# New inequalities for $n$-time differentiable functions

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**Abstract**

In this paper, we obtain several inequalities of Ostrowski type that the absolute values of n-time differentiable functions are convex. Also, some applications to special means of real numbers are provided.

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## 1 Introduction

In 1938 Ostrowski [14] obtained a bound for the absolute value of the difference of a function to its average over a finite interval. The theorem is as follows.

**Theorem 1.1.** Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on $[a, b]$ and let $|f'(t)| \leq M$ for all $t \in (a, b)$, then the following bound is valid

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq (b-a)M \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \tag{1.1}$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is sharp in the sence that it can not be replaced by a smaller one.

For applications of Ostrowski's inequality to some special means and some numerical quadrature rules, we refer the reader to the recent paper [9] by S.S. Dragomir and S. Wang who used integration by parts from $\int_a^b p(x,t)f'(t)dt$ to prove Ostrowski's inequality (1.1) where $p(x, t)$ is a peano kernel given by

$$p(x,t) = \begin{cases} t-a, & t \in [a,x] \\ t-b, & t \in (x,b]. \end{cases}$$

In [18], also A. Sofo and S.S Dragomir extended the result (1.1) in the $Lp$ norm.

Dragomir ([4]-[8]) further extended the result (1.1) to incorporate mappings of bounded variation, Lipschitzian and monotonic mappings.

Cerone *et al.* [2] as well as Dedić *et al.* [3] and Pearce *et al.* [15] further extended the result (1.1) by considering $n$-times differentiable mappings on an interior point $x \in [a, b]$. Furthermore, for recent results and generalizations concerning Ostrowski's inequality see [1], [10]-[13], [16] and [17].

In [2], Cerone, Dragomir and Roumeliotis proved the following results:

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**Lemma 1.2.** Let $f : [a, b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$. Then for all $x \in [a, b]$ we have the identity:

$$\begin{aligned}\int_a^b f(t)dt &= \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\\ &\quad +(-1)^n\int_a^b K_n(x,t)f^{(n)}(t)dt\end{aligned}$$

where the kernel $K_n : [a, b]^2 \to \mathbb{R}$ is given by

$$K_n(x,t)=\begin{cases}\frac{(t-a)^n}{n!} & if\ t\in[a,x]\\ \\ \frac{(t-b)^n}{n!} & if\ t\in(x,b],\end{cases}$$

$x \in [a, b]$ and $n$ natural number, $n \geq 1$.

**Corollary 1.3.** With the above assumptions, we have the representation:

$$\begin{aligned}\int_a^b f(t)dt &= \sum_{k=0}^{n-1}\left[\frac{1+(-1)^k}{(k+1)!}\right]\frac{(b-a)^{k+1}}{2^{k+1}}f^{(k)}\left(\frac{a+b}{2}\right)\\ &\quad +(-1)^n\int_a^b M_n(t)f^{(n)}(t)dt\end{aligned}$$

where

$$M_n(t)=\begin{cases}\frac{(t-a)^n}{n!} & if\ t\in\left[a,\frac{a+b}{2}\right]\\ \\ \frac{(t-b)^n}{n!} & if\ t\in\left(\frac{a+b}{2},b\right].\end{cases}$$

**Corollary 1.4.** With the above assumptions, we have the representation:

$$\begin{aligned}\int_a^b f(t)dt &= \sum_{k=0}^{n-1}\frac{(b-a)^{k+1}}{(k+1)!}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right]\\ &\quad +\int_a^b T_n(t)f^{(n)}(t)dt\end{aligned}$$

where

$$T_n(t)=\frac{1}{n!}\left[\frac{(b-t)^n+(-1)^n(t-a)^n}{2}\right],$$

$t \in [a, b]$.

In this paper, by using the some classical integral inequalities, Hölder and Power-Mean integral inequality, we establish some new inequalities for functions whose $n-th$ derivatives in absolute value are convex functions. Our established results generalize some of those results proved in recent papers for functions whose derivatives in absolute value are convex functions.

## 2 Main results

**Theorem 2.1.** For $n \geq 1$, let $f : [a,b] \to \mathbb{R}$ be $n-$time differentiable mapping and $a < b$. If $f^{(n)} \in L[a,b]$ and $\left|f^{(n)}\right|$ is convex on $[a,b]$, then

$$\begin{aligned} &\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \qquad (2.1)\\ \leq\ & \frac{1}{n!(b-a)}\left\{\left|f^{(n)}(a)\right|\left[\frac{(x-a)^{n+1}\left[(n+2)(b-x)+(x-a)\right]}{(n+1)(n+2)}+\frac{(b-x)^{n+2}}{(n+2)}\right]\right.\\ &\left.+\left|f^{(n)}(b)\right|\left[\frac{(b-x)^{n+1}\left[(n+2)(x-a)+(b-x)\right]}{(n+1)(n+2)}+\frac{(x-a)^{n+2}}{(n+2)}\right]\right\}. \end{aligned}$$

*Proof.* From Lemma 1.2 and using the properties of modulus, we write

$$\begin{aligned} &\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \qquad (2.2)\\ \leq\ & \left|\int_a^b K_n(x,t)f^{(n)}(t)dt\right|\\ =\ & \int_a^x \frac{(t-a)^n}{n!}\left|f^{(n)}(t)\right|dt+\int_x^b \frac{(b-t)^n}{n!}\left|f^{(n)}(t)\right|dt\\ =\ & \int_a^x \frac{(t-a)^n}{n!}\left|f^{(n)}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|dt\\ &+\int_x^b \frac{(b-t)^n}{n!}\left|f^{(n)}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|dt. \end{aligned}$$

Since $\left|f^{(n)}\right|$ is convex on $[a,b]$, we have

$$\begin{aligned} &\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|\\ \leq\ & \frac{1}{n!}\left\{\int_a^x (t-a)^n\left[\frac{b-t}{b-a}\left|f^{(n)}(a)\right|+\frac{t-a}{b-a}\left|f^{(n)}(b)\right|\right]dt\right.\\ &\left.+\int_x^b (b-t)^n\left[\frac{b-t}{b-a}\left|f^{(n)}(a)\right|+\frac{t-a}{b-a}\left|f^{(n)}(b)\right|\right]dt\right\}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \int_a^x (t-a)^n(b-t)dt &= \frac{(x-a)^{n+1}\left[(n+2)(b-x)+(x-a)\right]}{(n+1)(n+2)},\\ \int_a^x (t-a)^{n+1}dt &= \frac{(x-a)^{n+2}}{(n+2)},\\ \int_x^b (b-t)^{n+1}dt &= \frac{(b-x)^{n+2}}{(n+2)}, \end{aligned}$$

and

$$\int_x^b (b-t)^n (t-a)dt = \frac{(b-x)^{n+1}\left[(n+2)(x-a)+(b-x)\right]}{(n+1)(n+2)}.$$

This completes the proof. ∎

**Corollary 2.2.** With the above assumptions, if we choose $x = \frac{a+b}{2}$, then we get

$$\begin{aligned} &\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \left[\frac{1+(-1)^k}{(k+1)!}\right] \frac{(b-a)^{k+1}}{2^{k+1}} f^{(k)}\left(\frac{a+b}{2}\right) \right| \\ \leq\ & \frac{(b-a)^{n+1}}{2^n (n+1)!} \left[\frac{\left|f^{(n)}(a)\right| + \left|f^{(n)}(b)\right|}{2}\right]. \end{aligned}$$

**Corollary 2.3.** In Theorem 2.1, if we choose $x = a$ and $x = b$, respectively, we have

$$\begin{aligned} &\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} f^{(k)}(a) \right| \\ \leq\ & \frac{(b-a)^{n+1}}{(n+2)!} \left[(n+1)\left|f^{(n)}(a)\right| + \left|f^{(n)}(b)\right|\right] \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(-1)^k (b-a)^{k+1}}{(k+1)!} f^{(k)}(b) \right| \\ \leq\ & \frac{(b-a)^{n+1}}{(n+2)!} \left[\left|f^{(n)}(a)\right| + (n+1)\left|f^{(n)}(b)\right|\right]. \end{aligned} \tag{2.4}$$

**Corollary 2.4.** Let the conditions of Theorem 2.1 hold. Then the following result is valid. Namely,

$$\begin{aligned} &\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} \left[\frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2}\right] \right| \\ \leq\ & \frac{(b-a)^{n+1}}{(n+1)!} \left[\frac{\left|f^{(n)}(a)\right| + \left|f^{(n)}(b)\right|}{2}\right]. \end{aligned} \tag{2.5}$$

*Proof.* Summing the inequalities (2.3) and (2.4) and by using the triangle inequality, we have the inequality (2.5). ∎

**Corollary 2.5.** In Theorem 2.1, if we have $n = 1$, then

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\ \leq\ & \frac{1}{(b-a)^2} \left\{ \left[\frac{(x-a)^2\left[3(b-x)+(x-a)\right]}{6} + \frac{(b-x)^3}{3}\right] |f'(a)| \right. \\ & \left. + \left[\frac{(b-x)^2\left[3(x-a)+(b-x)\right]}{6} + \frac{(x-a)^3}{3}\right] |f'(b)| \right\}. \end{aligned}$$

**Theorem 2.6.** Let $f:[a,b]\to\mathbb{R}$ be $n-$time differentiable mapping and $a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^q$ is convex on $[a,b]$, then we have the following inequalities:

$$\begin{aligned}&\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \qquad (2.6)\\ \leq\ & \frac{1}{n!(b-a)^{\frac{1}{q}}}\left\{\frac{(x-a)^{np+1+\frac{1}{q}}}{np+1}\left[\frac{(2b-a-x)}{2}\left|f^{(n)}(a)\right|^q+\frac{(x-a)}{2}\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right.\\ &\left.+\frac{(b-x)^{np+1+\frac{1}{q}}}{np+1}\left[\frac{(b-x)}{2}\left|f^{(n)}(a)\right|^q+\frac{(b+x-2a)}{2}\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right\}\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* From Lemma 1.2, we have

$$\begin{aligned}&\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|\\ \leq\ &\left|\int_a^b K_n(x,t)f^{(n)}(t)dt\right|\\ =\ &\int_a^x\frac{(t-a)^n}{n!}\left|f^{(n)}(t)\right|dt+\int_x^b\frac{(b-t)^n}{n!}\left|f^{(n)}(t)\right|dt.\end{aligned}$$

By Hölder inequality, we obtain

$$\begin{aligned}&\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|\\ \leq\ &\frac{1}{n!}\left\{\left(\int_a^x(t-a)^{np}dt\right)^{\frac{1}{p}}\left(\int_a^x\left|f^{(n)}(t)\right|^q dt\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_x^b(b-t)^{np}dt\right)^{\frac{1}{p}}\left(\int_x^b\left|f^{(n)}(t)\right|^q dt\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Since $\left|f^{(n)}\right|^q$ is convex on $[a,b]$ and $t=\frac{b-t}{b-a}a+\frac{t-a}{b-a}b$, we have

$$\begin{aligned}&\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|\\ \leq\ &\frac{1}{n!}\left\{\left(\int_a^x(t-a)^{np}dt\right)^{\frac{1}{p}}\left(\int_a^x\left[\frac{b-t}{b-a}\left|f^{(n)}(a)\right|^q+\frac{t-a}{b-a}\left|f^{(n)}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_x^b(b-t)^{np}dt\right)^{\frac{1}{p}}\left(\int_x^b\left[\frac{b-t}{b-a}\left|f^{(n)}(a)\right|^q+\frac{t-a}{b-a}\left|f^{(n)}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\}\end{aligned}$$

$$= \frac{1}{n!}\left\{\frac{(x-a)^{np+1}}{np+1}\left[\frac{(x-a)(2b-a-x)}{2(b-a)}\left|f^{(n)}(a)\right|^q+\frac{(x-a)^2}{2(b-a)}\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\frac{(b-x)^{np+1}}{np+1}\left[\frac{(b-x)^2}{2(b-a)}\left|f^{(n)}(a)\right|^q+\frac{(b-x)(b+x-2a)}{2(b-a)}\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right\}.$$

This completes the proof. ∎

**Corollary 2.7.** Assume that $f$ is as in Teorem 2.6. If we choose $x=\frac{a+b}{2}$, then we have

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\left[\frac{1+(-1)^k}{(k+1)!}\right]\left(\frac{b-a}{2}\right)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \left(\frac{b-a}{2}\right)^{np+1+\frac{1}{q}}\frac{1}{(np+1)n!}$$
$$\times\left\{\left[\frac{3\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{\left|f^{(n)}(a)\right|^q+3\left|f^{(n)}(b)\right|^q}{4}\right]^{\frac{1}{q}}\right\}.$$

**Corollary 2.8.** With the above assumptions, if we choose $x=a$ and $x=b$, respectively, we have

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\frac{(b-a)^{k+1}}{(k+1)!}f^{(k)}(a)\right| \tag{2.7}$$
$$\leq \frac{(b-a)^{np+1+\frac{1}{q}}}{(np+1)n!}\left[\frac{\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q}{2}\right]^{\frac{1}{q}}$$

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\frac{(-1)^k(b-a)^{k+1}}{(k+1)!}f^{(k)}(b)\right| \tag{2.8}$$
$$\leq \frac{(b-a)^{np+1+\frac{1}{q}}}{(np+1)n!}\left[\frac{\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q}{2}\right]^{\frac{1}{q}}.$$

**Corollary 2.9.** Let the conditions of Theorem 2.6 hold. Then the following result is valid. Namely,

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\frac{(b-a)^{k+1}}{(k+1)!}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right]\right| \tag{2.9}$$
$$\leq \frac{(b-a)^{n+1}}{(n+1)!}\left[\frac{\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|}{2}\right].$$

*Proof.* Summing the inequalities (2.7) and (2.8) and by using the triangle inequality, we have the inequality (2.9). ∎

**Corollary 2.10.** In the inequalities (2.6), if we choose $n = 1$, then we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\
\leq\ & \frac{1}{(b-a)^{1+\frac{1}{q}}} \left\{ \frac{(x-a)^{p+1+\frac{1}{q}}}{p+1} \left[ \frac{(2b-a-x)}{2} \left|f'(a)\right|^q + \frac{(x-a)}{2} \left|f'(b)\right|^q \right]^{\frac{1}{q}} \right. \\
&\left. + \frac{(b-x)^{p+1+\frac{1}{q}}}{p+1} \left[ \frac{(b-x)}{2} \left|f'(a)\right|^q + \frac{(b+x-2a)}{2} \left|f'(b)\right|^q \right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

**Theorem 2.11.** Let $f : [a,b] \to \mathbb{R}$ be $n-$time differentiable mapping and $a < b$. If $f^{(n)} \in L[a,b]$ and $\left|f^{(n)}\right|^q$ is convex on $[a,b]$, $\frac{1}{p} + \frac{1}{q} = 1$, then we have

(2.10)

$$\begin{aligned}
&\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \left[ \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \\
\leq\ & \frac{1}{n!(b-a)^{\frac{1}{q}}(p+2)^{\frac{1}{q}}} \left( \frac{q-1}{nq+q-p-1} \right)^{1-\frac{1}{q}} \\
&\times \left\{ (x-a)^{n+1} \left[ \frac{(p+2)(b-x)+(x-a)}{(p+1)} \left|f^{(n)}(a)\right|^q + (x-a)^{p+1} \left|f^{(n)}(b)\right|^q \right]^{\frac{1}{q}} \right. \\
&\left. + (b-x)^{n+1} \left[ (b-x)^{p+1} \left|f^{(n)}(a)\right|^q + \frac{(p+2)(x-a)+(b-x)}{(p+1)} \left|f^{(n)}(b)\right|^q \right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

*Proof.* From Lemma 1.2 and using the properties of modulus, we have

$$\begin{aligned}
&\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \left[ \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \\
\leq\ & \left| \int_a^b K_n(x,t) f^{(n)}(t)dt \right| \\
=\ & \int_a^x \frac{(t-a)^n}{n!} \left| f^{(n)}(t) \right| dt + \int_x^b \frac{(b-t)^n}{n!} \left| f^{(n)}(t) \right| dt \\
=\ & \frac{1}{n!} \left\{ \int_a^x (t-a)^n \left| f^{(n)}(t) \right| dt + \int_x^b (b-t)^n \left| f^{(n)}(t) \right| dt \right\} \\
=\ & \frac{1}{n!} \left\{ \int_a^x \frac{(t-a)^n (t-a)^{\frac{p}{q}}}{(t-a)^{\frac{p}{q}}} \left| f^{(n)}(t) \right| dt + \int_x^b \frac{(b-t)^n (b-t)^{\frac{p}{q}}}{(b-t)^{\frac{p}{q}}} \left| f^{(n)}(t) \right| dt \right\}
\end{aligned}$$

By Hölder inequality, we obtain

$$\begin{aligned}
&\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \\
\leq\ & \frac{1}{n!}\left\{\left(\int_a^x\left[\frac{(t-a)^n}{(t-a)^{\frac{p}{q}}}\right]^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_a^x(t-a)^p\left|f^{(n)}(t)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_x^b\left[\frac{(b-t)^n}{(b-t)^{\frac{p}{q}}}\right]^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_x^b(b-t)^p\left|f^{(n)}(t)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Since $\left|f^{(n)}\right|^q$ is convex on $[a,b]$ and $t=\frac{b-t}{b-a}a+\frac{t-a}{b-a}b$, we have

$$\begin{aligned}
&\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \\
\leq\ & \frac{1}{n!}\left\{\left(\int_a^x(t-a)^{\frac{nq-p}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_a^x(t-a)^p\left[\frac{b-t}{b-a}\left|f^{(n)}(a)\right|^q+\frac{t-a}{b-a}\left|f^{(n)}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_x^b(b-t)^{\frac{nq-p}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_x^b(b-t)^p\left[\frac{b-t}{b-a}\left|f^{(n)}(a)\right|^q+\frac{t-a}{b-a}\left|f^{(n)}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\} \\
=\ & \frac{1}{n!(b-a)^{\frac{1}{q}}(p+2)^{\frac{1}{q}}}\left(\frac{q-1}{nq+q-p-1}\right)^{1-\frac{1}{q}} \\
&\times\left\{(x-a)^{n+1}\left[\frac{(p+2)(b-x)+(x-a)}{(p+1)}\left|f^{(n)}(a)\right|^q+(x-a)^{p+1}\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left.+(b-x)^{n+1}\left[(b-x)^{p+1}\left|f^{(n)}(a)\right|^q+\frac{(p+2)(x-a)+(b-x)}{(p+1)}\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

By using the fact that

$$\begin{aligned}
\int_a^x(t-a)^{\frac{nq-p}{q-1}}dt &= \frac{q-1}{nq+q-p-1}(x-a)^{\frac{nq+q-p-1}{q-1}}, \\
\int_x^b(b-t)^{\frac{nq-p}{q-1}}dt &= \frac{q-1}{nq+q-p-1}(b-x)^{\frac{nq+q-p-1}{q-1}}
\end{aligned}$$

we get the inequality (2.10), which completes the proof of the theorem. ∎

**Corollary 2.12.** Assume that $f$ is as in Teorem 2.11. If we choose $x = \frac{a+b}{2}$, then we have

$$\begin{aligned}
&\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \left[ \frac{1+(-1)^k}{(k+1)!} \right] \left( \frac{b-a}{2} \right)^{k+1} f^{(k)} \left( \frac{a+b}{2} \right) \right| \\
\leq\ & \frac{(b-a)^{n+1}}{n!2^{n+1+\frac{1}{q}}(p+2)^{\frac{1}{q}}} \left( \frac{q-1}{nq+q-p-1} \right)^{1-\frac{1}{q}} \\
&\times \left\{ \left[ \frac{p+3}{p+1} \left| f^{(n)}(a) \right|^q + \left( \frac{b-a}{2} \right)^p \left| f^{(n)}(b) \right|^q \right]^{\frac{1}{q}} \right. \\
&\left. + \left[ \left( \frac{b-a}{2} \right)^p \left| f^{(n)}(a) \right|^q + \frac{p+3}{p+1} \left| f^{(n)}(b) \right|^q \right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

**Corollary 2.13.** With the above assumptions, if we choose $x = a$ and $x = b$, respectively, we have

$$\begin{aligned}
&\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} f^{(k)}(a) \right| \qquad (2.11) \\
\leq\ & \frac{(b-a)^{n+1}}{n!(p+2)^{\frac{1}{q}}} \left( \frac{q-1}{nq+q-p-1} \right)^{1-\frac{1}{q}} \\
&\times \left[ (b-a)^p \left| f^{(n)}(a) \right|^q + \frac{1}{p+1} \left| f^{(n)}(b) \right|^q \right]^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
&\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(-1)^k (b-a)^{k+1}}{(k+1)!} f^{(k)}(b) \right| \qquad (2.12) \\
\leq\ & \frac{(b-a)^{n+1}}{n!(p+2)^{\frac{1}{q}}} \left( \frac{q-1}{nq+q-p-1} \right)^{1-\frac{1}{q}} \\
&\times \left[ \frac{1}{p+1} \left| f^{(n)}(a) \right|^q + (b-a)^p \left| f^{(n)}(b) \right|^q \right]^{\frac{1}{q}}.
\end{aligned}$$

**Corollary 2.14.** Let the conditions of Theorem 2.11 hold. Then the following result is valid.

Namely,

$$\begin{aligned}
&\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\frac{(b-a)^{k+1}}{(k+1)!}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right]\right| \qquad (2.13)\\
\leq\ & \frac{(b-a)^{n+1}}{n!(p+2)^{\frac{1}{q}}}\left(\frac{q-1}{nq+q-p-1}\right)^{1-\frac{1}{q}}\\
&\times\left\{\left[(b-a)^p\left|f^{(n)}(a)\right|^q+\frac{1}{p+1}\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right.\\
&\left.+\left[\frac{1}{p+1}\left|f^{(n)}(a)\right|^q+(b-a)^p\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

*Proof.* Summing the inequalities (2.11) and (2.12) and by using the triangle inequality, we have the inequality (2.13). ∎

**Corollary 2.15.** In the inequalities (2.10), if we choose $n=1$, then we have

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\\
\leq\ & \frac{1}{(b-a)^{\frac{1}{q}}(p+2)^{\frac{1}{q}}}\left(\frac{q-1}{2q-p-1}\right)^{1-\frac{1}{q}}\\
&\times\left\{(x-a)^2\left[\frac{(p+2)(b-x)+(x-a)}{(p+1)}\left|f'(a)\right|^q+(x-a)^{p+1}\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right.\\
&\left.+(b-x)^2\left[(b-x)^{p+1}\left|f'(a)\right|^q+\frac{(p+2)(x-a)+(b-x)}{(p+1)}\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

**Theorem 2.16.** For $n\geq 1$, let $f:[a,b]\to\mathbb{R}$ be $n-$time differentiable mapping and $a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^q$ is convex on $[a,b]$ and $q\geq 1$, then we have the following inequality:

$$\begin{aligned}
&\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \qquad (2.14)\\
\leq\ & \frac{1}{(n+1)!(b-a)^{\frac{1}{q}}(n+2)^{\frac{1}{q}}}\\
&\times\left\{(x-a)^{n+1}\left[[(n+2)(b-x)+(x-a)]\left|f^{(n)}(a)\right|^q+(n+1)(x-a)\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right.\\
&\left.+(b-x)^{n+1}\left[(n+1)(b-x)\left|f^{(n)}(a)\right|^q+[(n+2)(x-a)+(b-x)]\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

*Proof.* From Lemma 1.2 and using the properties of modulus, we obtain

$$\begin{aligned}
&\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \\
\leq\ &\left|\int_a^b K_n(x,t)f^{(n)}(t)dt\right| \\
=\ &\frac{1}{n!}\left\{\int_a^x (t-a)^n\left|f^{(n)}(t)\right|dt + \int_x^b (b-t)^n\left|f^{(n)}(t)\right|dt\right\}.
\end{aligned}$$

By Power-mean inequality, we obtain

$$\begin{aligned}
&\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \\
\leq\ &\frac{1}{n!}\left\{\left(\int_a^x (t-a)^n dt\right)^{1-\frac{1}{q}}\left(\int_a^x (t-a)^n\left|f^{(n)}(t)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_x^b (b-t)^n dt\right)^{1-\frac{1}{q}}\left(\int_x^b (b-t)^n\left|f^{(n)}(t)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Since $\left|f^{(n)}\right|^q$ is convex on $[a,b]$ and $t=\frac{b-t}{b-a}a+\frac{t-a}{b-a}b$, we have

$$\begin{aligned}
&\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \\
\leq\ &\frac{1}{n!}\left\{\left(\frac{(x-a)^{n+1}}{n+1}\right)^{1-\frac{1}{q}}\left(\int_a^x (t-a)^n\left[\frac{b-t}{b-a}\left|f^{(n)}(a)\right|^q+\frac{t-a}{b-a}\left|f^{(n)}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{(b-x)^{n+1}}{n+1}\right)^{1-\frac{1}{q}}\left(\int_x^b (b-t)^n\left[\frac{b-t}{b-a}\left|f^{(n)}(a)\right|^q+\frac{t-a}{b-a}\left|f^{(n)}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\} \\
=\ &\frac{1}{(n+1)!(b-a)^{\frac{1}{q}}(n+2)^{\frac{1}{q}}} \\
&\times\left\{(x-a)^{n+1}\left[[(n+2)(b-x)+(x-a)]\left|f^{(n)}(a)\right|^q+(n+1)(x-a)\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left.+(b-x)^{n+1}\left[(n+1)(b-x)\left|f^{(n)}(a)\right|^q+[(n+2)(x-a)+(b-x)]\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Hence the proof of the theorem is completed. ∎

**Corollary 2.17.** With the above assumptions, if we choose $x = \frac{a+b}{2}$, then we have

$$\begin{aligned}
&\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \left[ \frac{1+(-1)^k}{(k+1)!} \right] \left( \frac{b-a}{2} \right)^{k+1} f^{(k)} \left( \frac{a+b}{2} \right) \right| \\
\leq\ & \frac{(b-a)^{n+1}}{(n+1)!2^{n+1+\frac{1}{q}}(n+2)^{\frac{1}{q}}} \\
&\times \left\{ \left[ (n+3) \left| f^{(n)}(a) \right|^q + (n+1) \left| f^{(n)}(b) \right|^q \right]^{\frac{1}{q}} \right. \\
&\left. + \left[ (n+1) \left| f^{(n)}(a) \right|^q + (n+3) \left| f^{(n)}(b) \right|^q \right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

**Corollary 2.18.** In Theorem 2.16, if we choose $x = a$ and $x = b$, respectively, we have

$$\begin{aligned}
&\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} f^{(k)}(a) \right| \qquad (2.15) \\
\leq\ & \frac{(b-a)^{n+1}}{(n+1)!(n+2)^{\frac{1}{q}}} \left[ (n+1) \left| f^{(n)}(a) \right|^q + \left| f^{(n)}(b) \right|^q \right]^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
&\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(-1)^k(b-a)^{k+1}}{(k+1)!} f^{(k)}(b) \right| \qquad (2.16) \\
\leq\ & \frac{(b-a)^{n+1}}{(n+1)!(n+2)^{\frac{1}{q}}} \left[ \left| f^{(n)}(a) \right|^q + (n+1) \left| f^{(n)}(b) \right|^q \right]^{\frac{1}{q}}.
\end{aligned}$$

**Corollary 2.19.** Let the conditions of Theorem 2.16 hold. Then the following result is valid:

$$\begin{aligned}
&\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} \left[ \frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2} \right] \right| \qquad (2.17) \\
\leq\ & \frac{(b-a)^{n+1}}{2(n+1)!(n+2)^{\frac{1}{q}}} \\
&\times \left\{ \left[ (n+1) \left| f^{(n)}(a) \right|^q + \left| f^{(n)}(b) \right|^q \right]^{\frac{1}{q}} \right. \\
&\left. + \left[ \left| f^{(n)}(a) \right|^q + (n+1) \left| f^{(n)}(b) \right|^q \right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

*Proof.* Summing the inequalities (2.15) and (2.16) and by using the triangle inequality, we have the inequality (2.17). ∎

**Corollary 2.20.** In the inequalities (2.10), if we choose $n = 1$, then we have

$$\begin{aligned}&\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \\ \leq\ & \frac{1}{2(b-a)^{\frac{1}{q}}}\left\{(x-a)^2\left[\frac{(3b-2x-a)}{3}\left|f'(a)\right|^q + \frac{2(x-a)}{3}\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right. \\ &\left.+(b-x)^2\left[\frac{2(b-x)}{3}\left|f'(a)\right|^q + \frac{(b+2x-3a)}{3}\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

## 3 Applications to special means

We now consider the means for arbitrary real numbers $\alpha, \beta$ $(\alpha \neq \beta)$. We take

1. *Arithmetic mean* :
$$A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \quad \alpha,\beta \in \mathbb{R}^+.$$
2. *Logarithmic mean*:
$$L(\alpha,\beta) = \frac{\alpha-\beta}{\ln|\alpha| - \ln|\beta|}, \quad |\alpha| \neq |\beta|\,,\ \alpha,\beta \neq 0,\ \alpha,\beta \in \mathbb{R}^+.$$

Now using the results of Section 2, we give some applications for special means of real numbers.

**Proposition 3.1.** Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{Z}$, $|n| \geq 1$, then, the following inequality holds:

$$\begin{aligned}\left|L_n^n(a,b) - x^n\right| \ \leq\ & \frac{|n|}{(b-a)^2}\left\{\frac{\left[(x-a)^2(3b-a-2x) + 2(b-x)^3\right].a^{n-1}}{6}\right. \\ &\left.+\frac{\left[(b-x)^2(b-3a+2x) + 2(x-a)^3\right].b^{n-1}}{6}\right\}.\end{aligned}$$

*Proof.* The proof is obvious from Corollary 2.5 applied to the convex mapping $f(x) = x^n$, $x \in [a,b]$, $n \in \mathbb{Z}$. ∎

**Proposition 3.2.** Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{Z}$, $|n| \geq 1$, then, for all $q \geq 1$, the following inequality holds:

$$\begin{aligned}\left|L_n^n(a,b) - x^n\right| \ \leq\ & \frac{|n|}{2(b-a)^{\frac{1}{q}}}\left\{(x-a)^2\left[\frac{(3b-2x-a)\left(a^{n-1}\right)^q + 2(x-a)\left(b^{n-1}\right)^q}{3}\right]^{\frac{1}{q}}\right. \\ &\left.+(b-x)^2\left[\frac{2(b-x)\left(a^{n-1}\right)^q + (b+2x-3a)\left(b^{n-1}\right)^q}{3}\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

*Proof.* The proof is obvious from Corollary 2.20 applied to the convex mapping $f(x) = x^n$, $x \in [a,b]$, $n \in \mathbb{Z}$. ∎

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