Multiple solutions of critical singular degenerate elliptic system with concave-convex nonlinearities

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Abstract

This paper is devoted to study a class of critical singular degenerate elliptic system with concave-convex nonlinearities and sign-changing weight functions. The existence and multiplicity of nontrivial nonnegative solutions are obtained by the variational.

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1 Introduction and main results

Let $0 \in \Omega \subset R^N$ be a bounded domain with smooth boundary $\partial\Omega$. We are concerned with the following problem

$$\begin{cases} -\mathrm{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) - \mu\frac{|u|^{p-2}u}{|x|^{(a+1)p}} = \lambda a_1(x)|u|^{q-2}u + \frac{b(x)F_u(u,v)}{|x|^{bp^*}}, & \text{in } \Omega, \\ -\mathrm{div}(|x|^{-ap}|\nabla v|^{p-2}\nabla v) - \mu\frac{|v|^{p-2}v}{|x|^{(a+1)p}} = \lambda a_2(x)|v|^{q-2}v + \frac{b(x)F_v(u,v)}{|x|^{bp^*}}, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $N > p(a+1)$, $\lambda > 0$, $1 < q < p < N$, $0 \le \mu < \bar{\mu}$, $\bar{\mu} \doteq \left(\frac{N-p}{p} - a\right)^p$, $0 \le a < \frac{N-p}{p}$, $a \le b < a+1$, $p^* = p^*(a,b) = \frac{Np}{N-p(1+a-b)}$ is the critical Hardy-Sobolev exponent, $\Theta = \frac{p^*}{p^*-q}$, $a_1(x)$, $a_2(x) \in L^\Theta(\Omega)$, $b(x) \in L^\infty(\Omega)$ are allowed to change sign; $F \in C^1((R^+)^2, R^+)$ is positively homogeneous of degree p^*, that is, $F(tz) = t^{p^*}F(z)$ holds for all $z \in (R^+)^2$ and $t > 0$, here, $R^+ = [0, +\infty)$.

Problem (1) is related to the following Caffarelli-Kohn-Nirenberg inequality (see [4])

$$\left(\int_{R^N} \frac{|w|^{p^*}}{|x|^{bp^*}}\mathrm{d}x\right)^{\frac{p}{p^*}} \le C_{a,b}\int_{R^N} |x|^{-ap}|\nabla w|^p\mathrm{d}x, \quad \forall\, w \in C_0^\infty(R^N), \tag{2}$$

which is also named as Hardy-Sobolev inequality, where $C_{a,b}$ is a positive constant depending on a and b. Furthermore, the following Hardy inequality holds

$$\int_{R^N} \frac{|w|^p}{|x|^{p(a+1)}}\mathrm{d}x \le \frac{1}{\bar{\mu}}\int_{R^N} |x|^{-ap}|\nabla w|^p\mathrm{d}x, \quad \forall\, w \in C_0^\infty(R^N). \tag{3}$$

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Let $W_a^{1,p}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ under the norm

$$\|w\|_a = \left(\int_\Omega |x|^{-ap}|\nabla w|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

For $\mu < \bar{\mu}$, it follows from (3) that

$$\|w\|_{W_a^{1,p}(\Omega)} := \left(\int_\Omega \left(|x|^{-ap}|\nabla w|^p - \mu\frac{|w|^p}{|x|^{p(a+1)}}\right)\mathrm{d}x\right)^{\frac{1}{p}}$$

is equivalent to the norm $\|w\|_a$ of the space $W_a^{1,p}(\Omega)$. We define $E = W_a^{1,p}(\Omega) \times W_a^{1,p}(\Omega)$ endowed with the norm

$$\|z\|_E = \left(\|u\|^p_{W_a^{1,p}(\Omega)} + \|v\|^p_{W_a^{1,p}(\Omega)}\right)^{\frac{1}{p}}.$$

Using the inequality (2) and the boundedness of Ω, Xuan in [17] has proved that there exists $C > 0$ such that

$$\left(\int_\Omega |x|^{-\sigma}|w|^s \mathrm{d}x\right)^{\frac{p}{s}} \le C\int_\Omega |x|^{-ap}|\nabla w|^p \mathrm{d}x, \quad \forall\ w \in W_a^{1,p}(\Omega), \tag{4}$$

where $1 \le s \le \frac{Np}{N-p}$, $\sigma \le (a+1)s + N\left(1 - \frac{s}{p}\right)$. Moreover, the embedding $W_a^{1,p}(\Omega) \hookrightarrow L^s(\Omega, |x|^{-\sigma})$ is continuous and compact if $1 \le s < \frac{Np}{N-p}$ and $\sigma < (a+1)s + N\left(1 - \frac{s}{p}\right)$.

Set $u = v$, $a_1(x) = a_2(x)$ then problem (1) reduces to the following elliptic equation

$$\begin{cases} -\mathrm{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) - \mu\frac{|u|^{p-2}u}{|x|^{(a+1)p}} = \lambda a_1(x)|u|^{q-2}u + \frac{b(x)|u|^{p^*-2}u}{|x|^{bp^*}}, & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega. \end{cases} \tag{5}$$

In the case of $p = 2$, $1 < q < 2$, $a = b = 0$ and $a_1(x) = b(x) \equiv 1$, Chen in [5] proved problem (5) has at least two positive solutions for $\lambda > 0$ small enough. Subsequently, the results of [5] were extended to the case $0 \le a < \frac{N-2}{2}$ and $a \le b < a+1$ in [12]. For the case $1 < q < p$ and $a = b = 0$, the authors in [16] have obtain the existence of two positive solutions for problem (5). Moreover, the multiple results of positive solutions for problem (5) can be found in [15] and [9].

In a recent paper, Nyamoradi [14] have considered the following elliptic system

$$\begin{cases} -\mathrm{div}(|x|^{-2a}\nabla u) - \mu\frac{u}{|x|^{2(a+1)}} = \frac{\lambda}{q}G_u(u,v) + \frac{F_u(x,u,v)}{2^*|x|^{2^*b}}, & x \in \Omega, \\ -\mathrm{div}(|x|^{-2a}\nabla v) - \mu\frac{v}{|x|^{2(a+1)}} = \frac{\lambda}{q}G_v(u,v) + \frac{F_v(x,u,v)}{2^*|x|^{2^*b}}, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{6}$$

where $N \ge 3$, $\lambda > 0$, $1 < q < 2$, $0 \le \mu < \left(\frac{N-2}{2} - a\right)^2$, $0 \le a < \frac{N-2}{2}$, $a \le b < a+1$, $2^* = \frac{2N}{N-2(1+a-b)}$ is the critical Hardy-Sobolev exponent, $F \in C^1(\bar{\Omega} \times (R^+)^2, R^+)$ and $G \in C^1((R^+)^2, R^+)$ are positively homogeneous of degree 2^* and q respectively. Under some assumptions, by the variational approach involving the Nehari manifold, he proved problem (6) has at least one nontrivial solutions which extended the results of [13] and [10]. Recently, some authors also studied the singular elliptic systems with Hardy-Sobolev critical exponents (see [11-18]). However, as far as we know, very few

multiple results of problem (1) have been obtained in the literate for a more general term F when the weight functions are allowed to change sign. So our goal is extend the corresponding results in [13].

In the present paper, we consider problem (1) and obtain the existence and multiplicity of nontrivial nonnegative or positive solutions for problem (1) by applying the Ekeland's variation principle and the mountain pass theorem. The main results of this paper extend the corresponding results in [13]. Moreover, even in the elliptic equation case, our results also extend the corresponding results in [16].

Before stating our results, we give some notations and assumptions. Let $z = (u, v)$, $|z| = \sqrt{u^2 + v^2}$, $\|w\|_s = \left(\int_\Omega |w|^s dx\right)^{\frac{1}{s}}$ $(1 < s < \infty)$, $B(0, \delta) = \{x \in R^N : |x| < \delta\}$.

(a_1) $a_i(x) \in L^\Theta(\Omega)$ and $a_i^+ = \max\{a_i(x), 0\} \neq 0$, where $i = 1,\ 2$;

(a_2) There exist positive constants β_0 and δ_0 such that $B(0, 2\delta_0) \subset \Omega$ and $a_i(x) \geq \beta_0 (i = 1,\ 2)$ in $B(0, 2\delta_0)$.

(b_1) $b(x) \in L^\infty(\Omega)$ and $b^+ = \max\{b(x), 0\} \neq 0$;

(b_2) $b(0) = \|b\|_\infty$ and $b(x) > 0$ for all $x \in B(0, 2\delta_0)$;

(b_3) There exists $k > \frac{N - bp^*}{p-1}$ such that

$$b(x) = b(0) + o(|x|^k) \ \text{ as } \ x \to x_0.$$

In addition, we denote positive constants by $C,\ C_1,\ C_2,\ \cdots$. The main results of this paper are as follows.

Theorem 1. Let $N > p(a+1)$, $\lambda > 0$, $1 < q < p < N$, $0 \leq \mu < \bar{\mu}$, $0 \leq a < \frac{N-p}{p}$, $a \leq b < a + 1$. Suppose that $F \in C^1((R^+)^2, R^+)$ is positively homogeneous of degree p^* such that $F_u(0, v) = F_v(u, 0) = 0$ for all $u,\ v \in R^+$. If (a_1) and (b_1) hold, then there exists $\Lambda > 0$ such that problem (1) for all $\lambda \in (0, \Lambda)$ has at least one nontrivial nonnegative solution $z_1 = (u_1, v_1)$ satisfies that $u_1 \neq 0$, $v_1 \neq 0$ in Ω.

Theorem 2. Let $N > p^2(a+1)$, $p > 1$, $\lambda > 0$, $0 \leq a < \frac{N-p}{p}$, $a \leq b < a + 1$, $\frac{N(p-1)}{N-p(a+1)} < q < p$, $\mu < \frac{N^{p-1}(N(q-p+1)-pq(a+1))}{q^p}$. Suppose that $F \in C^1((R^+)^2, R^+)$ is positively homogeneous of degree p^* such that $F_u(u, 0) = F_u(0, v) = F_v(u, 0) = F_v(0, v) = 0$ for all $u,\ v \in R^+$, (a_1), (a_2), (b_1). If (b_2) and (b_3) hold, then there exists $\lambda^* > 0$ such that problem (1) for all $\lambda \in (0, \lambda^*)$ has at least two nontrivial nonnegative solutions $z_1 = (u_1, v_1)$ and $z_2 = (u_2, v_2)$ satisfies $u_1 \neq 0,\ v_1 \neq 0$, and $u_2 \neq 0,\ v_2 \neq 0$ in Ω.

From elliptic system reduce to elliptic equation, Theorem 2 can be described as follows,

Corollary 1. Let $N > p^2(a+1)$, $p > 1$, $\lambda > 0$, $0 \leq a < \frac{N-p}{p}$, $a \leq b < a + 1$, $\frac{N(p-1)}{N-p(a+1)} < q < p$, $\mu < \frac{N^{p-1}(N(q-p+1)-pq(a+1))}{q^p}$. Suppose that (a_1), (a_2), (b_1), (b_2) and (b_3) hold. Then there exists $\lambda^* > 0$ such that problem (5) for all $\lambda \in (0, \lambda^*)$ has at least two nontrivial nonnegative solutions u_λ and $\bar{u}_\lambda$ in Ω.

Remark 1. Theorem 1.1 and Theorem 1.2 in [13] are the special cases of our Theorem 1 and Theorem 2 corresponding to $p = 2$, $a_1(x) = a_2(x) = b(x) \equiv 1$ and $F(u, v) = \frac{1}{\alpha+\beta}|u|^\alpha |v|^\beta$,

$\alpha > 1,\ \beta > 1$ satisfy $\alpha + \beta = 2^*$, respectively. There are many homogeneous functions satisfying the conditions of our theorems. Some classical examples are

$$F(z) = \sum_{j=1}^{n} a_j u^{\alpha_j} v^{\beta_j}, \qquad F(z) = |z|_s^{p^*},$$

where $a_j > 0$, $\alpha_j > 1$, $\beta_j > 1$, $\alpha_j + \beta_j = p^*$, $|z|_s := (|u|^s + |v|^s)^{1/s}$ with $s > 1$.

Remark 2. In the elliptic equation case, our Corollary 1 extend the results of Theorem 1.2 in [16].

This paper is organized as follows. In section 2, we give the local Palais-Smale condition. The proof of Theorems 1 and 2 is provided in section 3.

2 The local Palais-Smale condition

In this section, we show that the corresponding functional of problem (1) satisfies the $(PS)_c$ condition. Let $w^{\pm} = \max\{\pm w, 0\}$ and $z^+ = (u^+, v^+)$, the corresponding functional of problem (1) is

$$I(z) = \frac{1}{p}\|z\|_E^p - \frac{\lambda}{q}\int_\Omega (a_1(x)(u^+)^q + a_2(x)(v^+)^q)\mathrm{d}x - \int_\Omega \frac{b(x)F(z^+)}{|x|^{bp^*}}\mathrm{d}x$$

for $(u, v) \in E$. Under the hypotheses of our theorems, it is obvious that I is a C^1 functional. The duality product between E and its dual space E^* is defined as

$$\begin{aligned}\langle I'(u,v),(\varphi,\psi)\rangle \;=\; & \int_\Omega \left(\frac{|\nabla u|^{p-2}\nabla u\nabla\varphi + |\nabla v|^{p-2}\nabla v\nabla\psi}{|x|^{ap}} - \mu\frac{(u^+)^{p-1}\varphi + (v^+)^{p-1}\psi}{|x|^{(a+1)p}}\right)\mathrm{d}x \\ & -\lambda\int_\Omega \left(a_1(x)(u^+)^{q-1}\varphi + a_2(x)(v^+)^{q-1}\psi\right)\mathrm{d}x \\ & -\int_\Omega \frac{b(x)(F_u(u^+,v^+)\varphi + F_v(u^+,v^+)\psi)}{|x|^{bp^*}}\mathrm{d}x\end{aligned}$$

for any $u,\ v,\ \varphi,\ \psi \in W_a^{1,p}(\Omega)$. By the definition of weak solution (u, v) of problem (1), it means that $(u, v) \in E$ satisfies $\langle I'(u, v), (\varphi, \psi)\rangle = 0$ for all $(\varphi, \psi) \in E$. It is well known that any critical point of I in E is a weak solution of problem (1). Hence, in order to obtain the nontrivial solutions of problem (1), we only need to look for the nontrivial critical points of I in E. In addition, since F is positively homogeneous of degree p^*, we have the so-called Euler identity

$$z \cdot \nabla F(z) = p^* F(z), \tag{7}$$

and

$$F(z) \le M|z|^{p^*} \quad \text{for all} \quad z \in (R^+)^2, \tag{8}$$

where $M = \max\limits_{\{z\in(R^+)^2:\ |z|=1\}} F(z) > 0$.

Definition 1 Let $c \in R$, E^* denote the dual space of the Banach space E.

(i) A sequence $\{z_n\} \subset E$ is called a $(PS)_c$ sequence of I if $I(z_n) \to c$ and $I^{'}(z_n) \to 0$ in E^* as $n \to \infty$;

(ii) We call I satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence $\{z_n\} \subset E$ of I has a convergent subsequence.

Lemma 1. Let $N > p(a+1)$, $\lambda > 0$, $1 < q < p < N$, $0 \le \mu < \bar{\mu}$, $0 \le a < \frac{N-p}{p}$, $a \le b < a+1$. Assume $F \in C^1((R^+)^2, R^+)$ is positively homogeneous of degree p^* such that $F_u(0,v) = F_v(u,0) = 0$ for all $u,\ v \in R^+$, (a_1) and (b_1) hold. If $\{z_n\} \subset E$ is a $(PS)_c$ sequence of I, then $\{z_n\}$ is bounded.

Proof. Let $A = \max\{\|a_1\|_\Theta, \|a_2\|_\Theta\}$. According to the hypothesis (a_1), for each $\varepsilon > 0$, by the Hölder inequality and the Young inequality, we imply from (4) that

$$\begin{aligned}
&\left|\frac{\lambda}{q}\int_\Omega (a_1(x)(u^+)^q + a_2(x)(v^+)^q)\mathrm{d}x\right| \\
&\le \frac{\lambda}{q}\left(\|a_1\|_\Theta \|u\|_{p^*}^q + \|a_2\|_\Theta \|v\|_{p^*}^q\right) \\
&\le \frac{\lambda}{q} A C^{\frac{q}{p}} \left(\|u\|^q_{W_a^{1,p}(\Omega)} + \|v\|^q_{W_a^{1,p}(\Omega)}\right) \\
&\le \varepsilon\left(\|u\|^p_{W_a^{1,p}(\Omega)} + \|v\|^p_{W_a^{1,p}(\Omega)}\right) + \frac{2}{q}(p-q)\left(\frac{C}{\varepsilon}\right)^{\frac{q}{p-q}}\left(\frac{\lambda A}{p}\right)^{\frac{p}{p-q}} \\
&= \varepsilon\|z\|_E^p + C(\varepsilon)\lambda^{\frac{p}{p-q}}, \qquad (9)
\end{aligned}$$

where $C(\varepsilon) = \frac{2}{q}(p-q)\left(\frac{C}{\varepsilon}\right)^{\frac{q}{p-q}}\left(\frac{A}{p}\right)^{\frac{p}{p-q}}$. Let $\{z_n\}$ be a $(PS)_c$ sequence of I. Using the hypotheses F is positively homogeneous of degree p^* such that $F_u(0,v) = F_v(u,0) = 0$ for all $u,\ v \in R^+$, we derive from (9) that

$$\begin{aligned}
&p^* I(z_n) - \langle I'(z_n), z_n\rangle \\
&= \left(\frac{p^*}{p} - 1\right)\|z_n\|_E^p + \frac{\lambda}{q}(q-p^*)\int_\Omega (a_1(x)(u_n^+)^q + a_2(x)(v_n^+)^q)\mathrm{d}x \\
&\ge \left(\frac{p^*}{p} - 1 - (p^*-q)\varepsilon\right)\|z_n\|_E^p - (p^*-q)C(\varepsilon)\lambda^{\frac{p}{p-q}}.
\end{aligned}$$

It follows that

$$\left(\frac{p^*}{p} - 1 - (p^*-q)\varepsilon\right)\|z_n\|_E^p \le p^* c + (p^*-q)C(\varepsilon)\lambda^{\frac{p}{p-q}} + o(\|z_n\|_E).$$

Let $\varepsilon < \frac{p^*-p}{p(p^*-q)}$, we obtain $\{z_n\}$ is bounded in E. □

Now we introduce the following version of the Brezis-Lieb lemma (see [2] or [3]).

Lemma 2. Assume $\Gamma \in C^1(R^2)$ with $\Gamma(0,0) = 0$ and $|\frac{\partial\Gamma}{\partial u}(z)|,\ |\frac{\partial\Gamma}{\partial v}(z)| \le C_1|z|^{s-1}$, for some $1 \le s < \infty$. Let $r \ge 0$ and $\{z_n\}$ be a bounded sequence in $L^s(\Omega, |x|^{-r}) \times L^s(\Omega, |x|^{-r})$ and such that $z_n \rightharpoonup z$ a.e. Ω. Then,

$$\int_\Omega \frac{\Gamma(z_n)}{|x|^r}\mathrm{d}x = \int_\Omega \frac{\Gamma(z_n - z)}{|x|^r}\mathrm{d}x + \int_\Omega \frac{\Gamma(z)}{|x|^r}\mathrm{d}x + o(1),$$

as $n \to \infty$.

Proof. According to the Fatou lemma, we have $\|z\|_{L^s(\Omega,|x|^{-r})} \leq \lim\limits_{n\to\infty} \inf \|z_n\|_{L^s(\Omega,|x|^{-r})}$. Fix $\xi > 0$, by the mean value theorem and the Young inequality, there exist $C(\xi) > 0$ and $\theta_n \in (0,1)$ such that

$$\begin{aligned} |\Gamma(z_n) - \Gamma(z_n - z)| &= \left|\frac{\partial \Gamma}{\partial u}(z_n - \theta_n z)u + \frac{\partial \Gamma}{\partial v}(z_n - \theta_n z)v\right| \\ &\leq C_1|z_n - \theta_n z|^{s-1}(|u| + |v|) \\ &\leq C_1\left(\frac{\xi}{2^{s-1}C_1}|z_n - \theta_n z|^s + C_0(\xi)|z|^s\right) \\ &\leq \xi|z_n - z|^s + \left(\xi(1-\theta_n)^s + C_1C_0(\xi)\right)|z|^s \\ &\leq \xi|z_n - z|^s + C(\xi)|z|^s, \end{aligned} \tag{10}$$

where $C(\xi) = \xi + C_1C_0(\xi) > 0$. Let

$$W_{\xi,n} = \left(\frac{|\Gamma(z_n) - \Gamma(z_n - z) - \Gamma(z)| - \xi|z_n - z|^s}{|x|^r}\right)^+.$$

According to the hypothesis of Lemma 2, there exists $C_2 > 0$ such that $|\Gamma(z)| \leq C_2|z|^s$. It implies from (10, $W_{\xi,n} \leq (C_2 + C(\xi))\frac{|z|^s}{|x|^r}$. By the dominated convergence theorem, $\int_\Omega W_{\xi,n}\mathrm{d}x \to 0$ as $n \to \infty$. Since

$$\frac{|\Gamma(z_n) - \Gamma(z_n - z) - \Gamma(z)|}{|x|^r} \leq W_{\xi,n} + \frac{\xi|z_n - z|^s}{|x|^r},$$

we obtain

$$\lim_{n\to\infty} \sup \int_\Omega \frac{|\Gamma(z_n) - \Gamma(z_n - z) - \Gamma(z)|}{|x|^r}\mathrm{d}x \leq \xi C_3$$

with $C_3 = \sup\limits_n \|z_n - z\|_{L^s(\Omega,|x|^{-r})} < \infty$. Letting $\xi \to 0$, we get the desire reslut. □

Lemma 3. Let $N > p(a+1)$, $\lambda > 0$, $1 < q < p < N$, $0 \leq \mu < \bar{\mu}$, $0 \leq a < \frac{N-p}{p}$, $a \leq b < a+1$. Assume $F \in C^1((R^+)^2, R^+)$ is positively homogeneous of degree p^* such that $F_u(0,v) = F_v(u,0) = 0$ for all $u,\ v \in R^+$, (a_1) and (b_1) hold. If $\{z_n\}$ is a $(PS)_c$ sequence of I with $z_n \rightharpoonup z$ in E as $n \to \infty$, then there exists a positive constant B depending on a, b, p, q, N, $\|a_1\|_\Theta$ and $\|a_2\|_\Theta$ such that

$$I'(z) = 0 \quad \text{and} \quad I(z) \geq -B\lambda^{\frac{p}{p-q}}.$$

Proof. Let $\{z_n\} = \{(u_n, v_n)\}$ be a $(PS)_c$ sequence of I with $z_n \rightharpoonup z = (u,v)$ in E. Then we have

$$I'(z_n) \to 0, \quad \text{strongly in } E^* \text{ as } \ n \to \infty.$$

Under the hypotheses of Lemma 3, we obtain I is a C^1 functional. According to the definition of the weak convergence, we obtain $I'(z) = 0$. In particular, we have $\langle I'(z), z\rangle = 0$, which implies from (7) that

$$p^* \int_\Omega \frac{b(x)F(u^+, v^+)}{|x|^{bp^*}}\mathrm{d}x = \|z\|_E^p - \lambda \int_\Omega (a_1(x)(u^+)^q + a_2(x)(v^+)^q)\mathrm{d}x.$$

It follows from (9) that

$$\begin{aligned} I(z) &= \left(\frac{1}{p}-\frac{1}{p^*}\right)\|z\|_E^p - \left(\frac{1}{q}-\frac{1}{p^*}\right)\lambda\int_\Omega (a_1(x)(u^+)^q + a_2(x)(v^+)^q)\mathrm{d}x. \\ &\geq \left(\frac{p^*-p}{pp^*} - \frac{p^*-q}{p^*}\varepsilon\right)\|z\|_E^p - \frac{p^*-q}{p^*}C(\varepsilon)\lambda^{\frac{p}{p-q}}. \end{aligned}$$

Take $\varepsilon = \frac{p^*-p}{p(p^*-q)}$, we have

$$I(z) > -B\lambda^{\frac{p}{p-q}},$$

where $B = B(a, b, p, q, N, \|a_1\|_\Theta, \|a_2\|_\Theta)$. □

Let

$$S_{\mu,a,b} = \inf_{w\in W_a^{1,p}(\Omega)\setminus\{0\}} \frac{\|w\|^p_{W_a^{1,p}(\Omega)}}{\left(\int_\Omega \frac{|w|^{p^*}}{|x|^{bp^*}}\mathrm{d}x\right)^{\frac{p}{p^*}}}$$

denote the best Sobolev constant for the imbedding $W_a^{1,p}(\Omega)$ into $L^{p^*}(\Omega, |x|^{-bp^*})$. Suppose that $0 \leq a < \frac{N-p}{p}$, $a \leq b < a+1$ and $0 \leq \mu < \bar{\mu}$, then $S_{\mu,a,b}$ is achieved on $\Omega = R^N$ by the function (see [11])

$$V_\eta(x) = \eta^{-\left(\frac{N-p}{p}-a\right)}U_{p,\mu}(\eta^{-1}x), \quad \forall\ \varepsilon > 0, \tag{11}$$

where $U_{p,\mu}(x) = U_{p,\mu}(|x|)$ is the unique radial solution of the following problem

$$\begin{cases} -\mathrm{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) - \mu\frac{u^{p-1}}{|x|^{p(a+1)}} = \frac{u^{p^*-1}}{|x|^{bp^*}}, & x \in R^N\setminus\{0\}, \\ u \in W_a^{1,p}(R^N), \quad u > 0, & x \in R^N\setminus\{0\}, \end{cases}$$

satisfying $U_{p,\mu}(1) = p^*\left(\frac{\bar{\mu}-\mu}{p}\right)^{\frac{1}{p^*-p}}$. Define

$$S_{\mu,a,b,F} := \inf_{z\in E\setminus\{0\}}\left\{\frac{\|z\|_E^p}{(\int_\Omega \frac{F(z^+)}{|x|^{bp^*}}\mathrm{d}x)^{\frac{p}{p^*}}} : \int_\Omega \frac{F(z^+)}{|x|^{bp^*}}\mathrm{d}x > 0\right\}.$$

We have the following lemma.

Lemma 4. Let $N > p(a+1)$, $\lambda > 0$, $1 < q < p < N$, $0 \leq \mu < \bar{\mu}$, $0 \leq a < \frac{N-p}{p}$, $a \leq b < a+1$. Assume $F \in C^1((R^+)^2, R^+)$ is positively homogeneous of degree p^* such that $F_u(0,v) = F_v(u,0) = 0$ for all $u,\ v \in R^+$. If (a_1) and (b_1) hold, then I satisfies the $(PS)_c$ condition with

$$c < c^* = \frac{p^*-p}{p}\|b\|_\infty^{-\frac{p}{p^*-p}}\left(\frac{S_{\mu,a,b,F}}{p^*}\right)^{\frac{p^*}{p^*-p}} - B\lambda^{\frac{p}{p-q}},$$

where B is the positive constant give in Lemma 3.

Proof. Let $\{z_n = (u_n, v_n)\} \subset E$ be a $(PS)_c$ sequence of I with $c < c^*$. By Lemma 1, we know $\{z_n\}$ is bounded. Up to a subsequence, there exists $z = (u, v) \in E$ such that

$$\begin{cases} z_n = (u_n, v_n) \rightharpoonup (u, v) = z, & \text{in } E, \\ z_n = (u_n, v_n) \rightharpoonup (u, v) = z, & \text{in } L^p(\Omega, |x|^{-p(a+1)}) \times L^p(\Omega, |x|^{-p(a+1)}), \\ z_n = (u_n, v_n) \rightharpoonup (u, v) = z, & \text{in } L^{p^*}(\Omega, |x|^{-bp^*}) \times L^{p^*}(\Omega, |x|^{-bp^*}), \\ z_n = (u_n, v_n) \to (u, v) = z, & \text{a.e. on } \Omega, \\ z_n = (u_n, v_n) \to (u, v) = z, & \text{in } L^s(\Omega) \times L^s(\Omega), \ \ 1 \le s < \frac{Np}{N-p}. \end{cases}$$

From Lemma 3, we have $I'(z) = 0$ and

$$\lambda \int_\Omega (a_1(x)(u_n^+)^q + a_2(x)(v_n^+)^q)\mathrm{d}x = \lambda \int_\Omega (a_1(x)(u^+)^q + a_2(x)(v^+)^q)\mathrm{d}x + o(1).$$

Let $\widetilde{z}_n = (\widetilde{u}_n, \widetilde{v}_n)$, where $\widetilde{u}_n = u_n - u$, $\widetilde{v}_n = v_n - v$. From Lemma 2, one has

$$\|\widetilde{z}_n\|_E^p = \|z_n\|_E^p - \|z\|_E^p + o(1).$$

Since $F \in C^1((R^+)^2, R^+)$ is positively homogeneous of degree p^* such that $F_u(0, v) = F_v(u, 0) = 0$ for all $u,\ v \in R^+$, it follows from Lemma 2 that

$$\int_\Omega \frac{F((\widetilde{z}_n)^+)}{|x|^{bp^*}}\mathrm{d}x = \int_\Omega \frac{F(z_n^+)}{|x|^{bp^*}}\mathrm{d}x - \int_\Omega \frac{F(z^+)}{|x|^{bp^*}}\mathrm{d}x + o(1),$$

which implies from (b_1) that

$$\int_\Omega \frac{b(x)F((\widetilde{z}_n)^+)}{|x|^{bp^*}}\mathrm{d}x = \int_\Omega \frac{b(x)F(z_n^+)}{|x|^{bp^*}}\mathrm{d}x - \int_\Omega \frac{b(x)F(z^+)}{|x|^{bp^*}}\mathrm{d}x + o(1).$$

Since $I(z_n) = c + o(1)$ and $I^{'}(z_n) = o(1)$, we obtain

$$\frac{1}{p}\|\widetilde{z}_n\|_E^p - \int_\Omega \frac{b(x)F((\widetilde{z}_n)^+)}{|x|^{bp^*}}\mathrm{d}x = c - I(z) + o(1), \tag{12}$$

and

$$\|\widetilde{z}_n\|_E^p - p^* \int_\Omega \frac{b(x)F((\widetilde{z}_n)^+)}{|x|^{bp^*}}\mathrm{d}x = o(1). \tag{13}$$

From (13), we may assume

$$\|\widetilde{z}_n\|_E^p \to p^* l, \qquad \int_\Omega \frac{b(x)F((\widetilde{z}_n)^+)}{|x|^{bp^*}}\mathrm{d}x \to l.$$

Assume $l > 0$, by the definition of $S_{\mu,a,b,F}$, we have

$$\begin{aligned} \|\widetilde{z}_n\|_E^p &\ge S_{\mu,a,b,F}\left(\int_\Omega \frac{F((\widetilde{z}_n)^+)}{|x|^{bp^*}}\mathrm{d}x\right)^{\frac{p}{p^*}} \\ &\ge S_{\mu,a,b,F}\left(\int_\Omega \frac{b(x)F((\widetilde{z}_n)^+)}{\|b\|_\infty |x|^{bp^*}}\mathrm{d}x\right)^{\frac{p}{p^*}} \\ &= S_{\mu,a,b,F}\|b\|_\infty^{-\frac{p}{p^*}}\left(\int_\Omega \frac{b(x)F((\widetilde{z}_n)^+)}{|x|^{bp^*}}\mathrm{d}x\right)^{\frac{p}{p^*}}. \end{aligned}$$

As $n \to \infty$, we deduce that

$$l \geq \|b\|_\infty^{-\frac{p}{p^*-p}} \left(\frac{S_{\mu,a,b,F}}{p^*}\right)^{\frac{p^*}{p^*-p}}.$$

It follows from (12) and Lemma 3 that

$$c = \left(\frac{p^*}{p}-1\right) l + I(z) \geq \frac{p^*-p}{p}\|b\|_\infty^{-\frac{p}{p^*-p}} \left(\frac{S_{\mu,a,b,F}}{p^*}\right)^{\frac{p^*}{p^*-p}} - B\lambda^{\frac{p}{p-q}} \doteq c^*,$$

which contradicts the fact $c < c^*$. Therefore, we have $l = 0$, which implies

$$z_n \to z \quad \text{in} \quad E.$$

Hence I satisfies the $(PS)_c$ condition with $c < c^*$. □

3 The proof of main results

Before the proof of Theorem 1, we first recall the following Lemma in [6].

Lemma 5. Let $r,\ s > 1$, $g \in L^s(\Omega)$ and $g^+ = \max\{g(x), 0\} \neq 0$. Then there exists $w_0 \in C_0^\infty(\Omega)$ such that $\int_\Omega g(x)(w_0^+)^r dx > 0$.

Proof of Theorem 1. According to the hypothesis (a_1), we obtain (9) holds. Applying the equivalence relation of the norm of the space $W_a^{1,p}(\Omega)$, by (2), (8) and (9), we have there exists $C_4 > 0$ such that

$$\begin{aligned} I(z) &\geq \left(\frac{1}{p}-\varepsilon\right)\|z\|_E^p - C(\varepsilon)\lambda^{\frac{p}{p-q}} - 2^{p^*-1}M\|b\|_\infty \int_\Omega \frac{|u|^{p^*}+|v|^{p^*}}{|x|^{bp^*}}\mathrm{d}x \\ &\geq \left(\frac{1}{p}-\varepsilon\right)\|z\|_E^p - 2^{p^*-1}(C_4C_{a,b})^{\frac{p^*}{p}} M\|b\|_\infty \|z\|_E^{p^*} - C(\varepsilon)\lambda^{\frac{p}{p-q}}. \end{aligned}$$

Let $\varepsilon < \frac{1}{p}$, there exist $\rho > 0$ and $\Lambda_1 > 0$, such that for any $\lambda \in (0, \Lambda_1)$

$$I(z) > 0 \ \text{ if } \ \|z\|_E = \rho \quad \text{and} \quad I(z) > -C_5 \quad \text{if} \quad \|z\|_E \leq \rho, \tag{14}$$

where $C_5 = C(\varepsilon)\lambda^{\frac{p}{p-q}}$.

From Lemma 5, we obtain there exist $\varphi_0,\ \psi_0 \in C_0^\infty(\Omega) \subset W_0^{1,p}(\Omega)$ such that

$$\int_\Omega a_1(x)(\varphi_0^+)^q \mathrm{d}x > 0, \qquad \int_\Omega a_2(x)(\psi_0^+)^q \mathrm{d}x > 0. \tag{15}$$

Therefore, one has

$$\begin{aligned} I(k\varphi_0, k\psi_0) &= \frac{1}{p}k^p\|(\varphi_0,\psi_0)\|_E^p - \frac{\lambda}{q}k^q\int_\Omega (a_1(x)(\varphi_0^+)^q + a_2(x)(\psi_0^+)^q)\mathrm{d}x \\ &\quad - k^{p^*}\int_\Omega \frac{b(x)F(\varphi_0^+,\psi_0^+)}{|x|^{bp^*}}\mathrm{d}x \\ &\leq \frac{1}{p}k^p\|(\varphi_0,\psi_0)\|_E^p - \frac{\lambda}{q}k^q\int_\Omega (a_1(x)(\varphi_0^+)^q + a_2(x)(\psi_0^+)^q)\mathrm{d}x \\ &\quad + \left(2^{p^*-1}(C_4C_{a,b})^{\frac{p^*}{p}} M\|b\|_\infty \|(\varphi_0,\psi_0)\|_E^{p^*}\right) k^{p^*}. \end{aligned}$$

Fix $\lambda \in (0, \Lambda_1)$, noticing that $p > q > 1$, from (15), we can obtain there exists $k = k(\lambda) > 0$ small enough such that $I(k\varphi_0, k\psi_0) < 0$. Thus we deduce that

$$c_1 = \inf_{z \in B_\rho(0)} I(z) < 0 < \inf_{z \in \partial B_\rho(0)} I(z).$$

Apply the Ekeland's variational principle in $\overline{B_\rho(0)}$ (see [7]), we obtain there exists a $(PS)_{c_1}$ sequence $\{z_n\} = \{(u_n, v_n)\} \subset \overline{B_\rho(0)}$ with c_1.

According to (8) and the Minkowski inequality, we have

$$\begin{aligned}\left(\int_\Omega \frac{F(z^+)}{|x|^{bp^*}}\mathrm{d}x\right)^{\frac{p}{p^*}} &\le M^{\frac{p}{p^*}} \left(\int_\Omega \frac{|z|^{p^*}}{|x|^{bp^*}}\mathrm{d}x\right)^{\frac{p}{p^*}} \\ &\le M^{\frac{p}{p^*}} \left(\left(\int_\Omega \frac{|u|^{p^*}}{|x|^{bp^*}}\mathrm{d}x\right)^{\frac{p}{p^*}} + \left(\int_\Omega \frac{|v|^{p^*}}{|x|^{bp^*}}\mathrm{d}x\right)^{\frac{p}{p^*}}\right) \\ &\le \frac{1}{S_{\mu,a,b}} M^{\frac{p}{p^*}} \|z\|_E^p.\end{aligned}$$

It implies

$$S_{\mu,a,b,F} \ge S_{\mu,a,b} M^{-\frac{p}{p^*}} > 0. \tag{16}$$

Hence we can choose $0 < \Lambda < \Lambda_1$ such that $c^* > 0$ for any $\lambda \in (0, \Lambda)$. In view of $c_1 < 0$, it follows from Lemma 4, I satisfies the $(PS)_{c_1}$ condition. Therefore, one has a subsequence still denoted by $\{z_n\}$ and $z_1 = (u_1, v_1) \in E$ such that $z_n \to z_1$ in E and

$$I(z_1) = c_1, \quad I'(z_1) = 0,$$

which implies z_1 is a solution of problem (1). Using the hypothesis $F_u(0, v^+) = F_v(u^+, 0) = 0$ for any $(u, v) \in E$, after a direct calculation, we derive that

$$\|u_1^-\|^p_{W_a^{1,p}(\Omega)} = \langle I'_u(u_1, v_1), -u_1^- \rangle = 0, \qquad \|v_1^-\|^p_{W_a^{1,p}(\Omega)} = \langle I'_v(u_1, v_1), -v_1^- \rangle = 0,$$

which implies $u_1^- = 0$ and $v_1^- = 0$. Hence we have $u_1 \ge 0$ and $v_1 \ge 0$.

Now we show $u_1 \ne 0$ and $v_1 \ne 0$. Since $I(z_1) = c_1 < 0 = I(0, 0)$, we have $u_1 \ne 0$ or $v_1 \ne 0$. Without loss of generality, we may assume $u_1 \ne 0$ and $v_1 = 0$. Then, for any $\lambda \in (0, \Lambda)$, we have

$$\begin{aligned}I(tu_1, t\psi_0^+) &= \frac{1}{p}t^p \|u_1\|^p_{W_a^{1,p}(\Omega)} + \frac{1}{p}t^p \|\psi_0^+\|^p_{W_a^{1,p}(\Omega)} - \frac{\lambda}{q}t^q \int_\Omega a_1(x) u_1^q \mathrm{d}x \\ &\quad - \frac{\lambda}{q}t^q \int_\Omega a_2(x)(\psi_0^+)^q \mathrm{d}x - t^{p^*} \int_\Omega \frac{b(x)F(u_1, \psi_0^+)}{|x|^{bp^*}}\mathrm{d}x.\end{aligned} \tag{17}$$

Since $(u_1, 0)$ is a critical point of I with $I(u_1, 0) = c_1$, it follows from (7) that

$$\|u_1\|^p_{W_a^{1,p}(\Omega)} - \lambda \int_\Omega a_1(x) u_1^q \mathrm{d}x - p^* \int_\Omega \frac{b(x)F(u_1, 0)}{|x|^{bp^*}}\mathrm{d}x = \langle I'_u(u_1, 0), u_1 \rangle = 0,$$

which implies

$$c_1 = I(u_1, 0) = \frac{p^* - p}{pp^*}\|u_1\|^p_{W^{1,p}_a(\Omega)} - \frac{p^* - q}{qp^*}\lambda\int_\Omega a_1(x)u_1^q \mathrm{d}x.$$

Let $t_1 = \min\left\{\left(\frac{p^*-p}{p^*-q}\right)^{\frac{1}{p-q}}, \left(\frac{p^*-q}{p^*}\right)^{\frac{1}{q}}\right\}$, for any $t \in (0, t_1)$, we have

$$\begin{aligned}
&\frac{1}{p}t^p\|u_1\|^p_{W^{1,p}_a(\Omega)} - \frac{\lambda}{q}t^q\int_\Omega a_1(x)u_1^q\mathrm{d}x \\
&= \frac{p^*}{p^*-q}t^q\left(\frac{p^*-q}{pp^*}t^{p-q}\|u_1\|^p_{W^{1,p}_a(\Omega)} - \frac{p^*-q}{qp^*}\lambda\int_\Omega a_1(x)u_1^q\mathrm{d}x\right) \\
&< \frac{p^*}{p^*-q}t^q\left(\frac{p^*-p}{pp^*}\|u_1\|^p_{W^{1,p}_a(\Omega)} - \frac{p^*-q}{qp^*}\lambda\int_\Omega a_1(x)u_1^q\mathrm{d}x\right) \\
&= \frac{p^*}{p^*-q}c_1t^q < c_1. \qquad (18)
\end{aligned}$$

According to $F \in C^1((R^+)^2, R^+)$ and (15), there exists $t_2 > 0$ such that for any $t \in (0, t_2)$

$$\begin{aligned}
&\frac{1}{p}t^p\|\psi_0^+\|^p_{W^{1,p}_a(\Omega)} - \frac{\lambda}{q}t^q\int_\Omega a_2(x)(\psi_0^+)^q\mathrm{d}x - t^{p^*}\int_\Omega \frac{b(x)F(u_1,\psi_0^+)}{|x|^{bp^*}}\mathrm{d}x \\
&\le \frac{1}{p}t^p\|\psi_0^+\|^p_{W^{1,p}_a(\Omega)} - \frac{\lambda}{q}t^q\int_\Omega a_2(x)(\psi_0^+)^q\mathrm{d}x + C_6t^{p^*} \\
&< 0, \qquad (19)
\end{aligned}$$

where $C_6 = 2^{p^*-1}(C_4C_{a,b})^{\frac{p^*}{p}}M\|b\|_\infty\|(u_1,\psi_0)\|^{p^*}_E + 1 > 0$. Let $t_3 = \min\{t_1, t_2\}$, from (17)-(19), for any $t \in (0, t_3)$, we obtain

$$I(tu_1, t\psi_0^+) < c_1.$$

Noticing that $\|(u_1,\psi_0)\|_E < \infty$, thus we can choose t so small that $(tu_1, t\psi_0^+) \in B_\rho(0)$. Hence we obtain

$$\inf_{z\in B_\rho(0)} I(z) < c_1,$$

which is a contradiction with $c_1 = \inf_{z\in B_\rho(0)} I(z)$. Therefore, we have $u_1 \neq 0$ and $v_1 = 0$ are not established. Similarly, we obtain $u_1 = 0$ and $v_1 \neq 0$ is impossible. Hence we have $u_1 \neq 0$ and $v_1 \neq 0$. The proof of Theorem 1 is completed. □

Assume $\alpha(\mu)$ and $\beta(\mu)$ are zeros of the function

$$g(t) = (p-1)t^p - (N - p(a+1))t^{p-1} + \mu, \quad t \ge 0,$$

then $\alpha(\mu)$ and $\beta(\mu)$ satisfy

$$0 \le \alpha(\mu) < \frac{N-p(a+1)}{p} < \beta(\mu) < \frac{N-p(a+1)}{p-1}. \qquad (20)$$

Define a cut-off function $\varphi(x) \in C_0^\infty(\Omega)$ satisfies $0 \le \varphi(x) \le 1$, $\varphi(x) = 1$ for $x \in B(0, \delta_0)$, $\varphi(x) = 0$ for $x \in R^N \setminus B(0, 2\delta_0)$. Set $u_\eta(x) = \varphi(x)V_\eta(x)$, we have the following estimates (see [11]).

Lemma 6. As $\eta \to 0$, we have

$$\int_\Omega \frac{|u_\eta|^{p^*}}{|x|^{bp^*}}\mathrm{d}x = (S_{\mu,a,b})^{\frac{p^*}{p^*-p}} + O\left(\eta^{(\beta(\mu)+b)p^*-N}\right),$$

$$\int_\Omega \left(\frac{|\nabla u_\eta|^p}{|x|^{ap}} - \mu\frac{|u_\eta|^p}{|x|^{(a+1)p}}\right)\mathrm{d}x = (S_{\mu,a,b})^{\frac{p^*}{p^*-p}} + O\left(\eta^{(\beta(\mu)+a+1)p-N}\right),$$

$$\int_\Omega |u_\eta|^s \mathrm{d}x \ge \begin{cases} C_7\eta^{N-\frac{s(N-(a+1)p)}{p}}, & \frac{N}{\beta(\mu)} < s < \frac{Np}{N-p(a+1)}, \\ C_7\eta^{\frac{s((\beta(\mu)+a+1)p-N)}{p}}|\ln\eta|, & s = \frac{N}{\beta(\mu)}, \\ C_7\eta^{\frac{s((\beta(\mu)+a+1)p-N)}{p}}, & 1 \le s < \frac{N}{\beta(\mu)}. \end{cases}$$

Lemma 7. Let $N > p^2(a+1)$, $p > 1$, $\lambda > 0$, $0 \le a < \frac{N-p}{p}$, $a \le b < a+1$, $\frac{N(p-1)}{N-p(a+1)} < q < p$, $\mu < \frac{N^{p-1}(N(q-p+1)-pq(a+1))}{q^p}$. Suppose that $F \in C^1((R^+)^2, R^+)$ is positively homogeneous of degree p^* such that $F_u(u,0) = F_u(0,v) = F_v(u,0) = F_v(0,v) = 0$ for all $u,\ v \in R^+$. If (a_1), (a_2), (b_1), (b_2) and (b_3) hold, then there exists $\Lambda^* > 0$, for any $\lambda \in (0, \Lambda^*)$, we can find $z_\lambda \in E$ such that $\sup_{t\ge 0} I(tz_\lambda) < c^*$.

Proof. For convenience, we consider the functional $J : E \to R$ defined by

$$J(z) = \frac{1}{p}\|z\|_E^p - \int_\Omega \frac{b(x)F(z^+)}{|x|^{bp^*}}\mathrm{d}x.$$

for all $(u,v) \in E$. Since F is positively homogeneous of degree p^*, we have (7) and (8) hold. It follows from $F \in C^1((R^+)^2, R^+)$ and (8) that there exists $(e_1, e_2) \in \{z \in (R^+)^2 : |z| = 1\}$ such that $F(e_1, e_2) = M$. Define

$$h(t) = J(te_1u_\eta, te_2u_\eta) = \frac{1}{p}t^p\|u_\eta\|^p_{W_a^{1,p}(\Omega)} - Mt^{p^*}\int_\Omega \frac{b(x)|u_\eta|^{p^*}}{|x|^{bp^*}}\mathrm{d}x, \quad \forall\ t \ge 0.$$

Assume that $h(t)$ reach its maximum at t_η, then

$$0 = h'(t_\eta) = t_\eta^{p-1}\left(\|u_\eta\|^p_{W_a^{1,p}(\Omega)} - p^*Mt_\eta^{p^*-p}\int_\Omega \frac{b(x)|u_\eta|^{p^*}}{|x|^{bp^*}}\mathrm{d}x\right),$$

which implies that

$$t_\eta = \left(\frac{\|u_\eta\|^p_{W_a^{1,p}(\Omega)}}{p^*M\int_\Omega \frac{b(x)|u_\eta|^{p^*}}{|x|^{bp^*}}\mathrm{d}x}\right)^{\frac{1}{p^*-p}}.$$

Using the definition of u_η and (b_2), we obtain $t_\eta < \infty$. Therefore,

$$\sup_{t\geq 0} J(te_1u_\eta, te_2u_\eta) = J(t_\eta e_1u_\eta, t_\eta e_2u_\eta) = \Phi(\eta) + \Psi(\eta), \tag{21}$$

where

$$\Phi(\eta) = \frac{1}{p}t_\eta^p \|u_\eta\|^p_{W_a^{1,p}(\Omega)} - M\|b\|_\infty t_\eta^{p^*} \int_\Omega \frac{|u_\eta|^{p^*}}{|x|^{bp^*}}\mathrm{d}x,$$

$$\Psi(\eta) = Mt_\eta^{p^*} \int_\Omega \frac{(\|b\|_\infty - b(x))|u_\eta|^{p^*}}{|x|^{bp^*}}\mathrm{d}x.$$

Using the fact

$$\max_{t\geq 0}\left(\frac{A_1}{p}t^p - \frac{A_2}{p^*}t^{p^*}\right) = \frac{p^*-p}{pp^*}\left(\frac{A_1}{A_2^{p/p^*}}\right)^{\frac{p^*}{p^*-p}}.$$

According to Lemma 6, (16) and (20), we obtain

$$\begin{aligned}
\Phi(\eta) &\leq \frac{p^*-p}{pp^*}\left(\frac{\|u_\eta\|^p_{W_a^{1,p}(\Omega)}}{\left(M\|b\|_\infty \int_\Omega \frac{|u_\eta|^{p^*}}{|x|^{bp^*}}\mathrm{d}x\right)^{p/p^*}}\right)^{\frac{p^*}{p^*-p}} \\
&= \frac{p^*-p}{pp^*}(Mp^*\|b\|_\infty)^{-\frac{p}{p^*-p}}(S_{\mu,a,b})^{\frac{p^*}{p^*-p}} + O\left(\eta^{(\beta(\mu)+a+1)p-N}\right) \\
&\leq \frac{p^*-p}{p}\|b\|_\infty^{-\frac{p}{p^*-p}}\left(\frac{S_{\mu,a,b,F}}{p^*}\right)^{\frac{p^*}{p^*-p}} + O\left(\eta^{(\beta(\mu)+a+1)p-N}\right).
\end{aligned} \tag{22}$$

Set $\tau = \frac{N-p(a+1)}{p}$, arguing as in [11], there exist C_8 and C_9 such that

$$C_8 \leq U_{p,\mu}(x)\left(|x|^{\frac{\alpha(\mu)}{\tau}} + |x|^{\frac{\beta(\mu)}{\tau}}\right)^\tau \leq C_9.$$

Hence,

$$0 \leq U_{p,\mu}(x) \leq C_9|x|^{-\beta(\mu)}. \tag{23}$$

It follows from (b_3) that there is $\rho_0 \in (0, \delta_0)$ such that

$$0 \leq b(0) - b(x) \leq |x|^k, \quad \forall\ x \in B(0,\rho_0),$$

where $k \geq \frac{N-bp^*}{p-1}$. In addition, it implies from (20) that $k > (\beta(\mu)+b)p^* - N$. Noticing that $b(0) = \|b\|_\infty$ and $t_\eta < \infty$, we deduce from (23) that

$$\Psi(\eta) = Mt_\eta^{p^*}\eta^{-\tau p^*}\int_\Omega \frac{(\|b\|_\infty - b(x))\left(\varphi(x)U_{p,\mu}(\eta^{-1}x)\right)^{p^*}}{|x|^{bp^*}}\mathrm{d}x$$

$$
\begin{aligned}
&\leq C_9 M t_\eta^{p^*} \|b\|_\infty \eta^{(\beta(\mu)-\tau)p^*} \int_{R^N \setminus B(0,\rho_0)} |x|^{-(\beta(\mu)+b)p^*} \mathrm{d}x \\
&\quad + C_9 M t_\eta^{p^*} \eta^{(\beta(\mu)-\tau)p^*} \int_{B(0,\rho_0)} |x|^{k-(\beta(\mu)+b)p^*} \mathrm{d}x \\
&= C_9 M N \omega_N t_\eta^{p^*} \|b\|_\infty \eta^{(\beta(\mu)-\tau)p^*} \int_{\rho_0}^{+\infty} r^{N-(\beta(\mu)+b)p^*-1} \mathrm{d}r \\
&\quad + C_9 M N \omega_N t_\eta^{p^*} \eta^{(\beta(\mu)-\tau)p^*} \int_0^{\rho_0} r^{k+N-(\beta(\mu)+b)p^*-1} \mathrm{d}r \\
&= C_9 M N \omega_N t_\eta^{p^*} \|b\|_\infty \eta^{(\beta(\mu)-\tau)p^*} \frac{\rho_0^{-((\beta(\mu)+b)p^*-N)}}{(\beta(\mu)+b)p^*-N} \\
&\quad + C_9 M N \omega_N t_\eta^{p^*} \eta^{(\beta(\mu)-\tau)p^*} \frac{\rho_0^{k-((\beta(\mu)+b)p^*-N)}}{k-((\beta(\mu)+b)p^*-N)} \\
&= O\left(\eta^{(\beta(\mu)-\tau)p^*}\right), \qquad (24)
\end{aligned}
$$

where $\omega_N = \frac{2\pi^{N/2}}{N\Gamma(N/2)}$ denotes the volume of the unit ball $B(0,1)$ in R^N. In view of $(\beta(\mu)-\tau)p^* > (\beta(\mu)-\tau)p = (\beta(\mu)+a+1)p-N$, we deduce from (21), (22) and (24) that

$$
\sup_{t\geq 0} J(te_1u_\eta, te_2u_\eta) \leq \frac{p^*-p}{p}\|b\|_\infty^{-\frac{p}{p^*-p}} \left(\frac{S_{\mu,a,b,F}}{p^*}\right)^{\frac{p^*}{p^*-p}} + O\left(\eta^{(\beta(\mu)+a+1)p-N}\right). \qquad (25)
$$

By the inequality (16), we can choose δ_1 such that

$$
c^* = \frac{p^*-p}{p}\|b\|_\infty^{-\frac{p}{p^*-p}} \left(\frac{S_{\mu,a,b,F}}{p^*}\right)^{\frac{p^*}{p^*-p}} - B\lambda^{\frac{p}{p-q}} > 0 \quad \text{for all} \quad 0<\lambda<\delta_1.
$$

Using the definitions of I and u_η, from (a_2) and (b_2), we have

$$
I(te_1u_\eta, te_2u_\eta) \leq \frac{1}{p}t^p\|u_\eta\|^p_{W_a^{1,p}(\Omega)} = \frac{1}{p}t^p\left((S_{\mu,a,b})^{\frac{p^*}{p^*-p}} + O\left(\eta^{(\beta(\mu)+a+1)p-N}\right)\right)
$$

for all $t \geq 0$ and $\lambda > 0$. It follows that there exist $T \in (0,1)$ and $\eta_1 > 0$ such that for all $0<\lambda<\delta_1$ and $0<\eta<\eta_1$

$$
\sup_{0\leq t\leq T} I(te_1u_\eta, te_2u_\eta) \leq c^*. \qquad (26)
$$

By means of $p > q$, we obtain $\frac{N}{q} > \frac{N}{p} \geq \frac{N-p(a+1)}{p}$. Arguing as in [8], it implies from (20) that

$$
\begin{aligned}
\beta(\mu) > \frac{N}{q} &\Leftrightarrow g\left(\frac{N}{q}\right) < g(\beta(\mu)) \\
&\Leftrightarrow (p-1)\left(\frac{N}{q}\right)^p - (N-p(a+1))\left(\frac{N}{q}\right)^{p-1} + \mu < 0 \\
&\Leftrightarrow 0 \leq \mu < \frac{N^{p-1}(N(q-p+1)-pq(a+1)}{q^p}. \qquad (27)
\end{aligned}
$$

Noticing that $0 \le \mu < \frac{N^{p-1}(N(q-p+1)-pq(a+1)}{q^p}$, using the definitions of I and u_η, it follows from (25), (27) and Lemma 6 that

$$\begin{aligned}
&\sup_{t\ge T} I(te_1u_\eta, te_2u_\eta)\\
&= \sup_{t\ge T}\left(J(te_1u_\eta, te_2u_\eta) - \frac{\lambda}{q}t^q\int_\Omega (a_1(x)+a_2(x))|u_\eta|^q \mathrm{d}x\right)\\
&\le \frac{p^*-p}{p}\|b\|_\infty^{-\frac{p}{p^*-p}}\left(\frac{S_{\mu,a,b,F}}{p^*}\right)^{\frac{p^*}{p^*-p}} + O\left(\eta^{(\beta(\mu)+a+1)p-N}\right) - \frac{2\lambda\beta_0T^q}{q}\int_{B(0,\delta_0)}|u_\eta|^q\mathrm{d}x\\
&\le \frac{p^*-p}{p}\|b\|_\infty^{-\frac{p}{p^*-p}}\left(\frac{S_{\mu,a,b,F}}{p^*}\right)^{\frac{p^*}{p^*-p}} + O\left(\eta^{(\beta(\mu)+a+1)p-N}\right)\\
&-\frac{2C_{10}\lambda\beta_0T^q}{q}\eta^{N-\frac{q(N-p(a+1))}{p}}. \qquad (28)
\end{aligned}$$

According to (27), we obtain $\frac{((\beta(\mu)+a+1)p-N)(p-q)}{p(q\beta(\mu)+qa-N)} > 0$. Let $\eta = \lambda^{\frac{p}{(p-q)((\beta(\mu)+a+1)p-N)}}$, for some positive constants C_{11}, C_{12} and for any $\lambda < \left(\frac{C_{12}}{B+C_{11}}\right)^{\frac{((\beta(\mu)+a+1)p-N)(p-q)}{p(q\beta(\mu)+qa-N)}}$, we have

$$C_{11}\eta^{(\beta(\mu)+a+1)p-N} - C_{12}\lambda\eta^{N-\frac{q(N-p(a+1))}{p}} = C_{11}\lambda^{\frac{p}{p-q}} - C_{12}\lambda^{\frac{((\beta(\mu)+a+1)p-q\beta(\mu))p}{(\beta(\mu)+a+1)p-N)(p-q)}} < -B\lambda^{\frac{p}{p-q}},$$

which implies that there exists $\eta_2 > 0$ such that

$$O\left(\eta^{(\beta(\mu)+a+1)p-N}\right) - \frac{2C_{10}\lambda\beta_0T^q}{q}\eta^{N-\frac{q(N-p(a+1))}{p}} < -B\lambda^{\frac{p}{p-q}}, \qquad (29)$$

for all $0 < \lambda < \delta_2$ and $\eta = \lambda^{\frac{p}{(p-q)((\beta(\mu)+a+1)p-N)}}$. Combining (28) with (29), for any $0 < \lambda < \delta_2$ and $\eta = \lambda^{\frac{p}{(p-q)((\beta(\mu)+a+1)p-N)}}$, one has

$$\sup_{t\ge T} I(te_1u_\eta, te_2u_\eta) \le c^*. \qquad (30)$$

Set $\Lambda^* = \min\left\{\delta_1, \delta_2, \eta_1^{\frac{(p-q)((\beta(\mu)+a+1)p-N)}{p}}\right\}$ and let $\lambda \in (0,\Lambda^*)$, $\eta = \lambda^{\frac{p}{(p-q)((\beta(\mu)+a+1)p-N)}}$ and $z_\lambda = (e_1u_\eta, e_2u_\eta)$, we deduce from (26) and (30) that

$$\sup_{t\ge 0} I(tz_\lambda) \le c^* \qquad \text{for all } \ 0 < \lambda < \Lambda^*.$$

□

Proof of Theorem 2. Choose $\lambda^* = \min\{\Lambda, \Lambda^*\}$, from the proof of Theorem 1, we have already obtained problem (1) for any $\lambda \in (0,\lambda^*)$ has a nontrivial nonnegative solution z_1 with $I(z_1) < 0$. Now we only need to find the second nontrivial nonnegative solution of problem (1). According to (a_1), we can obtain (9) and (14) hold. It follows from (b_1) and Lemma 5, there exists $\varphi_0 \in C_0^\infty(\Omega)$ such that

$$\int_\Omega \frac{b(x)(\varphi_0^+)^{p^*}}{|x|^{bp^*}}\mathrm{d}x > 0.$$

Since F is positively homogeneous of degree p^*, we have (8) holds. It follows from $F \in C^1((R^+)^2, R^+)$ and (8), there exists $(e_1, e_2) \in \{z \in (R^+)^2 : |z| = 1\}$ such that $F(e_1, e_2) = M$. Let $z_0 = (e_1\varphi_0, e_2\varphi_0)$, from (9), we have

$$I(tz_0) \leq \left(\frac{1}{p} + \varepsilon\right) t^p \|z_0\|_E^p - Mt^{p^*} \int_\Omega \frac{b(x)(\varphi_0^+)^{p^*}}{|x|^{bp^*}} \mathrm{d}x + C(\varepsilon)\lambda^{\frac{p}{p-q}},$$

which implies

$$I(tz_0) \to -\infty \quad \text{as} \quad t \to +\infty.$$

Hence, there exists a positive number t_4 such that $\|t_4 z_0\|_E > \rho$ and $I(t_4 z_0) < 0$ for any $\lambda \in (0, \lambda^*)$. It implies from (14) that the functional I has the mountain pass geometry. Define

$$\Gamma = \{\gamma \in C([0,1], E) |\ \gamma(0) = 0,\ \gamma(1) = t_4 z_0\}, \quad c_2 = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)).$$

From Lemma 7, we have $c_2 < c^*$. Applying Lemma 4, we know I satisfies the $(PS)_{c_2}$ condition. By the mountain pass theorem (see [1]), we obtain problem (1) has the second solution $z_2 = (u_2, v_2)$ with $I(z_2) > 0$. Noticing that $F_u(0, v_2^+) = F_v((u_2^+, 0) = 0$ and after a direct calculation, we obtain

$$\|u_2^-\|^p_{W_a^{1,p}(\Omega)} = \langle I'_u(z_2), -u_2^- \rangle = 0, \qquad \|v_2^-\|^p_{W_a^{1,p}(\Omega)} = \langle I'_v(z_2), -v_2^- \rangle = 0,$$

which implies $u_2^- = 0$ and $v_2^- = 0$. Hence we have $u_2 \geq 0$ and $v_2 \geq 0$.

Next, we show $u_2 \neq 0$ and $v_2 \neq 0$. Since $I(z_2) > 0 = I(0,0)$, we have $u_2 \neq 0$ or $v_2 \neq 0$. Without loss of generality, we may assume $u_2 \neq 0$ and $v_2 = 0$. Using the hypothesis $F_u(u, 0) = F_v(0, v) = 0$ for all $u,\ v \in R^+$, it is easy to obtain u_2 satisfies

$$\begin{cases} -\mathrm{div}(|x|^{-ap}|\nabla u_2|^{p-2}\nabla u_2) - \mu \frac{|u_2|^{p-2}u_2}{|x|^{(a+1)p}} = \lambda a_1(x) u_2^{q-1}, & \text{in } \Omega, \\ u_2 = 0, & \text{on } \partial\Omega. \end{cases} \tag{31}$$

Acting on (31) with $u_2 \in W_0^{1,p}(\Omega)$ yields

$$\|u_2\|^p_{W_a^{1,p}(\Omega)} = \lambda \int_\Omega a_1(x) u_2^q \mathrm{d}x.$$

Therefore, we have

$$I(u_2, 0) = \frac{1}{p}\|u_2\|^p_{W_a^{1,p}(\Omega)} - \frac{\lambda}{q}\int_\Omega a_1(x) u_2^q \mathrm{d}x = \left(\frac{1}{p} - \frac{1}{q}\right)\|u_2\|^p_{W_a^{1,p}(\Omega)} < 0,$$

which is a contradiction with $I(u_2, 0) = I(z_2) > 0$. Therefore, we have $u_2 \neq 0$ and $v_2 = 0$ are not established. Similarly, we obtain $u_2 = 0$ and $v_2 \neq 0$ are impossible. Hence we have $u_2 \neq 0$ and $v_2 \neq 0$. Since $I(z_1) < 0 < I(z_2)$, we have z_2 is another nontrivial nonnegative solution of problem (1). The proof of Theorem 2 is completed. □

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