Some Qi-type integral inequalities involving several weight functions

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Abstract

We prove some integral inequalities related to Feng Qi's inequality from [*Several integral inequalities*, J. Inequal. Pure Appl. Math. **1** (2000), No. 2, 7p., Article No. 19] and obtain a few corollaries.

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1 Introduction

In the paper [24], Feng Qi proved the following integral inequality: if $f : [a, b] \to \mathbb{R}$ is n-times continuously differentiable such that $f^{(i)}(a) \geq 0$ for $i = 0, \ldots, n-1$ and $f^{(n)}(x) \geq n!$ for all $x \in [a, b]$, then

$$\left(\int_a^b f(t)\,\mathrm{d}t \right)^{n+1} \leq \int_a^b f^{n+2}(t)\,\mathrm{d}t \tag{1.1}$$

In the following years, many variants and generalizations of this inequality have been investigated (in particular, versions for real exponents). For a detailed account of the results that have already been established, the reader is referred to the list of references at the end of this paper and further references therein.

Here we will prove some further inequalities related to Feng Qi's inequality and obtain a few corollaries which might be interesting. The basic methods of proof are the same as in [24] (exploiting the connection between a function's monotonicity behavior and the sign of its derivative, finite induction for a suitably defined set of auxiliary functions, etc.).

Since we will also be dealing with one-sided derivatives, let us introduce some notations and recall some facts, which will be used later without further mention.

For a given interval $[a, b]$ we denote by $D_+[a, b]$ (resp. $D_-[a, b]$) the set of all continuous functions $h : [a, b] \to \mathbb{R}$ such that the right-derivative $h'_+(x)$ (resp. the left-derivative $h'_-(x)$) exists for all $x \in (a, b)$.

The usual sum and product rules also apply for one-sided derivatives. Furthermore, a function $h \in D_+[a, b]$ is increasing (decreasing) if and only if $h'_+(x) \geq 0$ ($h'_+(x) \leq 0$) for all $x \in (a, b)$ and an analogous statement holds for functions in $D_-[a, b]$ (see [31], p.358).

Finally, if $h \in D_+[a, b]$, I is an interval which contains the range of h and $f : I \to \mathbb{R}$ is differentiable, then $f \circ h \in D_+[a, b]$ with $(f \circ h)'_+(x) = f'(h(x))h'_+(x)$ for all $x \in (a, b)$ (and analogously for functions in $D_-[a, b]$). This is proved in the same manner as the usual chain-rule.

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2 A generalization of Qi's inequality

The first result is a generalization of (1.1) to a setting where several weight functions and an additional exponent α are involved.

Proposition 2.1. Let $n \in \mathbb{N}$ with $n \geq 2$ and let $g : [a,b] \to \mathbb{R}$ be a continuous function with $g \geq 0$. Further, let $h_1, \dots, h_n \in D_+[a,b]$ or $h_1, \dots, h_n \in D_-[a,b]$ with $h_i \geq 0$ for each i. Let also $\alpha \leq \frac{n}{n-1}$ and $f : [a,b] \to \mathbb{R}$ be a strictly positive, $(n-1)$-times differentiable function such that $f^{(i)} \geq 0$ for all $i = 1, \dots, n-1$ and $(n+1-\alpha) f^{(n-1)} f^{n(1-\alpha)} \geq n! h_1 h_2^2 \dots h_n^n$.

Suppose that there exists a partition of $\{1, \dots, n\}$ into two disjoint subsets I and J such that the following conditions hold:

(i) $g \prod_{i \in I} h_i$ is increasing,

(ii) $\left(\prod_{i \in I} h_i^i\right)\left(\prod_{j \in \underline{J}_k} h_j^{j-k}\right)$ is decreasing for all $k = 1, \dots, n-1$, where $\underline{J}_k$ denotes the set $\{j \in J : j \geq k+1\}$.

(iii) $\prod_{j \in \overline{J}_k} h_j^{k+1-j}$ is increasing for $k = 0, \dots, n-2$, where $\overline{J}_k$ denotes the set $\{j \in J : j \leq k+1\}$.

Let $M := \inf\{g(x) : x \in [a,b]\}$ and $m_i := \inf\{h_i(x) : x \in [a,b]\}$ for $i = 1, \dots, n$. Then we have

$$(b-a) K M^n \prod_{i=1}^n m_i^{n-i} + \left(\int_a^b f^\alpha(t) g(t) \prod_{i=1}^n h_i(t) \, \mathrm{d}t \right)^n \leq \int_a^b f^{n+1}(t) g^n(t) \prod_{i=1}^n h_i^{n-i}(t) \, \mathrm{d}t,$$

where $K = f^{n+1}(a)$ if $\alpha \geq 0$ and $K = f^{n+1-\alpha}(a) f^\alpha(b)$ if $\alpha < 0$.

Note that the conditions (ii) and (iii) are satisfied in particular if $h_1 h_2^2 \dots h_n^n$ is decreasing and h_j is increasing and strictly positive for each $j \in J$.

Note further that the above inequality not only generalizes Qi's inequality, but it also sharpens the trivial estimate

$$(b-a) K M^n \prod_{i=1}^n m_i^{n-i} \leq \int_a^b f^{n+1}(t) g^n(t) \prod_{i=1}^n h_i^{n-i}(t) \, \mathrm{d}t.$$

Proof. Let $h_1, \dots, h_n \in D_+[a,b]$ (the case of left-derivatives is treated analogously).

We first define the function F by setting

$$F(x) := \left(\int_a^x f^\alpha(t) g(t) \prod_{i=1}^n h_i(t) \, \mathrm{d}t \right)^n - \int_a^x f^{n+1}(t) g^n(t) \prod_{i=1}^n h_i^{n-i}(t) \, \mathrm{d}t \quad \forall x \in [a,b].$$

Then F is differentiable with

$$F'(x) = n \left(\int_a^x f^\alpha(t) g(t) \prod_{i=1}^n h_i(t) \, \mathrm{d}t \right)^{n-1} f^\alpha(x) g(x) \prod_{i=1}^n h_i(x) - f^{n+1}(x) g^n(x) \prod_{i=1}^n h_i^{n-i}(x). \tag{2.1}$$

Next we define

$$G(x) := n\left(\int_a^x f^\alpha(t)\prod_{j\in J} h_j(t)\,\mathrm{d}t\right)^{n-1}\left(\prod_{i\in I} h_i^i(x)\right)\left(\prod_{j\in J} h_j(x)\right) - f^{n+1-\alpha}(x)\prod_{j\in J} h_j^{n-j}(x)$$

and claim that

$$F'(x) \le f^\alpha(x)g^n(x)G(x)\prod_{i\in I} h_i^{n-i}(x) \quad \forall x\in[a,b]. \tag{2.2}$$

To see this, note that assumption (i) implies

$$\int_a^x f^\alpha(t)g(t)\prod_{i=1}^n h_i(t)\,\mathrm{d}t \le g(x)\left(\prod_{i\in I} h_i(x)\right)\int_a^x f^\alpha(t)\prod_{j\in J} h_j(t)\,\mathrm{d}t.$$

Combining this with (2.1) we obtain

$$\begin{aligned} F'(x) &\le ng^n(x)f^\alpha(x)\left(\int_a^x f^\alpha(t)\prod_{j\in J} h_j(t)\,\mathrm{d}t\right)^{n-1}\left(\prod_{i\in I} h_i^{n-1}(x)\right)\left(\prod_{i=1}^n h_i(x)\right)\\ &\quad - f^{n+1}(x)g^n(x)\prod_{i=1}^n h_i^{n-i}(x) \end{aligned}$$

which can be easily simplified to (2.2).

We denote by φ the characteristic function of J in $\{1,\dots,n\}$, i. e. $\varphi(i)=1$ for $i\in J$ and $\varphi(i)=0$ for $i\in I$, and for each $k\in\{1,\dots,n-1\}$ we define

$$\begin{aligned} H_k(x) &:= \left(\prod_{i=k}^n i\right)\left(\int_a^x f^\alpha(t)\prod_{j\in J} h_j(t)\,\mathrm{d}t\right)^{k-1}\left(\prod_{i\in I} h_i^i(x)\right)\left(\prod_{i=k}^n h_i^{\varphi(i)(i-k+1)}(x)\right)\\ &\quad - (n+1-\alpha)f^{n(1-\alpha)+(k-1)\alpha}(x)f^{(n-k)}(x)\prod_{i=1}^{k-1} h_i^{(k-i-1)\varphi(i)}(x) \quad \text{for all } x\in[a,b]. \end{aligned}$$

We will show inductively that $H_k\le 0$ for all $k\in\{1,\dots,n-1\}$. For $k=1$ we have $H_1 = n!\prod_{i=1}^n h_i^i - (n+1-\alpha)f^{(n-1)}f^{n(1-\alpha)}$, which is negative by assumption.

Now suppose that $k\in\{1,\dots,n-2\}$ and $H_k\le 0$. The function H_{k+1} belongs to $D_+[a,b]$ and

satisfies

$$
\begin{aligned}
&(H_{k+1})'_+(x) = \\
&f^\alpha(x)\left(\prod_{i=k}^{n} i\right)\left(\int_a^x f^\alpha(t)\prod_{j\in J} h_j(t)\,\mathrm{d}t\right)^{k-1}\left(\prod_{j\in J} h_j(x)\right)\left(\prod_{i\in I} h_i^i(x)\right)\left(\prod_{i=k+1}^{n} h_i^{\varphi(i)(i-k)}(x)\right)\\
&+\left(\prod_{i=k+1}^{n} i\right)\left(\int_a^x f^\alpha(t)\prod_{j\in J} h_j(t)\,\mathrm{d}t\right)^{k}\left(\left(\prod_{i\in I} h_i^i\right)\left(\prod_{i=k+1}^{n} h_i^{\varphi(i)(i-k)}\right)\right)'_+(x)\\
&-(n+1-\alpha)\Bigg((n(1-\alpha)+k\alpha)f^{n(1-\alpha)+k\alpha-1}(x)f'(x)f^{(n-k-1)}(x)\prod_{i=1}^{k} h_i^{\varphi(i)(k-i)}(x)\\
&+f^{n(1-\alpha)+k\alpha}(x)f^{(n-k)}(x)\prod_{i=1}^{k} h_i^{\varphi(i)(k-i)}(x)\\
&+f^{n(1-\alpha)+k\alpha}(x)f^{(n-k-1)}(x)\left(\prod_{i=1}^{k} h_i^{\varphi(i)(k-i)}\right)'_+(x)\Bigg)
\end{aligned}
$$

for each $x \in (a,b)$. By our assumption on f we have $f' \geq 0$ and $f^{(n-k-1)} \geq 0$. We also have $\left(\prod_{i=1}^{k} h_i^{\varphi(i)(k-i)}\right)'_+ \geq 0$ because of (iii) and

$$
\left(\left(\prod_{i\in I} h_i^i\right)\left(\prod_{i=k+1}^{n} h_i^{\varphi(i)(i-k)}\right)\right)'_+ \leq 0
$$

because of (ii).

Furthermore, the assumption $\alpha \leq \frac{n}{n-1}$ ensures that $n+1 > \alpha$ and $n(1-\alpha)+k\alpha \geq 0$. It follows that, for all $x \in (a,b)$,

$$
\begin{aligned}
&(H_{k+1})'_+(x) \leq \\
&f^\alpha(x)\left(\prod_{i=k}^{n} i\right)\left(\int_a^x f^\alpha(t)\prod_{j\in J} h_j(t)\,\mathrm{d}t\right)^{k-1}\left(\prod_{j\in J} h_j(x)\right)\left(\prod_{i\in I} h_i^i(x)\right)\left(\prod_{i=k+1}^{n} h_i^{\varphi(i)(i-k)}(x)\right)\\
&-(n+1-\alpha)f^{n(1-\alpha)+k\alpha}(x)f^{(n-k)}(x)\prod_{i=1}^{k} h_i^{\varphi(i)(k-i)}(x)\\
&=f^\alpha(x)H_k(x)\prod_{i=1}^{k-1} h_i^{\varphi(i)}(x) \leq 0.
\end{aligned}
$$

Thus H_{k+1} is decreasing and hence $H_{k+1}(x) \leq H_{k+1}(a) \leq 0$ for all $x \in [a,b]$, which finishes the induction.

Now we define

$$H(x) := nh_n^{\varphi(n)}(x)\left(\int_a^x f^\alpha(t)\prod_{j\in J}h_j(t)\,\mathrm{d}t\right)^{n-1}\left(\prod_{i\in I}h_i^i(x)\right)$$
$$- f^{n+1-\alpha}(x)\prod_{i=1}^{n-1}h_i^{\varphi(i)(n-i-1)}(x).$$

Then $H \in D_+[a,b]$ and similar to the induction step above one can show that

$$H'_+(x) \le f^\alpha(x)H_{n-1}(x)\prod_{i=1}^{n-2}h_i^{\varphi(i)}(x) \quad \forall x \in (a,b).$$

Since $H_{n-1} \le 0$ it follows that H is decreasing and hence $H(x) \le H(a) \le 0$ for every $x \in [a,b]$. Also, from the definition G one can easily see that $G = H\prod_{i=1}^{n-1}h_i^{\varphi(i)}$. Together with (2.2) we obtain

$$F'(x) \le f^\alpha(x)H(a)M^n\left(\prod_{i\in I}m_i^{n-i}\right)\left(\prod_{i=1}^{n-1}m_i^{\varphi(i)}\right) \quad \forall x \in [a,b]$$

Furthermore, it is easily checked that

$$H(a)\prod_{i=1}^{n-1}m_i^{\varphi(i)} \le -f^{n+1-\alpha}(a)\prod_{i\in J}m_i^{n-i}.$$

Also, f is increasing (since $f' \ge 0$) and hence f^α is increasing for $\alpha \ge 0$ and decreasing for $\alpha < 0$. Thus we get

$$F'(x) \le -KM^n\prod_{i=1}^{n}m_i^{n-i} \quad \forall x \in [a,b].$$

By the mean value theorem this implies

$$F(b) = F(b) - F(a) \le -(b-a)KM^n\prod_{i=1}^{n}m_i^{n-i},$$

which is the desired inequality. Q.E.D.

Let us explicitly note the following special case of Proposition 2.1.

Corollary 2.2. Let g,h,p be nonnegative functions on $[a,b]$ such that $h,p \in D_+[a,b]$ or $h,p \in D_-[a,b]$, g is continuous, p is decreasing and h and gp are increasing.

Suppose further that $n \in \mathbb{N}$, $n \ge 2$, and f is a strictly positive, $(n-1)$-times differentiable function on $[a,b]$ with $f^{(i)} \ge 0$ for $i = 1,\dots,n-1$ and $(n+1-\alpha)f^{(n-1)}f^{n(1-\alpha)} \ge n!hp^\nu$ for some $\nu \in \{2,\dots,n\}$ and $\alpha \le \frac{n}{n-1}$. Let $M := \inf\{g(x) : x \in [a,b]\}$. Then we have

$$(b-a)KM^nh^{n-1}(a)p^{n-\nu}(b) + \left(\int_a^b f^\alpha(t)g(t)h(t)p(t)\,\mathrm{d}t\right)^n$$
$$\le \int_a^b f^{n+1}(t)g^n(t)h^{n-1}(t)p^{n-\nu}(t)\,\mathrm{d}t,$$

where $K = f^{n+1}(a)$ if $\alpha \geq 0$ and $K = f^{n+1-\alpha}(a)f^{\alpha}(b)$ if $\alpha < 0$.

Proof. Put $h_1 := h$, $h_\nu := p$ and $h_i := 1$ for $i \in \{2, \dots, n\}\backslash\{\nu\}$ as well as $I := \{\nu\}, J := \{1, \dots, n\}\backslash I$ and apply Proposition 2.1. Q.E.D.

This yields in particular the following Corollaries.

Corollary 2.3. Let $\alpha \leq \frac{n}{n-1}$. If f is a strictly positive, $(n-1)$-times differentiable function on $[a, b]$ (where $n \geq 2$) such that $f^{(i)} \geq 0$ for $i = 1, \dots, n-1$, then

$$(b-a)Kg^n(a) + (n+1-\alpha)\frac{A}{n!}\left(\int_a^b f^{\alpha}(t)g(t)\,\mathrm{d}t\right)^n \leq \int_a^b f^{n+1}(t)g^n(t)\,\mathrm{d}t$$

holds for every nonnegative function g on $[a, b]$ which is continuous and increasing, where $A := \inf\{f^{(n-1)}(x)f^{n(1-\alpha)}(x) : x \in [a, b]\}$ and K is defined as before.

Proof. Put $c := ((n+1-\alpha)A/n!)^{-1/(n(1-\alpha)+1)}$ and apply Corollary 2.2 to the function cf (with $h = p = 1$). Q.E.D.

Corollary 2.4. Let f be a strictly positive function on $[a, b]$ which is n-times differentiable ($n \geq 2$) with $f^{(i)} \geq 0$ for all $i = 1, \dots, n$. Let g be a nonnegative function on $[a, b]$ which is continuous and increasing and let $\alpha \leq \frac{n}{n-1}$. Then we have

$$\begin{aligned}&(b-a)Kg^n(a)(f^{(n-1)}(a))^{n-1} + (n+1-\alpha)\frac{C^{n(1-\alpha)}}{n!}\left(\int_a^b f^{\alpha}(t)g(t)f^{(n-1)}(t)\,\mathrm{d}t\right)^n\\ &\leq \int_a^b f^{n+1}(t)g^n(t)(f^{(n-1)}(t))^{n-1}\,\mathrm{d}t,\end{aligned}$$

where K is defined as above and $C = f(a)$ if $\alpha \leq 1$, $C = f(b)$ if $\alpha > 1$.

Proof. Set $p := 1$, $h := (n+1-\alpha)C^{n(1-\alpha)}f^{(n-1)}/n!$ and $\nu := 2$ and apply Corollary 2.2. Q.E.D.

Corollary 2.5. Let f be a strictly positive, n-times differentiable function on $[a, b]$ ($n \geq 2$) satisfying $f^{(i)} \geq 0$ for $i = 1, \dots, n-2$, $f^{(n-1)}(x) > 0$ for all $x \in [a, b]$, and $f^{(n)} \leq 0$. Let $h \in D_+[a, b]$ or $h \in D_-[a, b]$ be increasing with $0 \leq h \leq 1$ and let $\alpha \leq \frac{n}{n-1}$. Then we have

$$\begin{aligned}&(b-a)K\frac{h^{n-1}(a)}{f^{(n-1)}(a)} + (n+1-\alpha)\frac{C^{n(1-\alpha)}}{n!}\left(\int_a^b f^{\alpha}(t)h(t)\,\mathrm{d}t\right)^n\\ &\leq \int_a^b \frac{f^{n+1}(t)h^{n-1}(t)}{f^{(n-1)}(t)}\,\mathrm{d}t,\end{aligned}$$

where K and C are defined as in the previous Corollary.

Proof. Put $p(t) := ((n+1-\alpha)C^{n(1-\alpha)}f^{(n-1)}(t)/n!)^{1/n}$ for $t \in [a, b]$, $g := 1/p$ and $\nu := n$ and apply Corollary 2.2. Q.E.D.

Corollary 2.6. Let f be a strictly positive, n-times differentiable function on $[a,b]$ ($n \geq 2$) satisfying $f^{(i)} \geq 0$ for $i = 1, \ldots, n$. Let g be a continuous, increasing function on $[a,b]$ with $g \geq 0$ and let $\alpha \in (-\infty, 1]$. Put $\beta := n(1-\alpha)+\alpha$. Then we have

$$(b-a)Lg^n(a)(f^{(n-1)}(a))^{n-1} + \frac{n+1-\alpha}{n!}\left(\int_a^b f^\beta(t)g(t)f^{(n-1)}(t)\,\mathrm{d}t\right)^n$$
$$\leq \int_a^b f^{n\beta+1}(t)g^n(t)(f^{(n-1)}(t))^{n-1}\,\mathrm{d}t,$$

where $L = f^{n\beta+1}(a)$ for $\alpha \geq 0$ and $L = f^{n\beta+1-\alpha}(a)f^\alpha(b)$ for $\alpha < 0$.

Proof. Put $h := (n+1-\alpha)f^{(n-1)}f^{n(1-\alpha)}/n!$, $p := 1$ and $\nu := 2$ and apply Corollary 2.2. Q.E.D.

3 An integral inequality for 1/f

Proposition 2.1 already provides a Qi-type integral inequality for $1/f$ (consider the case $\alpha = -1$). The next result is another inequality in the spirit of (1.1) for $1/f$.

Proposition 3.1. Let $f : [a,b] \to \mathbb{R}$ be twice differentiable such that $f', f'' \geq 0$ and $f(t) > 0$ for all $t \in [a,b]$. Let further g be an increasing, continuous function on $[a,b]$ with $g \geq 0$ and $h \in D_+[a,b]$ or $h \in D_-[a,b]$ be an increasing, nonnegative function.

If $n \in \mathbb{N}$ and $f^{n+1}(a)(f'(a))^n h(a) \geq n!/(n+1)^{n-1}$, then

$$(b-a)\frac{f^{n+1}(a)g^{n+1}(a)h(a)}{f(b)} + \left(\int_a^b \frac{g(t)}{f(t)}\,\mathrm{d}t\right)^{n+1} \leq \int_a^b f^n(t)g^{n+1}(t)h(t)\,\mathrm{d}t.$$

Note that the conditions $f' \geq 0$ and $f'' \geq 0$ just mean that f is increasing and convex.

Proof. Let $h \in D_+[a,b]$ (the other case is completely analogous). We define

$$F(x) := \left(\int_a^x \frac{g(t)}{f(t)}\,\mathrm{d}t\right)^{n+1} - \int_a^x f^n(t)g^{n+1}(t)h(t)\,\mathrm{d}t \quad \forall x \in [a,b]$$

and

$$G(x) := (n+1)\left(\int_a^x \frac{1}{f(t)}\,\mathrm{d}t\right)^n - f^{n+1}(x)h(x) \quad \forall x \in [a,b]$$

and claim that

$$F' \leq \frac{g^{n+1}}{f}G. \tag{3.1}$$

To see this first note that

$$F'(x) = \frac{g(x)}{f(x)}\left((n+1)\left(\int_a^x \frac{g(t)}{f(t)}\,\mathrm{d}t\right)^n - f^{n+1}(x)g^n(x)h(x)\right). \tag{3.2}$$

Since g is increasing we have

$$\int_a^x \frac{g}{f} \leq g(x)\int_a^x \frac{1}{f}.$$

Combining this with (3.2) gives (3.1).

We further have $G \in D_+[a,b]$ and

$$G'_+(x) = \frac{n(n+1)}{f(x)}\left(\int_a^x \frac{1}{f(t)}\,\mathrm{d}t\right)^{n-1} - (n+1)f^n(x)f'(x)h(x) - f^{n+1}(x)h'_+(x).$$

Since h is increasing we have $h'_+ \geq 0$ and hence

$$G'_+(x) \leq \frac{n+1}{f(x)}\left(n\left(\int_a^x \frac{1}{f(t)}\,\mathrm{d}t\right)^{n-1} - f^{n+1}(x)f'(x)h(x)\right). \tag{3.3}$$

Next we define functions $H_1, \ldots, H_n$ on $[a,b]$ by

$$H_k(x) := \left(\prod_{i=k}^{n} i\right)\left(\int_a^x \frac{1}{f(t)}\,\mathrm{d}t\right)^{k-1} - (n+1)^{n-k}f^{n+1}(x)(f')^{n-k+1}(x)h(x).$$

We will show inductively that $H_k \leq 0$ for all $k = 1, \ldots, n$. First note that $H_1 = n! - (n+1)^{n-1}f^{n+1}(f')^n h$ and by our assumptions h, f and f' are increasing functions, thus H_1 is decreasing. Hence $H_1 \leq H_1(a)$ and again by assumption we have $H_1(a) \leq 0$.

Now suppose that $1 \leq k < n$ and $H_k \leq 0$. We have $H_{k+1} \in D_+[a,b]$ and

$$\begin{aligned}(H_{k+1})'_+(x) &= \frac{1}{f(x)}\left(\prod_{i=k}^{n} i\right)\left(\int_a^x \frac{1}{f(t)}\,\mathrm{d}t\right)^{k-1} \\ &- (n+1)^{n-k}f^n(x)(f')^{n-k+1}(x)h(x) - (n+1)^{n-k-1}f^{n+1}(x)((f')^{n-k}h)'_+(x).\end{aligned}$$

Since f' and h are increasing, it follows that

$$(H_{k+1})'_+(x) \leq \frac{H_k(x)}{f(x)} \leq 0 \quad \forall x \in (a,b)$$

and hence $H_{k+1} \leq H_{k+1}(a) \leq 0$.

So in particular $H_n \leq 0$ and from (3.3) it follows that $G'_+ \leq (n+1)H_n/f$. Thus $G'_+ \leq 0$ and consequently, $G \leq G(a) = -f^{n+1}(a)h(a)$.

Using (3.1) and the fact that g is increasing and $1/f$ decreasing, we obtain the inequality $F' \leq -f^{n+1}(a)g^{n+1}(a)h(a)/f(b)$. The mean value theorem now implies $F(b) = F(b) - F(a) \leq -(b-a)f^{n+1}(a)g^{n+1}(a)h(a)/f(b)$, which is equivalent to the desired inequality. Q.E.D.

Let us now collect some corollaries to the above result.

Corollary 3.2. Let f be a twice differentiable, strictly positive function on $[a,b]$ such that $f'(a) > 0$ and $f'' \geq 0$ and let g be a nonnegative function on $[a,b]$ which is continuous and increasing. Then we have for each $n \in \mathbb{N}$

$$\begin{aligned}&(b-a)\frac{g^{n+1}(a)}{f(b)} + (f'(a))^n \frac{(n+1)^{n-1}}{n!}\left(\int_a^b \frac{g(t)}{f(t)}\,\mathrm{d}t\right)^{n+1} \\ &\leq \frac{1}{f^{n+1}(a)}\int_a^b f^n(t)g^{n+1}(t)\,\mathrm{d}t.\end{aligned}$$

Proof. For a given $n \in \mathbb{N}$, put $c := (n!((n+1)^{n-1} f^{n+1}(a)(f'(a))^n)^{-1})^{1/(2n+1)}$ and apply Proposition 3.1 to the functions cf and g (and $h := 1$). Q.E.D.

Corollary 3.3. If f and g are as in Corollary 3.2, then

$$\int_a^b \frac{g}{f} \le \frac{1}{ef(a)f'(a)} \liminf_{n\to\infty} \left(\int_a^b f^n(t) g^{n+1}(t)\,\mathrm{d}t \right)^{1/(n+1)}$$
$$\le \frac{1}{ef(a)f'(a)} \limsup_{n\to\infty} \left(\int_a^b f^n(t) g^{n+1}(t)\,\mathrm{d}t \right)^{1/(n+1)} \le \frac{f(b)g(b)}{ef(a)f'(a)}.$$

Proof. Since f and g are increasing, we have

$$\left(\int_a^b f^n(t) g^{n+1}(t)\,\mathrm{d}t \right)^{1/(n+1)} \le (b-a)^{1/(n+1)} (f(b))^{n/(n+1)} g(b) \quad \forall n \in \mathbb{N}.$$

The righthand side of this inequality converges to $f(b)g(b)$ for $n \to \infty$, thus

$$\limsup_{n\to\infty} \left(\int_a^b f^n(t) g^{n+1}(t)\,\mathrm{d}t \right)^{1/(n+1)} \le f(b)g(b).$$

Corollary 3.2 further implies that

$$\int_a^b \frac{g(t)}{f(t)}\,\mathrm{d}t \le \left(\int_a^b f^n(t) g^{n+1}(t)\,\mathrm{d}t \right)^{1/(n+1)} \left(\frac{n!}{(n+1)^{n-1}} \right)^{1/(n+1)} \frac{1}{f(a)(f'(a))^{n/(n+1)}}$$

for each $n \in \mathbb{N}$.

Using the well-known limits $\lim_{n\to\infty} \sqrt[n]{n} = 1$ and $\lim_{n\to\infty} n/\sqrt[n]{n!} = e$, we obtain

$$\lim_{n\to\infty} \left(\frac{n!}{(n+1)^{n-1}} \right)^{1/(n+1)} = \lim_{n\to\infty} \frac{\sqrt[n+1]{(n+1)!}}{n+1}\, \sqrt[n+1]{n+1} = \frac{1}{e}.$$

It follows that

$$\int_a^b \frac{g(t)}{f(t)}\,\mathrm{d}t \le \frac{1}{ef(a)f'(a)} \liminf_{n\to\infty} \left(\int_a^b f^n(t) g^{n+1}(t)\,\mathrm{d}t \right)^{1/(n+1)}.$$

Q.E.D.

In fact, the inequality $\int_a^b \frac{g(t)}{f(t)}\,\mathrm{d}t \le f(b)g(b)/(ef(a)f'(a))$ also holds under weaker assumptions.

Proposition 3.4. Let $f : [a,b] \to \mathbb{R}$ be a strictly positive, differentiable function such that f' is increasing and $f'(a) > 0$. Let $g : [a,b] \to \mathbb{R}$ be increasing and nonnegative. Then we have

$$\int_a^b \frac{g(t)}{f(t)}\,\mathrm{d}t \le \frac{g(b)}{f'(a)} \log\left(\frac{f(b)}{f(a)} \right) \le \frac{f(b)g(b)}{ef(a)f'(a)}. \tag{3.4}$$

For the proof we need the following well-known Lemma (see for instance [25, 9] and references therein).

Lemma 3.5. For every $x > 0$ we have $x^e \leq e^x$. Equality holds if and only if $x = e$.

Proof of Proposition 10. The monotonicity of g and f' implies

$$\int_a^b \frac{g(t)}{f(t)}\,\mathrm{d}t \leq \frac{g(b)}{f'(a)}\int_a^b \frac{f'(t)}{f(t)}\,\mathrm{d}t = \frac{g(b)}{f'(a)}\log\left(\frac{f(b)}{f(a)}\right)$$

and Lemma 3.5 implies

$$\log\left(\frac{f(b)}{f(a)}\right) \leq \frac{f(b)}{ef(a)}$$

which concludes the proof. Q.E.D.

4 Further variants of Qi's inequality

In the last section of this paper, we present some other variants of Qi's inequality (1.1). We start with some results in which we have the same exponent n for the integral and the function f.

Proposition 4.1. Let f be an $(n-1)$-times differentiable function on $[a,b]$ $(n \geq 2)$ such that $f^{(i)} \geq 0$ for $i = 0,\ldots,n-2$. Let further $g \in D_+[a,b]$ or $g \in D_-[a,b]$ be a strictly positive, increasing function and let $\alpha \in (n,\infty)$. Suppose that $g^{\alpha-n}f^{(n-1)} \geq \frac{n!}{n-1}f$. Then we have

$$(b-a)f^n(a)g^\alpha(a) + \left(\int_a^b f(t)g(t)\,\mathrm{d}t\right)^n \leq \int_a^b f^n(t)g^\alpha(t)\,\mathrm{d}t.$$

Proof. The proof is similar to the previous ones. Again we assume that $g \in D_+[a,b]$, define

$$F(x) := \left(\int_a^x f(t)g(t)\,\mathrm{d}t\right)^n - \int_a^x f^n(t)g^\alpha(t)\,\mathrm{d}t,$$
$$G(x) := n\left(\int_a^x f(t)\,\mathrm{d}t\right)^{n-1} - f^{n-1}(x)g^{\alpha-n}(x)$$

and show as before that

$$F' \leq fg^nG. \tag{4.1}$$

Next, using the fact that $g'_+ \geq 0$ and $\alpha > n$, we obtain

$$G'_+(x) \leq (n-1)\left(nf(x)\left(\int_a^x f(t)\,\mathrm{d}t\right)^{n-2} - f^{n-2}(x)f'(x)g^{\alpha-n}(x)\right). \tag{4.2}$$

If $n = 2$ this implies $G'_+ \leq 2f - f'g^{\alpha-2}$, which by assumption is nonpositive. Thus we obtain $G \leq G(a) = -f(a)g^{\alpha-2}(a)$.

Now consider the case $n \geq 3$. We define functions $H_1,\ldots,H_{n-2}$ on $[a,b]$ by

$$H_k(x) := n\left(\prod_{i=k+1}^{n-2} i\right)\left(\int_a^x f(t)\,\mathrm{d}t\right)^k - f^{k-1}(x)f^{(n-k-1)}(x)g^{\alpha-n}(x).$$

We will show that $H_k \le 0$ for $k = 1, \dots, n-2$. First, we have $(H_1)'_+ = fn!/(n-1) - f^{(n-1)}g^{\alpha-n} - f^{(n-2)}(\alpha-n)g^{\alpha-n-1}g'_+$. Since $f^{(n-2)}, g'_+ \ge 0$ and $\alpha > n$ this implies $(H_1)'_+ \le fn!/(n-1) - f^{(n-1)}g^{\alpha-n}$, and thus by our assumption on f and g we have $(H_1)'_+ \le 0$. Hence $H_1 \le H_1(a) \le 0$.

Now suppose that $H_k \le 0$ for some $k \le n-3$. We have

$$\begin{aligned}
(H_{k+1})'_+(x) &= nf(x)\left(\prod_{i=k+1}^{n-2} i\right)\left(\int_a^x f(t)\,\mathrm{d}t\right)^k \\
&\quad - kf^{k-1}(x)f'(x)f^{(n-k-2)}(x)g^{\alpha-n}(x) - f^k(x)f^{(n-k-1)}(x)g^{\alpha-n}(x) \\
&\quad - f^k(x)f^{(n-k-2)}(x)(\alpha-n)g^{\alpha-n-1}(x)g'_+(x) \\
&\le nf(x)\left(\prod_{i=k+1}^{n-2} i\right)\left(\int_a^x f(t)\,\mathrm{d}t\right)^k - f^k(x)f^{(n-k-1)}(x)g^{\alpha-n}(x) \\
&= f(x)H_k(x) \le 0.
\end{aligned}$$

Hence $H_{k+1} \le H_{k+1}(a) \le 0$.

It follows from (4.2) that $G'_+ \le (n-1)fH_{n-2} \le 0$ and thus $G \le G(a) = -f^{n-1}(a)g^{\alpha-n}(a)$.

Using this together with (4.1) and the fact that f and g are increasing we obtain $F' \le -f^n(a)g^\alpha(a)$. The mean value theorem therefore implies $F(b) = F(b) - F(a) \le -(b-a)f^n(a)g^\alpha(a)$, finishing the proof. Q.E.D.

This yields the following corollary.

Corollary 4.2. If $n \ge 2$ and f is an $(n-1)$-times differentiable function on $[a,b]$ such that $f^{(i)} \ge 0$ for $i = 0, \dots, n-2$, $f^{(n-1)}$ is strictly positive and $f/f^{(n-1)}$ is bounded, then

$$(b-a)f^n(a) + \frac{n-1}{n!A}\left(\int_a^b f(t)\,\mathrm{d}t\right)^n \le \int_a^b f^n(t)\,\mathrm{d}t,$$

where $A := \left\|f/f^{(n-1)}\right\|_\infty$ and $\|\cdot\|_\infty$ denotes the usual sup-norm.

Proof. Define g to be constant on $[a,b]$ with value $(n!A/(n-1))^{1/n}$ and $\alpha := 2n$ and apply Proposition 4.1. Q.E.D.

An analogous result to Proposition 4.1 for the case $\alpha \le n$ reads as follows.

Proposition 4.3. Let f be an $(n-1)$-times differentiable function on $[a,b]$ $(n \ge 2)$ such that $f^{(i)} \ge 0$ for $i = 0, \dots, n-2$. Let g be a strictly positive, continuous and increasing function on $[a,b]$ and let $\alpha \in (-\infty, n]$ with $f^{(n-1)} \ge \frac{n!}{n-1} f \|g\|_\infty^{n-\alpha}$. Then we have

$$(b-a)f^n(a)K + \left(\int_a^b f(t)g(t)\,\mathrm{d}t\right)^n \le \int_a^b f^n(t)g^\alpha(t)\,\mathrm{d}t,$$

where $K = g^\alpha(a)$ for $\alpha \ge 0$ and $K = g^\alpha(b)$ for $\alpha < 0$.

Proof. The proof is similar to the last one, so we will only sketch it. First define F exactly as in the previous proof and let

$$G(x) := n\|g\|_\infty^{n-\alpha}\left(\int_a^x f(t)\,\mathrm{d}t\right)^{n-1} - f^{n-1}(x).$$

It is easy to prove that

$$F' \le fg^\alpha G. \tag{4.3}$$

Next we define again functions H_k (where $k = 1, \dots, n-2$) by

$$H_k(x) := n\|g\|_\infty^{n-\alpha}\left(\prod_{i=k+1}^{n-2} i\right)\left(\int_a^x f(t)\,\mathrm{d}t\right)^k - f^{k-1}(x)f^{(n-k-1)}(x).$$

Similar to our previous proofs one can show inductively that $H_k \le 0$ for $k = 1, \dots, n-2$ and $G' = (n-1)fH_{n-2}$.

It follows that G is decreasing and hence $G \le G(a) = -f^{n-1}(a)$. Using this together with (4.3), the monotonicity of f and g, and the mean value theorem, one obtains

$$F(b) \le -(b-a)f^n(a)K$$

and the proof is finished. Q.E.D.

Here is yet another result of the above type.

Proposition 4.4. Let f be an $(n-1)$-times differentiable function on $[a,b]$ ($n \ge 2$) such that $f^{(i)} \ge 0$ for $i = 0, \dots, n-2$. Let $g, h \in D_+[a,b]$ or $g, h \in D_-[a,b]$ with $g, h \ge 0$. Assume further that g is increasing and h and gh are decreasing. Let $f^{(n-1)} \ge \frac{n!}{n-1} f g^{n-l} h^n$ for some $l \in \{1, \dots, n\}$. Then we have

$$(b-a)f^n(a)g^l(a) + \left(\int_a^b f(t)g(t)h(t)\,\mathrm{d}t\right)^n \le \int_a^b f^n(t)g^l(t)\,\mathrm{d}t.$$

Proof. We will only sketch the proof for the case $g, h \in D_+[a,b]$. First define

$$F(x) := \left(\int_a^x f(t)g(t)h(t)\,\mathrm{d}t\right)^n - \int_a^x f^n(t)g^l(t)\,\mathrm{d}t,$$
$$G(x) := n\left(\int_a^x f(t)g(t)h(t)\,\mathrm{d}t\right)^{n-1} h(x) - f^{n-1}(x)g^{l-1}(x).$$

Then $F' = fgG$.

In the following, we will only treat the case $2 \le l \le n-1$. The boundary cases are treated

similarly. We define

$$H_k(x) := n\left(\prod_{i=2}^{k}(n-i)\right)\left(\int_a^x f(t)g(t)h(t)\,\mathrm{d}t\right)^{n-k-1} h^{k+1}(x)$$
$$- f^{(k)}(x)f^{n-k-2}(x)g^{l-k-1}(x),$$
$$J_s(x) := n\left(\prod_{i=2}^{l+s-1}(n-i)\right)\left(\int_a^x f(t)g(t)h(t)\,\mathrm{d}t\right)^{n-l-s} h^{l+s}(x)g^s(x)$$
$$- f^{(l+s-1)}(x)f^{n-l-s-1}(x)$$

for $k = 1, \dots, l-1$ and $s = 0, \dots, n-1-l$.

Then $J_0 = H_{l-1}$. Moreover, using $h' \le 0$ and $g' \ge 0$, it is easy to see that $G'_+ \le (n-1)fgH_1$, $(H_k)'_+ \le fgH_{k+1}$ for each k and $(J_s)'_+ \le fJ_{s+1}$ for each s (note that $h^{l+s}g^s$ is also decreasing (and hence has a negative right-derivative) since $h^{l+s}g^s = (gh)^s h^l$).

We further have $(J_{n-l-1})'_+ \le \frac{n!}{n-1} fh^n g^{n-l} - f^{(n-1)}$, whence by assumption $(J_{n-l-1})'_+ \le 0$.

Using the usual monotonicity and induction arguments, we obtain that all the functions J_s and H_k are negative. It follows in particular that $G'_+ \le 0$ and hence $G \le G(a) = -f^{n-1}(a)g^{l-1}(a)$.

Since $F' = fgG$ and f and g are increasing, we obtain $F' \le -f^n(a)g^l(a)$. From this and the mean value theorem we can deduce the desired inequality. Q.E.D.

An immediate corollary is the following inequality.

Corollary 4.5. Let f be an $(n-1)$-times differentiable function on $[a,b]$ $(n \ge 2)$ such that $f^{(i)} \ge 0$ for $i = 0, \dots, n-2$ and g a strictly positive, increasing function on $[a,b]$ with $g \in D_+[a,b]$ or $g \in D_-[a,b]$. If $f^{(n-1)}g \ge \frac{n!}{n-1} f$, then

$$(b-a)f^n(a)g(a) + \left(\int_a^b f(t)\,\mathrm{d}t\right)^n \le \int_a^b f^n(t)g(t)\,\mathrm{d}t.$$

Proof. Put $h := 1/g$ and $l = 1$ and apply Proposition 4.4. Q.E.D.

This yields a further corollary which is related to Corollary 2.5.

Corollary 4.6. Let f be a strictly positive, n-times differentiable function on $[a,b]$ $(n \ge 2)$ such that $f^{(i)} \ge 0$ for $i = 1, \dots, n-2$. Suppose further that $f^{(n-1)}$ is strictly positive and $f/f^{(n-1)}$ is increasing. Then we have

$$(b-a)\frac{f^{n+1}(a)}{f^{(n-1)}(a)} + \frac{n-1}{n!}\left(\int_a^b f(t)\,\mathrm{d}t\right)^n \le \int_a^b \frac{f^{n+1}(t)}{f^{(n-1)}(t)}\,\mathrm{d}t.$$

Proof. Put $g := n!/(n-1)(f/f^{(n-1)})$ and apply Corollary 4.5. Q.E.D.

Note that Corollary 2.5 implies (for $h = 1$ and $\alpha = 1$) the stronger inequality

$$(b-a)\frac{f^{n+1}(a)}{f^{(n-1)}(a)} + \frac{1}{(n-1)!}\left(\int_a^b f(t)\,\mathrm{d}t\right)^n \le \int_a^b \frac{f^{n+1}(t)}{f^{(n-1)}(t)}\,\mathrm{d}t$$

under the stronger assumption that $f^{(n)} \leq 0$ (the assumption that $f/f^{(n-1)}$ is increasing is equivalent to $f^{(n)} \leq f^{(n-1)}f'/f$).

Finally, we also have the following generalization of (1.1).

Proposition 4.7. Let $n \geq 2$, $k \in \{1, \ldots, n-1\}$ and $f : [a,b] \to \mathbb{R}$ a k-times differentiable function such that $f^{(i)} \geq 0$ for $i = 0, \ldots, k-1$. Let $g : [a,b] \to \mathbb{R}$ be continuous and increasing with $g \geq 0$. Let $h \in D_+[a,b]$ or $h \in D_-[a,b]$ be increasing and strictly positive. Suppose that $f^{(k)}(x) \geq (x-a)^{n-k-1}h(x)\frac{(n-1)!}{(n-k-1)!}$ for all $x \in [a,b]$. Then we have

$$(b-a)f^{n+1}(a)g^n(a)h^{n-1}(a) + \left(\int_a^b f(t)g(t)h(t)\,\mathrm{d}t\right)^n \leq \int_a^b f^{n+1}(t)g^n(t)h^{n-1}(t)\,\mathrm{d}t.$$

Proof. We will only give a sketch of the proof. Assume that $h \in D_+[a,b]$ and define

$$F(x) := \left(\int_a^x f(t)g(t)h(t)\,\mathrm{d}t\right)^n - \int_a^x f^{n+1}(t)g^n(t)h^{n-1}(t)\,\mathrm{d}t,$$
$$G(x) := n\left(\int_a^x f(t)h(t)\,\mathrm{d}t\right)^{n-1} - f^n(x)h^{n-2}(x).$$

Using the monotonicity of g, we see as before that $F' \leq g^n fhG$.

Next we define

$$H_s(x) := \left(\prod_{i=1}^{s}(n-i)\right)\left(\int_a^x f(t)h(t)\,\mathrm{d}t\right)^{n-s-1} - f^{n-s-1}(x)f^{(s)}(x)h^{n-s-2}(x)$$

for $s = 1, \ldots, k$.

Using similar arguments as before we obtain that $G'_+ \leq nfhH_1$ and $(H_s)'_+ \leq fhH_{s+1}$ for all s.

We further have, due to the monotonicity of f and h,

$$H_k(x) \leq f^{n-k-1}(x)h^{n-k-2}(x)\left(\frac{(n-1)!}{(n-k-1)!}(x-a)^{n-k-1}h(x) - f^{(k)}(x)\right).$$

Hence our assumption implies $H_k \leq 0$.

It follows inductively that $H_s \leq 0$ for all s. Hence $G'_+ \leq 0$ and thus $G \leq G(a) = -f^n(a)h^{n-2}(a)$.

This implies $F' \leq -f^{n+1}(a)g^n(a)h^{n-2}(a)$ and the mean value theorem gives us the desired conclusion. Q.E.D.

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