{array}\right),$$

$$x_2(t;\varepsilon,0)=\frac{0.4e^{-\frac{2t^2}{\varepsilon}}}{e^{\frac{1}{2\varepsilon}}\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)+0.8\sqrt{\varepsilon}}\left(\begin{array}{c}\left(\begin{array}{c}(1.2-2.5t)\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)e^{\frac{4t^2+1}{2\varepsilon}}\\+(1.2-2.5t)e^{\frac{4t^2+1}{2\varepsilon}}\operatorname{erf}\left(\frac{1-2t}{\sqrt{2}\sqrt{\varepsilon}}\right)\\+\sqrt{\varepsilon}\left((1-2.t)e^{\frac{2t^2}{\varepsilon}}+e^{\frac{2t}{\varepsilon}}\right)\end{array}\right)\\\left(\theta\left(\frac{1}{2}\left(-2t+\frac{e^{\frac{1}{2\varepsilon}}\left(\sqrt{2\pi}(1-2t)\operatorname{erf}\left(\frac{1-2t}{\sqrt{2}\sqrt{\varepsilon}}\right)+2e^{-\frac{(1-2t)^2}{2\varepsilon}}\sqrt{\varepsilon}\right)}{e^{\frac{1}{2\varepsilon}}\sqrt{2\pi}\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)+2\sqrt{\varepsilon}}+1\right)\right)\right)\\+0.5\end{array}\right)$$

$$+0.4\left(\begin{array}{c}\left(\begin{array}{c}(1.2-2.5t)e^{\frac{4t^2+1}{2\varepsilon}}\operatorname{erf}\left(\frac{1-2t}{\sqrt{2}\sqrt{\varepsilon}}\right)\\+(2.5t-1.2)\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)e^{\frac{4t^2+1}{2\varepsilon}}\\+\sqrt{\varepsilon}\left((2.t-1)e^{\frac{2t^2}{\varepsilon}}+e^{\frac{2t}{\varepsilon}}\right)\end{array}\right)\\\left(\theta\left(t+\frac{e^{\frac{1}{2\varepsilon}}\left(\sqrt{2\pi}(1-2t)\operatorname{erf}\left(\frac{1-2t}{\sqrt{2}\sqrt{\varepsilon}}\right)+2e^{-\frac{(1-2t)^2}{2\varepsilon}}\sqrt{\varepsilon}\right)}{e^{\frac{1}{2\varepsilon}}\sqrt{2\pi}\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)+4\sqrt{\varepsilon}}-\frac{1}{2}\right)\right)\\+0.5\end{array}\right).$$

The graph of the lower part and the upper part of the $\alpha-$cuts of the solution in Example 4.6 at $\alpha=0$ are shown in Figure 11. The α-cut region of the solution given in Example 4.6 is shown in Figure 12. Here $a_1(t)=2(2t-1), a_2(t)=-4$ and $[t_0,t_1]=[0,1]$. Let $t^*=\frac{t_0+t_1}{2}$. Since $a_1(t^*)=0$, $\frac{d}{dt}a_1(t^*)<0$, and $a_2(t)<0$ then there exist an internal fuzzy boundary layer in the vicinity of $t^*=\frac{t_0+t_1}{2}$ just as the classical case in [9].

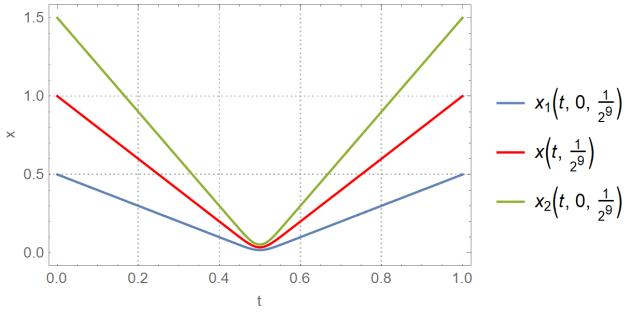


FIGURE 11. The graph of the lower part and the upper part of the $\alpha-$cuts of the solution in Example 4.6 at $\alpha=0$.

Example 4.7. Consider the internal layer SPTPFBVP

$$\varepsilon x''(t)+tx'(t)-x(t)=0$$
$$x(-1)=(0.5,1,1.5)$$
$$x(1)=(1.8,2,2.2)$$

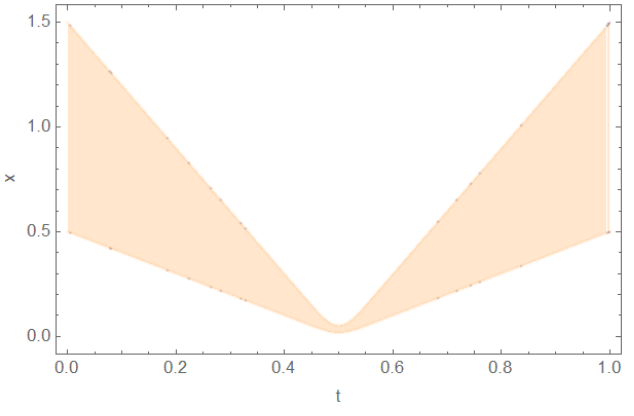


FIGURE 12. The α-cut region of the solution given in Example 4.6.

The exact solution of the corresponding crisp problem is

$$x(t)=\frac{2e^{\frac{-1}{2\varepsilon}}\sqrt{\varepsilon}\left(t+3e^{-\frac{t^2-1}{2\varepsilon}}\right)+\left(\sqrt{2\pi}t\,\mathrm{erf}(\frac{1}{\sqrt{2\varepsilon}})+3\sqrt{2\pi}\,\mathrm{erf}(\frac{t}{\sqrt{2\varepsilon}})\right)}{2\sqrt{2\pi}\,\mathrm{erf}(\frac{1}{\sqrt{2\varepsilon}})+4e^{\frac{-1}{2\varepsilon}}\sqrt{\varepsilon}}.$$

The endpoints of the α-cuts of the fuzzy solutions are

$$x_1(t;\varepsilon,\alpha)=\frac{1}{2}\left(\begin{array}{c}\left(\dfrac{e^{\frac{1}{2\varepsilon}}\left(\sqrt{2\pi}t\,\mathrm{erf}\left(\frac{t}{\sqrt{2\varepsilon}}\right)+2\sqrt{\varepsilon}e^{-\frac{t^2}{2\varepsilon}}\right)}{\sqrt{2\pi}e^{\frac{1}{2\varepsilon}}\,\mathrm{erf}\left(\frac{1}{\sqrt{2\varepsilon}}\right)+2\sqrt{\varepsilon}}-t\right)\\ \left(1.5-\theta\left(\frac{1}{2}\left(\dfrac{e^{\frac{1}{2\varepsilon}}\left(\sqrt{2\pi}t\,\mathrm{erf}\left(\frac{t}{\sqrt{2\varepsilon}}\right)+2e^{-\frac{t^2}{2\varepsilon}}\sqrt{\varepsilon}\right)}{e^{\frac{1}{2\varepsilon}}\sqrt{2\pi}\,\mathrm{erf}\left(\frac{1}{\sqrt{2\varepsilon}}\right)+2\sqrt{\varepsilon}}-t\right)\right)\right)\end{array}\right)$$
$$+0.9\left(\begin{array}{c}\left(\dfrac{e^{\frac{1}{2\varepsilon}}\left(\sqrt{2\pi}t\,\mathrm{erf}\left(\frac{t}{\sqrt{2\varepsilon}}\right)+2\sqrt{\varepsilon}e^{-\frac{t^2}{2\varepsilon}}\right)}{\sqrt{2\pi}e^{\frac{1}{2}\varepsilon}\,\mathrm{erf}\left(\frac{1}{\sqrt{2\varepsilon}}\right)+2\sqrt{\varepsilon}}+t\right)\\ \theta\left(\frac{1}{2}\left(t+\dfrac{e^{\frac{1}{2\varepsilon}}\left(\sqrt{2\pi}t\,\mathrm{erf}\left(\frac{t}{\sqrt{2\varepsilon}}\right)+2e^{-\frac{t^2}{2\varepsilon}}\sqrt{\varepsilon}\right)}{e^{\frac{1}{2\varepsilon}}\sqrt{2\pi}\,\mathrm{erf}\left(\frac{1}{\sqrt{2\varepsilon}}\right)+2\sqrt{\varepsilon}}\right)\right)\end{array}\right),$$

$$x_2(t;\varepsilon,\alpha)=\frac{1}{2}\left(\begin{array}{c}\left(\dfrac{e^{\frac{1}{2}\varepsilon}\left(\sqrt{2\pi}t\,\mathrm{erf}\left(\frac{t}{\sqrt{2\varepsilon}}\right)+2e^{-\frac{t^2}{2\varepsilon}}\sqrt{\varepsilon}\right)}{e^{\frac{1}{2}\varepsilon}\sqrt{2\pi}\,\mathrm{erf}\left(\frac{1}{\sqrt{2\varepsilon}}\right)+2\sqrt{\varepsilon}}-t\right)\\ \left(\theta\left(\frac{1}{2}\left(\dfrac{e^{\frac{1}{2}\varepsilon}\left(\sqrt{2\pi}t\,\mathrm{erf}\left(\frac{t}{\sqrt{2\varepsilon}}\right)+2e^{-\frac{t^2}{2\varepsilon}}\sqrt{\varepsilon}\right)}{e^{\frac{1}{2}\varepsilon}\sqrt{2\pi}\,\mathrm{erf}\left(\frac{1}{\sqrt{2\varepsilon}}\right)+2\sqrt{\varepsilon}}-t\right)\right)+0.5\right)\end{array}\right)$$
$$+\left(\begin{array}{c}\left(\dfrac{e^{\frac{1}{2}\varepsilon}\left(\sqrt{2\pi}t\,\mathrm{erf}\left(\frac{t}{\sqrt{2\varepsilon}}\right)+2\sqrt{\varepsilon}e^{-\frac{t^2}{2\varepsilon}}\right)}{\sqrt{2\pi}e^{\frac{1}{2}\varepsilon}\,\mathrm{erf}\left(\frac{1}{\sqrt{2\varepsilon}}\right)+2\sqrt{\varepsilon}}+t\right)\\ \left(0.4\theta\left(\frac{1}{2}\left(t+\dfrac{e^{\frac{1}{2}\varepsilon}\left(\sqrt{2\pi}t\,\mathrm{erf}\left(\frac{t}{\sqrt{2\varepsilon}}\right)+2e^{-\frac{t^2}{2\varepsilon}}\sqrt{\varepsilon}\right)}{e^{\frac{1}{2}\varepsilon}\sqrt{2\pi}\,\mathrm{erf}\left(\frac{1}{\sqrt{2\varepsilon}}\right)+2\sqrt{\varepsilon}}\right)\right)+1.8\right)\end{array}\right).$$

The graph of the lower part and the upper part of the $\alpha-$cuts of the solution in Example 4.7 at $\alpha=0$ are shown in Figure 13. The α-cut region of the solution given in Example 4.7 is shown in Figure 14. Here $a_1(t)=t, a_2(t)=-1$ and $[t_0,t_1]=[-1,1]$. Let $t^*=\frac{t_0+t_1}{2}$. Since $a_1(t^*)=0$, $\frac{d}{dt}a_1(t^*)<0$, and $a_2(t)<0$ then there exist an internal fuzzy boundary layer in the vicinity of $t^*=\frac{t_0+t_1}{2}$ just as the classical case in [9].

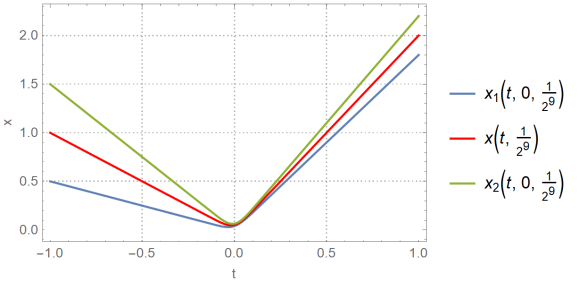


FIGURE 13. The graph of the lower part and the upper part of the $\alpha-$cuts of the solution in Example 4.7 at $\alpha=0$.

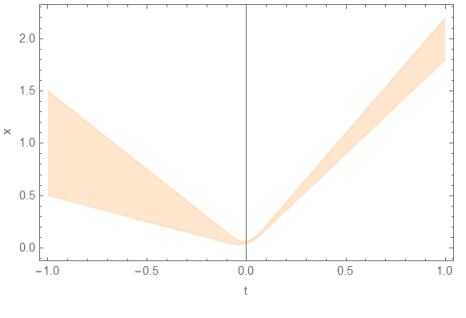


FIGURE 14. The α-cut region of the solution given in Example 4.7.

5 Conclusion

In this study, we firstly introduce the concept of singularly perturbed two-point fuzzy boundary value problems (SPTPFBVPs), and we provide an algorithm for finding the α-cuts of solutions to an SPTPFBVP with fuzzy boundary values using the Zadeh's extension principle. Furthermore, we obtain some properties of the α-cuts of the solution to problem (3.1)-(3.3) in Theorem 3.1 and Theorem 3.2. We apply the algorithm to seven different fuzzified singularly perturbed two-point boundary value problems, which correspond to the crisp ones discussed in [9]. Similar to the solutions of classical singularly perturbed problems in [9], we observe the existence of a fuzzy boundary layer near $t = t_0$ in Example 4.1, and a fuzzy boundary layer near $t = t_1$ in Examples 4.2-4.3. In Examples 4.4-4.5, there are dual fuzzy boundary layers near $t = t_0$ and $t = t_1$. In Examples 4.6-4.7, there are internal fuzzy boundary layers near $t = \frac{t_0+t_1}{2}$. Additionally, it can be observed that the crisp solutions of the problems lie between the α-cuts of the solutions $x_1(t;\varepsilon,\alpha)$ and $x_2(t;\varepsilon,\alpha)$ in each example.

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