Common fixed points of set mappings endowed with directed graphs

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Abstract

The aim of this article is to obtain common fixed points of set mappings defined on a family of sets endowed with a graph satisfying $\Theta-$contraction. Our results generalize and extend various results in the existing literature. We also provide some non trivial examples to support the validity of our main results. Some applications to the construction of common fixed points of set mappings in ε-chainable metric space are also discussed.

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1 Introduction and preliminaries

In 1922, Banach [9] presented a revolutionary contraction principle (Banach contraction principle) in which Picard iteration process was used for the evaluation of a fixed point. This principle guarantees the existence and uniqueness of fixed points of certain self-mappings on metric spaces, and provides a constructive method to find those fixed points. The Banach contraction principle was also used to establish the existence of a unique solution for a nonlinear integral equation. For instance, it has been used to show the existence of solutions of nonlinear Volterra integral equations, nonlinear integro-differential equations in Banach spaces and to show the convergence of algorithms in computational mathematics. Because of its importance and usefulness for mathematical theory, it has become a very popular tool in solving existence problems in many directions. Due to its importance and simplicity, several authors have obtained many interesting extensions and generalizations of the Banach contraction principle (see [1-14]). Following the Banach contraction principle, Nadler [21] proved a fixed point theorem for set-valued mapping using the Hausdorff metric. Recall that if (X, d) is a metric space and $CB(X)$ denotes the class of all nonempty closed and bounded subsets of X and $A, B \in CB(X)$, then

$$H(A, B) = \max\{\sup_{b\in B} d(b, A), \sup_{a\in A} d(a, B)\},$$

where $d(x, B) = \inf\{d(x, b) : b \in B\}$.

In 2008, Jachymski [20] proved some fixed point results in metric spaces endowed with a graph and generalized simultaneously Banach's contraction principle from metric and partially ordered metric spaces. Consistent with Jachymski, let (X, d) be a metric space and let Δ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, i.e., $\Delta \subseteq E(G)$. Also assume

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that the graph G has no parallel edges and, thus, one can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph (see [20]) by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for each $i = 1, \cdots, N$.

Jachymski [20] introduced the following definition of G-contraction:

Definition 1.1. [20] Let (X, d) be a metric space. A mapping $F : X \to X$ is called Banach G-contraction (or simply G-contraction) if

(a) For each $x, y \in X$ with $(x, y) \in E(G)$, we have $(F(x), F(y)) \in E(G)$ (that is F preserves edges).

(b)There exists $\alpha \in (0, 1)$ such that for all $x, y \in X$ with $(x, y) \in E(G)$, we have

$$d(F(x), F(y)) \leq \alpha d(x, y). \tag{1.1}$$

Notice that a graph G is connected if there is a directed path between any two vertices and it is weakly connected if $\widetilde{G}$ is connected, where $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Denote by G^{-1} the graph obtained from G by reversing the direction of edges. Thus, we have

$$V\left(G^{-1}\right) = V\left(G\right) \text{ and } E\left(G^{-1}\right) = \left\{(x, y) \in X \times X : (y, x) \in E\left(G\right)\right\}.$$

It is more convenient to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric, under this convention; we have that

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

By a subgraph of G we mean a graph H satisfying $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, such that $V(H)$ contains the vertices of all edges of $E(H)$. If $E(G)$ is symmetric, then for $x \in V(G)$, the subgraph G_x consisting of all edges and vertices that are contained in some path in G starting at x. Then G_x is called the component of G containing x. In this case $V(G_x) = [x]_G$, where $[x]_G$ denotes the equivalence class of x with respect to the relation R defined on $V(G)$ by the rule:

$$yRz \text{ iff there is a path in } G \text{ from } y \text{ to } z.$$

Clearly G_x is connected for all $x \in G$.

Let $\Psi = \{G : G \text{ is a directed graph with } V(G) = X \text{ and } \Delta \subseteq E(G)\}$.

Recall that if $F : X \to X$ is a mapping, then we denote by f_F, the set of all fixed points of F. Let $X_F := \{x \in X : (x, F(x)) \in E(G)\}$.

Jachymski [19] introduced the following property:

(P) For any sequence $\{x_n\}$ in X, if $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$, then $(x_n, x) \in E(G)$, for all $n \in \mathbb{N}$.

Theorem 1.2. [19] Let (X, d) be a complete metric space and let G be a directed graph such that $V(G) = X$ and let $F : X \to X$ be a G-contraction. Suppose that $E(G)$ and the triplet (X, d, G) have property (P). Then the following statements hold:

(1) $f_F \neq \varnothing$ if and only if $X_F \neq \varnothing$;

(2) if $X_F \neq \varnothing$ and G is weakly connected, then f is a Picard operator, i.e., $f_F = \{x^*\}$ and sequence $\{F^n(x)\} \to x^*$ as $n \to \infty$, for all $x \in X$;

(3) for any $x \in X_F$, $F\mid_{[x]_{\widetilde{G}}}$ is a Picard operator;

(4) if $X_F \times X_F \subseteq E(G)$, then F is a weakly Picard operator, i.e.,$f_F \neq \varnothing$ and, for each $x \in X$, we have sequence $\{F^n(x)\} \to x^* \in f_F$ as $n \to \infty$.

Throughout this paper, we assume that a directed graph G has no parallel edge and G is a weighted graph in the sense that each vertex x is assigned the weight $d(x, x) = 0$ and each edge (x, y) is assigned the weight $d(x, y)$. Since d is a metric on X, the weight assigned to each vertex x to vertex y need not be zero and, whenever a zero weight is assigned to some edge (x, y), it reduces to a loop (x, x) having weight 0. Further, in Pompeiu-Hausdorff metric induced by metric d, the Pompeiu-Hausdorff weight assigned to each $U, V \in CB(X)$ need not be zero (that is, $H(U, V) \neq 0$) and, whenever a zero Pompeiu-Hausdorff weight is assigned to some $U, V \in CB(X)$, then it reduces to $U = V$.

In 2014, Abbas et al. [1] considered the set mapping $F : CB(X) \to CB(X)$ instead of a mapping F from X to X or from X to $CB(X)$ and they gave the definition of fixed point of such mapping $F : CB(X) \to CB(X)$ as:

A set $A \in CB(X)$ is said to be a fixed point of F if $F(A) = A$. The set of all fixed points of F is denoted by $Fix(F)$. Indeed Abbas et al. [1] gave the following definition:

Definition 1.3. [1] Let A and B be two nonempty subsets of X with a given graph G. Then:

(a) $E(A, B)$ denotes the set of edges from A to B, there is an edge between A and B, we mean there is an edge between some $a \in A$ and $b \in B$.

(b) $P(A, B)$ denotes the set of paths from A to B, there is a path between A and B, we mean that there is a path between some $a \in A$ and $b \in B$.

On $CB(X)$, a relation R is defined in the following way: for $A, B \in CB(X)$, we have ARB if and only if, there is a path between A and B.

For mappings $F : CB(X) \to CB(X)$, the set X_T is defined as

$$X_F := \{U \in CB(X) : (U, F(U)) \subset E(G)\}.$$

A subset Υ of $CB(X)$ is said to be complete if for any set $X, Y \in \Upsilon$, there is an edge between X and Y.

Abbas et al. [1] used the property P^* stated as follows:

(P^*) For any sequence $\{X_n\}$ in $CB(X)$ with $X_n \to X$ as $n \to \infty$ and there is an edge between X_n and X_{n+1} for $n \in \mathbb{N}$, then there is a subsequence $\{X_{n_k}\}$ of $\{X_n\}$ with an edge between X_{n_k} and X for $k \in \mathbb{N}$.

Recently, Jleli and Samet [27] introduced a new type of contraction called Θ-contraction and established some new fixed point theorems for such contraction in the context of generalized metric spaces.

Let $\Theta : (0, \infty) \to (1, \infty)$ be a function satisfying the following:

(Θ_1) Θ is nondecreasing;

(Θ_2) for each sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \Theta(\alpha_n) = 1$ if and only if $\lim_{n\to\infty}(\alpha_n) = 0$;

(Θ_3) there exists $0 < h < 1$ and $l \in (0, \infty]$ such that $\lim_{\alpha\to 0^+} \frac{\Theta(\alpha)-1}{\alpha^h} = l$.

Definition 1.4. Let (X, d) be a metric space. A mapping $F : X \to X$ is said to be Θ-contraction if there exists a function Θ satisfying (Θ_1)-(Θ_3) and a constant $\lambda \in (0, 1)$ such that

$$Fx \neq Fy \Longrightarrow \Theta(d(Fx, Fy)) \leq [\Theta(d(x, y))]^\lambda \tag{1.2}$$

for all $x, y \in X$.

Theorem 1.5. [27] Let (X, d) be a complete metric space and $F : X \to X$ be a Θ-contraction. Then F has a unique fixed point.

To be consistent with Samet et al. [27], we denote by Ω the set of all functions $\Theta : (0, \infty) \to (1, \infty)$ satisfying the above conditions (Θ_1)-(Θ_3).

The following definition of an $\varepsilon-$chainable metric space is much needed in the sequel.

Definition 1.6. A metric space (X, d) is called an $\varepsilon-$chainable metric space for some $\varepsilon > 0$, if for $x, y \in X$, there is $n \in \mathbb{N}$ and a sequence $\{x_i\}_{i=0}^n$ such that

$$x_0 = x,\ x_n = y \text{ and } d(x_{i-1}, x_i) < \varepsilon \text{ for } i = 1, ..., n.$$

For more information on $\varepsilon-$chainable metric space, we refer the reader to [10].

We also need of the following lemma of Nadler [21] (see also, [7]).

Lemma 1.7. [21] Let (X, d) be a metric space. If $U, V \in CB(X)$ with $H(U, V) < \varepsilon$, then for each $u \in U$ there exists an element $v \in V$ such that $d(u, v) < \varepsilon$.

The aim of this article is to prove some common fixed point results for noncommutative Θ-contractive mappings, defined on the family of closed and bounded subsets of a metric space, endowed with a graph.

2 Main results

In this section, we define the notion of graphic Θ-contraction and establish some common fixed point theorems for set mappings from $CB(X)$ to $CB(X)$.

Definition 2.1. Let $F, S : CB(X) \to CB(X)$ be two set mappings. The pair (F, S) is said to be a graphic Θ-contraction if the following conditions hold:

(i) For every U in $CB(X)$, $(U, F(U)) \subset E(G)$ and $(U, S(U)) \subset E(G)$.

(ii) There exists some $\lambda \in (0, 1)$ such that for some $\Theta \in \Omega$, there is an edge between A and B with $F(A) \neq S(B)$, then

$$\Theta(H(F(A), S(B))) \leq [\Theta(M_{F,S}(A, B))]^\lambda , \tag{2.1}$$

where

$$M_{F,S}(A, B) = \max \left\{ \begin{array}{c} H(A, B), H(A, F(A)), H(B, S(B)), \\ \underline{H(A, S(B)) + H(B, F(A))} \\ 2 \end{array} \right\} \tag{2.2}$$

holds.

It is obvious that if a pair (F, S) of set mappings on $CB(X)$ is a graphic θ-contraction for a graph G, then (F, S) is also graphic Θ-contraction for the graphs G^{-1} and $\widetilde{G}$.

Now we establish the following result regarding common fixed points of set mappings $F, S : CB(X) \to CB(X)$, under some new contractive conditions.

Theorem 2.2. Let (X, d) be a complete metric space endowed with a directed graph G such that $V(G) = X$, $\Delta \subseteq E(G)$ and let $F, S : CB(X) \to CB(X)$ be a graphic Θ-contraction pair such that the relation R on $CB(X)$ is transitive. If Θ or one of mappings F, S is continuous. Then the following statements hold:

(a) $Fix(F) \neq \varnothing$ or $Fix(S) \neq \varnothing$ if and only if $Fix(F) \cap Fix(S) \neq \varnothing$.

(b) $Fix(F) \cap Fix(S) \neq \varnothing$ provided that G is weakly connected and satisfies the property (P^*).

(c) If $Fix(F) \cap Fix(S)$ is complete, then the Pompeiu-Hausdorff weight assigned to the $U, V \in Fix(F) \cap Fix(S)$ is 0.

(d) $Fix(F) \cap Fix(S)$ is complete if and only if $F(F) \cap F(S)$ is a singleton.

Proof. Suppose $Fix(F) \neq \varnothing$ and $U \in Fix(F)$. By the assumption, we have $(U, S(U)) \subset E(G)$. We now show that $U \in Fix(S)$. Assume on contrary that $U \notin Fix(S)$. As the pair (F, S) is graphic Θ-contraction and $(U, U) = (U, S(U)) \subset E(G)$, we have

$$\begin{aligned} \Theta(H(U, S(U))) &= \Theta(H(F(U), S(U))) \\ &\leq [\Theta(M_{F,S}(U, U))]^{\lambda}, \end{aligned}$$

where

$$\begin{aligned} M_{F,S}(U, U) &= \max\left\{ \begin{array}{c} H(U, U), H(U, F(U)), H(U, S(U)), \\ \frac{H(U,S(U))+H(U,F(U))}{2} \end{array} \right\} \\ &= H(U, S(U)). \end{aligned}$$

Thus, we have

$$\Theta(H(U, S(U))) \leq [\Theta(H(U, S(U)))]^{\lambda},$$

which is a contradiction because $\lambda \in (0, 1)$. Hence, the Pompeiu-Hausdorff weight assigned to U and $S(U)$ is zero and so $U \in Fix(S)$. Therefore, $U \in Fix(F) \cap Fix(S) \neq \varnothing$. Similarly, if $U \in Fix(S)$, then we have $U \in Fix(F)$. The converse is obvious.

Now we prove (b): Let A_0 be an arbitrary element in $CB(X)$. If $A_0 \in Fix(F)$ or $A_0 \in Fix(S)$, then done by (a). We assume that $A_0 \notin Fix(F)$ and $A_0 \notin Fix(S)$. The inclusion $(A_0, F(A_0)) \subset E(G)$ implies that there exists some x_0 in A_0 such that there is an edge between x_0 and some $x_1 \in F(A_0)$. Let $A_1 = F(A_0)$, again the inclusion $(A_1, S(A_1)) \subset E(G)$ implies that there is an edge between x_1 and some $x_2 \in S(A_1)$. Let $A_2 = S(A_1)$. Continuing this way, we take

$$A_1 = F(A_0),\ A_2 = S(A_1), ...,\ A_{2k+1} = F(A_{2k}),\ A_{2k+2} = S(A_{2k+1}),$$

for $k \in \mathbb{N}$. We may assume that $A_{2k} \neq A_{2k+1}$, for all $k \in \mathbb{N}$. If not, then $A_{2k_0} = A_{2k_0+1}$ for some k_0, $F(A_{2k_0}) = A_{2k_0+1} = A_{2k_0}$, and thus $A_{2k_0} \in Fix(F)$. Hence $A_{2k_0} \in Fix(F) \cap Fix(S)$ by (a). As $(A_{2k}, A_{2k+1}) \subset E(G)$, from (2.1), we have

$$\begin{aligned} 1 &< \Theta(H(A_{2k+1}, A_{2k+2})) = \Theta(H(F(A_{2k}), S(A_{2k+1}))) \\ &\leq [\Theta(M_{F,S}(A_{2k}, A_{2k+1}))]^{\lambda}, \end{aligned}$$

where

$$\begin{aligned} M_{F,S}(A_{2k}, A_{2k+1}) &= \max\{H(A_{2k}, A_{2k+1}), H(A_{2k}, F(A_{2k})), H(A_{2k+1}, S(A_{2k+1})), \\ &\quad \frac{H(A_{2k}, S(A_{2k+1})) + H(A_{2k+1}, F(A_{2k}))}{2}\} \\ &= \max\{H(A_{2k}, A_{2k+1}), H(A_{2k}, A_{2k+1}), H(A_{2k+1}, A_{2k+2}), \\ &\quad \frac{H(A_{2k}, A_{2k+2}) + H(A_{2k+1}, A_{2k+1})}{2}\} \\ &\leq \max\{H(A_{2k}, A_{2k+1}), H(A_{2k}, A_{2k+1}), \\ &\quad \frac{H(A_{2k}, A_{2k+1}) + H(A_{2k+1}, A_{2k+2})}{2}\} \\ &= \max\{H(A_{2k}, A_{2k+1}), H(A_{2k+1}, A_{2k+2})\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} 1 &< \Theta(H(A_{2k+1}, A_{2k+2})) \leq [\Theta(\max\{H(A_{2k}, A_{2k+1}), H(A_{2k+1}, A_{2k+2})\})]^{\lambda} \\ &= [\Theta(H(A_{2k}, A_{2k+1}))]^{\lambda}, \end{aligned}$$

for all $k \in \mathbb{N}$. Otherwise we get a contraction to the fact that $\lambda \in (0, 1)$. Again from (2.1), we have

$$\begin{aligned} 1 &< \Theta(H(A_{2k+2}, A_{2k+3})) = \Theta(H(S(A_{2k+1}), F(A_{2k+2}))) \\ &= \Theta(H(F(A_{2k+2}), S(A_{2k+1}))) \\ &\leq [\Theta(M_{F,S}(A_{2k+2}, A_{2k+1}))]^{\lambda}, \end{aligned}$$

where

$$\begin{aligned} & M_{F,S}(A_{2k+2}, A_{2k+1}) \\ = \ & \max\{H(A_{2k+2}, A_{2k+1}), H(A_{2k+2}, F(A_{2k+2})), H(A_{2k+1}, S(A_{2k+1})), \\ & \frac{H(A_{2k+2}, S(A_{2k+1})) + H(A_{2k+1}, F(A_{2k+2}))}{2}\} \\ = \ & \max\{H(A_{2k+2}, A_{2k+1}), H(A_{2k+2}, A_{2k+3}), H(A_{2k+1}, A_{2k+2}), \\ & \frac{H(A_{2k+2}, A_{2k+2}) + H(A_{2k+1}, A_{2k+3})}{2}\} \\ \leq \ & \max\{H(A_{2k+2}, A_{2k+1}), H(A_{2k+2}, A_{2k+3}), \\ & \frac{H(A_{2k+1}, A_{2k+2}) + H(A_{2k+2}, A_{2k+3})}{2}\} \\ = \ & \max\{H(A_{2k+2}, A_{2k+1}), H(A_{2k+2}, A_{2k+3})\}, \end{aligned}$$

which gives that

$$\begin{aligned} 1 &< \Theta(H(A_{2k+2}, A_{2k+3})) \leq \Theta(\max\{H(A_{2k+1}, A_{2k+2}), H(A_{2k+2}, A_{2k+3})\}) \\ &= \Theta(H(A_{2k+1}, A_{2k+2})), \end{aligned}$$

for all $k \in \mathbb{N}$. Otherwise we get a contraction to the fact that $\lambda \in (0, 1)$. Hence

$$1 < \Theta(H(A_n, A_{n+1})) \leq [\Theta(H(A_{n-1}, A_n))]^{\lambda}$$

for all $n \in \mathbb{N}$. Therefore

$$\begin{aligned} 1 &< \Theta\left(H(A_n, A_{n+1})\right) \leq \left[\Theta\left(H(A_{n-1}, A_n)\right)\right]^{\lambda} \\ &\leq \left[\Theta\left(H(A_{n-2}, A_{n-1})\right)\right]^{\lambda^2} \\ &\quad . \\ &\quad . \\ &\quad . \\ &\leq \left[\Theta\left(H(A_0, A_1)\right)\right]^{\lambda^n}. \end{aligned} \tag{2.3}$$

By (Θ_2), we get

$$\lim_{n\to\infty} \Theta\left(H(A_n, A_{n+1})\right) = 1 \Longleftrightarrow \lim_{n\to\infty} H(A_n, A_{n+1}) = 0. \tag{2.4}$$

From the condition (Θ_3), there exist $0 < h < 1$ and $l \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\Theta(H(A_n, A_{n+1})) - 1}{[H(A_n, A_{n+1})]^h} = l.$$

Suppose that $l < \infty$. In this case, let $B = \frac{l}{2} > 0$. From the definition of the limit, there exists $n_1 \in \mathbb{N}$ such that

$$\left|\frac{\Theta(H(A_n, A_{n+1})) - 1}{[H(A_n, A_{n+1})]^h} - l\right| \leq B$$

for all $n > n_1$. This implies that

$$\frac{\Theta(H(A_n, A_{n+1})) - 1}{[H(A_n, A_{n+1})]^h} \geq l - B = \frac{l}{2} = B$$

for all $n > n_1$. Then

$$n[H(A_n, A_{n+1})]^h \leq Cn[\Theta(H(A_n, A_{n+1})) - 1]$$

for all $n > n_1$, where $C = \frac{1}{B}$. Now we suppose that $l = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_1 \in \mathbb{N}$ such that

$$B \leq \frac{\Theta(H(A_n, A_{n+1})) - 1}{[H(A_n, A_{n+1})]^h}$$

for all $n > n_1$. This implies that

$$n[H(A_n, A_{n+1})]^h \leq Cn[\Theta(H(A_n, A_{n+1})) - 1]$$

for all $n > n_1$, where $C = \frac{1}{B}$. Thus, in all cases, there exist $A > 0$ and $n_1 \in \mathbb{N}$ such that

$$n[H(A_n, A_{n+1})]^h \leq Cn[\Theta(H(A_n, A_{n+1})) - 1] \tag{2.5}$$

for all $n > n_1$. Thus by (2.3), we get

$$n[H(A_n, A_{n+1})]^h \leq Cn([\Theta(H(A_0, A_1))]^{\lambda^n} - 1).$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} n[H(A_n, A_{n+1})]^h = 0.$$

Thus, there exists $n_2 \in \mathbb{N}$ such that

$$n[H(A_n, A_{n+1})]^h \leq 1 \tag{2.6}$$

for all $n > n_2$. Consequently we have

$$H(A_n, A_{n+1}) \leq \frac{1}{n^{\frac{1}{h}}} \tag{2.7}$$

for all $n > n_2$.

Now we prove that $\{x_n\}$ is a Cauchy sequence. For $m > n > n_2$, we have

$$\begin{aligned} H(A_n, A_m) &\leq H(A_n, A_{n+1}) + H(A_{n+1}, A_{n+2}) + ... + H(A_{m-1}, A_m) \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/h}}. \end{aligned}$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/h}}$, we get $H(A_n, A_m) \to 0$ as $n, m \to \infty$. Therefore $\{A_n\}$ is a Cauchy sequence in X. Since $(CB(X), H)$ is complete, we have $A_n \to U$ as $n \to \infty$ for some $U \in CB(X)$. Now if we prove that U is the fixed point of F. Then by (a), we can get U as the common fixed point of F and S : Let us show that by considering the following cases.

Case 1. We suppose that F is continuous. We have

$$H(U, F(U)) = \lim_{n\to\infty} H(A_{2n}, F(A_{2n})) = \lim_{n\to\infty} H(A_{2n}, A_{2n+1}) = 0.$$

This proves that U is a fixed point of F.

Case 2. Θ is continuous. In this case, we consider two following subcases.

Subcase 2.1. For each $n \in \mathbb{N}$, there exists $i_n \in \mathbb{N}$ such that $A_{2i_n+1} = F(U)$ and $i_n > i_{n-1}$, where $i_0 = 1$. Then we have

$$U = \lim_{n\to\infty} A_{2i_n+1} = \lim_{n\to\infty} F(U) = F(U).$$

This proves that U is a fixed point of F.

Subcase 2.2. We suppose on the contrary that Pompeiu-Hausdorff weight assign to the U and $F(U)$ is not zero that is $H(U, F(U)) > 0$. Now, for A_{2n+1} in $CB(X)$ $n \in \mathbb{N}$, we have $(A_{2n+1}, A_{2n+2}) = (A_{2n+1}, S(A_{2n+1})) \subset E(G)$. By property (P^*), there is a subsequence $\{A_{2n_k+1}\}$ of $\{A_{2n+1}\}$ with an edge between A_{2n_k+1} and U for $k \in \mathbb{N}$. Now

$$\begin{aligned} 1 &< \Theta(H(F(U), A_{2n_k+2})) = \Theta(H(F(U), S(A_{2n_k+1}))) \\ &\leq [\Theta(M_{F,S}(U, A_{2n_k+1}))]^{\lambda}, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} M_{F,S}(U, A_{2n_k+1}) &= \max\{H(U, A_{2n_k+1}), H(U, F(U)), H(A_{2n_k+1}, S(A_{2n_k+1})), \\ &\quad \frac{H(U, S(A_{2n_k+1})) + H(A_{2n_k+1}, F(U))}{2}\} \\ &= \max\{H(U, A_{2n_k+1}), H(U, F(U)), H(A_{2n_k+1}, A_{2n_k+2}), \\ &\quad \frac{H(U, A_{2n_k+2}) + H(A_{2n_k+1}, F(U))}{2}\}. \end{aligned}$$

If $H(U, F(U)) > 0$, then by the fact

$$\lim_{n\to\infty} H(U, A_{2n_k+1}) = \lim_{n\to\infty} H(A_{2n_k+1}, A_{2n_k+2}) = \lim_{n\to\infty} H(U, A_{2n_k+2}) = 0,$$

there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$M_{F,S}(U, A_{2n_k+1}) = H(U, F(U)).$$

Then from (2.8), we get

$$\Theta(H(F(U), A_{2n_k+2})) \leq [\Theta(H(U, F(U)))]^k, \tag{2.9}$$

for all $n \geq n_0$. On taking limit as $k \to \infty$ in (2.9) and using the continuity of Θ, we have

$$\Theta(H(F(U), U)) \leq [\Theta(H(U, F(U)))]^\lambda$$

which is a contradiction as $\lambda \in (0, 1)$. Hence $U = F(U)$. By using (a), we obtain that U is a common fixed point of F and S.

To prove (c), let $U, V \in Fix(F) \cap Fix(S)$. Suppose that the Pompeiu-Hausdorff weight assign to the U and V is not zero. As (F, S) is a graphic Θ-contraction, we obtain that

$$\Theta(H(U, V)) = \Theta(H(F(U), S(V))) \leq [\Theta(M_{F,S}(U, U))]^\lambda,$$

where

$$\begin{aligned} \Theta(M_{F,S}(U, U)) &= \Theta(\max\{H(U, V), H(U, F(U)), H(V, S(V)), \\ &\quad \frac{H(U, S(V)) + H(V, F(U))}{2}\}) \\ &= \Theta(\max\{H(U, V), H(U, U), H(V, V), \\ &\quad \frac{H(U, V) + H(V, U)}{2}\}) \\ &= \Theta(H(U, V)). \end{aligned}$$

Thus we have

$$\Theta(H(U, V)) \leq [\Theta(H(U, V))]^\lambda$$

which is a contradiction as $\lambda \in (0, 1)$. Hence (c) is proved. Finally we prove (d). We suppose the set $Fix(F) \cap Fix(S)$ is complete, then from (c), for any $U, V \in Fix(F) \cap Fix(S)$, we have $U = V$. Thus $Fix(S) \cap Fix(F)$ is singleton. Conversely, if $Fix(F) \cap Fix(S)$ is singleton, then since $\Delta \subseteq E(G)$, so it is obvious that $F(F) \cap F(S)$ is complete set. Hence the proof is completed. Q.E.D.

Now, we give the following example which support our deduced results.

Example 2.3. Let $X = \{0, 1, 2, ..., n-1\} = V(G)$, $n > 3$ and

$$\begin{aligned} E(G) &= \{(0,0),(1,1),(2,2),...,(n-1,n-1),\\ &\quad (0,1),(0,2),...,(0,n-1),\\ &\quad (1,2),(1,3),...,(1,n-1),\\ &\quad \cdot \quad \cdot \quad \cdot \\ &\quad \cdot \quad \cdot \\ &\quad (n-2,n-1)\}. \end{aligned}$$

Let $V(G)$ be endowed with metric $d : X \times X \to \mathbb{R}^+$ defined by

$$\begin{aligned} d(0,0) &= d(1,1) = ... = d(n-1,n-1) = 0,\\ d(0,1) &= d(1,0) = d(0,2) = d(2,0) = \frac{2}{n+2},\\ d(1,2) &= d(2,1) = ... = d(n-2,n-1) = d(n-1,n-2) = \frac{n+2}{n+3}. \end{aligned}$$

The Hausdorff weight (for $n = 5$) assigned to $A, B \in CB(X)$ are shown in the figure below:

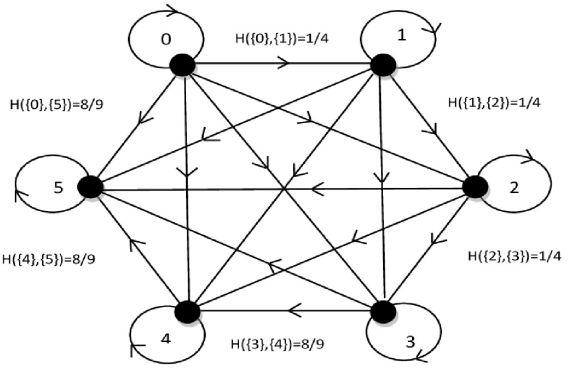


FIGURE 1. Hausdorff weight

Furthermore,

$$H(A,B) = \begin{cases} \frac{2}{n+2} & \text{if } A, B \subseteq \{0,1,2\} \text{ with } A \neq B,\\ \frac{n+2}{n+3} & \text{if } A \text{ or } B \text{ (or both)} \nsubseteq \{0,1,2\} \text{ with } A \neq B,\\ 0 & \text{if } A = B. \end{cases}$$

Define $F, S : CB(X) \to CB(X)$ as follows:

$$\begin{aligned} F(U) &= \begin{cases} \{0\}, & \text{if } U \subseteq \{0,1,2\}, \\ \{0,1\}, & \text{if } U \not\subseteq \{0,1,2\} \end{cases} \\ S(U) &= \begin{cases} \{0\}, & \text{if } U \subseteq \{0,1,2\}, \\ \{0,2\}, & \text{if } U \not\subseteq \{0,1,2\}. \end{cases} \end{aligned}$$

Note that, for all $V \in CB(X)$, $(V, F(V)) \subseteq E(G)$ and $(V, S(V)) \subseteq E(G)$. Take $\Theta(t) = e^{\sqrt{te^t}}$, $t > 0$ and . We can easily show that $\Theta \in \Omega$ and the pair of mappings (F, S) is a graphic Θ-contraction. That is for all $A, B \in CB(X)$ with $F(A) \neq S(B)$ the following holds:

$$e^{\sqrt{H(F(A),S(B))e^{H(F(A),S(B))}}} \leq e^{\lambda\sqrt{M_{F,S}(A,B)e^{M_{F,S}(A,B)}}}$$

for some $\lambda \in (0,1)$. The above condition is equivalent to

$$H(F(A), S(B))e^{H(F(A),S(B))} \leq \lambda^2 M_{F,S}(A,B)\, e^{M_{F,S}(A,B)}.$$

So, we have to check that

$$\frac{H(F(A), S(B))e^{H(F(A),S(B))}}{e^{M_{F,S}(A,B)}} \leq \lambda^2 M_{F,S}(A,B)$$

that is

$$H(F(A), S(B))e^{H(F(A),S(B))-M_{F,S}(A,B)} \leq \lambda^2 M_{F,S}(A,B).$$

We consider the following cases:

(I) If $A \subseteq \{0,1,2\}$ and $B \not\subseteq \{0,1,2\}$, then we have

$$\begin{aligned} H(F(A), S(B))e^{H(F(A),S(B))-M_{F,S}(A,B)} &\leq H(F(A), S(B))e^{H(F(A),S(B))-H(B,S(B))} \\ &= \frac{2}{n+2} e^{\frac{2}{n+2}-\frac{n+2}{n+3}} \\ &= \frac{2}{n+2} e^{-(\frac{n^2+2n-2}{(n+2)(n+3)})} \\ &< \frac{n+2}{n+3} e^{-1} = e^{-1} H(B, S(B)) \\ &\leq (e^{-\frac{1}{2}})^2 M_{F,S}(A,B) \end{aligned}$$

(II) If $A \not\subseteq \{0,1,2\}$ and $B \subseteq \{0,1,2\}$, then we have

$$\begin{aligned} H(F(A), S(B))e^{H(F(A),S(B))-M_{F,S}(A,B)} &\leq H(F(A), S(B))e^{H(F(A),S(B))-H(A,F(A))} \\ &= \frac{2}{n+2} e^{\frac{2}{n+2}-\frac{n+2}{n+3}} \\ &= \frac{2}{n+2} e^{-(\frac{n^2+2n-2}{(n+2)(n+3)})} \\ &< \frac{n+2}{n+3} e^{-1} = e^{-1} H(A, F(A)) \\ &\leq (e^{-\frac{1}{2}})^2 M_{F,S}(A,B) \end{aligned}$$

(III) In case $A, B \not\subseteq \{0,1,2\}$, we have

$$\begin{aligned} H(F(A),S(B))e^{H(F(A),S(B))-M_{F,S}(A,B)} &\leq H(F(A),S(B))e^{H(F(A),S(B))-H(A,B)} \\ &= \frac{2}{n+2}e^{\frac{2}{n+2}-\frac{n+2}{n+3}} \\ &= \frac{2}{n+2}e^{-(\frac{n^2+2n-2}{(n+2)(n+3)})} \\ &< \frac{n+2}{n+3}e^{-1} = e^{-1}H(A,B) \\ &\leq (e^{-\frac{1}{2}})^2 M_{F,S}(A,B) \end{aligned}$$

Hence for every $A, B \in CB(X)$ having an edge between A and B, (1) is satisfied. Thus all the conditions of Theorem 2.1 are satisfied. Moreover, $\{0\}$ is the common fixed point of F and S, and $Fix(F) \cap Fix(S)$ is complete.

The next example shows that the given graph $(V(G), E(G))$ is not always a complete graph.

Example 2.4. Let $X = \{0,1,2,...,n-1\} = V(G)$, $n > 3$ and

$$\begin{aligned} E(G) &= \{(0,0),(1,1),(2,2),...,(n-1,n-1), \\ &\quad (0,1),(0,2),...,(0,n-1)\}. \end{aligned}$$

On $V(G)$, the metric $d : X \times X \to \mathbb{R}^+$ and Pompeiu-Hausdorff metric $H : CB(X) \to \mathbb{R}^+$ are defined as in Example 2.3. The Pompeiu-Hausdorff weights (for $n = 5$) assigned to $A, B \in CB(X)$ are shown in the Figure 2.

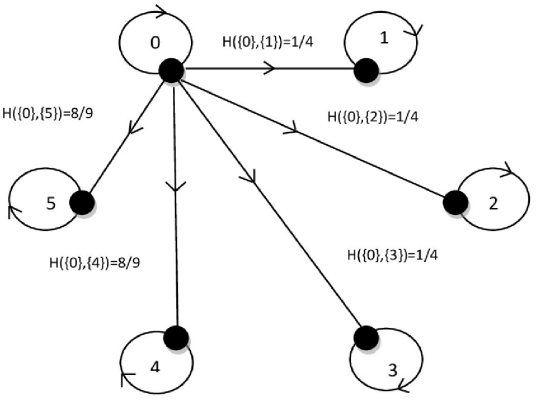


FIGURE 2. Pompeiu-Hausdorff weights

We can define $F, S : CB(X) \to CB(X)$ as follows:

$$\begin{aligned} F(U) &= \begin{cases} \{0\}, & \text{if } U \subseteq \{0,1,2\}, \\ \{0,1,2\}, & \text{if } U \subsetneq \{0,1,2\} \end{cases} \\ S(U) &= \begin{cases} \{0\}, & \text{if } U \subseteq \{0,1,2\}, \\ \{0,2\}, & \text{if } U \subsetneq \{0,1,2\}. \end{cases} \end{aligned}$$

Note that, for all $V \in CB(X)$, $(V, F(V)) \subseteq E(G)$ and $(V, S(V)) \subseteq E(G)$.
Take $\Theta(t) = e^{\sqrt{te^t}}$, $t > 0$ and . We can easily show that $\Theta \in \Omega$ and the pair of mappings (F, S) is a graphic Θ-contraction. That is for all $A, B \in CB(X)$ with $F(A) \neq S(B)$ the following holds:

$$e^{\sqrt{H(F(A),S(B))e^{H(F(A),S(B))}}} \leq e^{\lambda\sqrt{M_{F,S}(A,B)e^{M_{F,S}(A,B)}}}$$

for some $\lambda \in (0,1)$. The above condition is equivalent to

$$H(F(A), S(B))e^{H(F(A),S(B))} \leq \lambda^2 M_{F,S}(A,B)\, e^{M_{F,S}(A,B)}.$$

So, we have to check that

$$\frac{H(F(A), S(B))e^{H(F(A),S(B))}}{e^{M_{F,S}(A,B)}} \leq \lambda^2 M_{F,S}(A,B)$$

that is

$$H(F(A), S(B))e^{H(F(A),S(B))-M_{F,S}(A,B)} \leq \lambda^2 M_{F,S}(A,B).$$

(i) If $A \subseteq \{0,1,2\}$ and $B \not\subseteq \{0,1,2\}$ with $(A,B) \subseteq E(G)$, then we have

$$\begin{aligned} H(F(A), S(B))e^{H(F(A),S(B))-M_{F,S}(A,B)} &\leq H(F(A), S(B))e^{H(F(A),S(B))-H(B,S(B))} \\ &= \frac{2}{n+2}e^{\frac{2}{n+2}-\frac{n+2}{n+3}} \\ &= \frac{2}{n+2}e^{-(\frac{n^2+2n-2}{(n+2)(n+3)})} \\ &< \frac{n+2}{n+3}e^{-1} = e^{-1}H(B, S(B)) \\ &\leq (e^{-\frac{1}{2}})^2 M_{F,S}(A,B) \end{aligned}$$

(ii) When $A \not\subseteq \{0,1,2\}$ and $B \subseteq \{0,1,2\}$ with $(A,B) \subseteq E(G)$, implies that

$$\begin{aligned} H(F(A), S(B))e^{H(F(A),S(B))-M_{F,S}(A,B)} &\leq H(F(A), S(B))e^{H(F(A),S(B))-H(A,F(A))} \\ &= \frac{2}{n+2}e^{\frac{2}{n+2}-\frac{n+2}{n+3}} \\ &= \frac{2}{n+2}e^{-(\frac{n^2+2n-2}{(n+2)(n+3)})} \\ &< \frac{n+2}{n+3}e^{-1} = e^{-1}H(A, F(A)) \\ &\leq (e^{-\frac{1}{2}})^2 M_{F,S}(A,B) \end{aligned}$$

(iii) In case $A, B \not\subseteq \{0,1,2\}$ with $(A,B) \subseteq E(G)$, we have

$$\begin{aligned} H(F(A), S(B))e^{H(F(A),S(B))-M_{F,S}(A,B)} &\leq H(F(A), S(B))e^{H(F(A),S(B))-H(A,B)} \\ &= \frac{2}{n+2}e^{\frac{2}{n+2}-\frac{n+2}{n+3}} \\ &= \frac{2}{n+2}e^{-(\frac{n^2+2n-2}{(n+2)(n+3)})} \\ &< \frac{n+2}{n+3}e^{-1} = e^{-1}H(A,B) \\ &\leq (e^{-\frac{1}{2}})^2 M_{F,S}(A,B) \end{aligned}$$

Hence for any $A, B \in CB(X)$ that have an edge between A and B, (2.1) and (2.2) are satisfied. Thus all the conditions of Theorem 2.2 are satisfied. Moreover, F and S have a common fixed point and $Fix(F) \cap Fix(S)$ is complete in $CB(X)$.

An an application of Theorem 2.2, we now establish the following theorem for mappings satisfying the inequalities 2.1 and 2.2 on an $\varepsilon-$chainable metric space .

Theorem 2.5. Let (X,d) be a $\varepsilon-$chainable complete metric space for some $\varepsilon > 0$, and $F, S : CB(X) \to CB(X)$ be set mappings. Suppose that for all $A, B \in CB(X)$,

$$0 < H(A,B) < \varepsilon$$

and there exists some $\lambda \in (0,1)$ and $\Theta \in \Omega$ satisfying

$$\Theta(H(F(A), S(B))) \leq [\Theta(M_{F,S}(A,B))]^{\lambda}$$

where

$$\begin{aligned} M_{F,S}(A,B) &= \max\{H(A,B), H(A,F(A)), H(B,S(B)), \\ &\quad \frac{H(A,S(B)) + H(B,F(A))}{2}\}. \end{aligned}$$

If Θ or one of the pair (F,S) is continuous, then F and S have a common fixed point.

Proof. By Lemma 1.7, from $H(A,B) < \varepsilon$, for each $a \in A$, we have an element $b \in B$, such that $d(a,b) < \varepsilon$. If we consider the graph G as $V(G) = X$ and

$$E(G) = \{(a,b) \in X \times X : 0 < d(a,b) < \varepsilon\},$$

then the $\varepsilon-$chainability of (X,d) implies that G is connected. For $(A,B) \subset E(G)$ with $F(A) \neq S(B)$ we have from the hypothesis

$$\Theta(H(F(A), S(B))) \leq [\Theta(M_{F,S}(A,B))]^{\lambda}$$

where

$$\begin{aligned} M_{F,S}(A,B) &= \max\{H(A,B), H(A,F(A)), H(B,S(B)), \\ &\quad \frac{H(A,S(B)) + H(B,F(A))}{2}\} \end{aligned}$$

implies that pair (F,S) is graphic $\Theta-$contraction. Q.E.D.

Also, G has property (P^*). Indeed, if $\{X_n\}$ in $CB(X)$ with $X_n \to X$ and $H(X_n, X_{n+1}) < \varepsilon$ for $n \in \mathbb{N}$, then $H(X_{n_k}, X) < \varepsilon$ for sufficiently large $n \in \mathbb{N}$ and for subsequence $\{X_{n_k}\}$ of $\{X_n\}$. Therefore $(X_{n_k}, X) \subset E(G)$ for $n \in \mathbb{N}$. So by Theorem 2.2 (c), F and S have a common fixed point.

Consequently we have the following result:

Corollary 2.6. Let (X, d) be a complete metric space endowed with a directed graph G such that $V(G) = X$ and $\Delta \subseteq E(G)$. Suppose that the mapping $T : CB(X) \to CB(X)$ satisfies the following:

(a) For every V in $CB(X)$, $(V, T(V)) \subset E(G)$.

(b) There exists some $\lambda \in (0, 1)$ such that, for $\Theta \in \Omega$ the existence of an edge between A and B with $T(A) \neq T^2(B)$ implies that

$$\Theta(H\left(T(A), T^2(B)\right)) \leq [\Theta(M_T(A, B))]^{\lambda},$$

where

$$\begin{aligned} M_T(A, B) \quad = \quad & \max\{H(A, B), H(A, T(A)), H(B, T^2(B)), \\ & \frac{H(A, T^2(B)) + H(B, T(A))}{2}\}). \end{aligned}$$

Then following statements hold:

(a) If $Fix(T)$ is complete, then the Pompeiu-Hausdorff weight assigned to the $U, V \in Fix(T)$ is 0.

(b) If the weakly connected graph G satisfies the property (P^*), then T has a fixed point.

(c) $Fix(T)$ is complete if and only if $Fix(T)$ is a singleton.

Proof. Take $T = F$ and $S = T^2$ in (2.1) and (2.2), then Corollary 2.6 follows from Theorem 2.2.

Remark 2.7. (1) If $E(G) := X \times X$, then clearly G is connected and our Theorem 2.2 improves and generalizes Theorem 2.1 in [1], Theorem 2.1 in [10] and Theorem 3.1 in [20](2) Theorem 2.2

with the graph G improves, extends and generalizes Corollary 2.1 in [27] from single valued to two set mappings in the setting of complete metric spaces.

(3) If $E(G) := X \times X$, then clearly G is connected and our Theorem 2.5 extends and generalizes Theorem 2.5 in [10], Theorem 3.2 in [21], Theorem 5.1 in [13] and Theorem 3.1 in [20].

(4) If $E(G) := X \times X$, then clearly G is connected and our Corollary 2.6 improves and generalizes Theorem 3.2 in [21] and Theorem 3.1 in [20].

(5) If we take $S = T$ in graph Θ -contraction pair, then we obtain the fixed point results for graph $\Theta-$contraction of a single map.

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