# Higher chromatic analogues of twisted $K$-theory

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**Abstract**

In this paper we introduce a new family of twisted $K(n)$-local homology theories. These theories are given by the spectra $R_n = E_n^{hS\mathbb{G}_n}$, twisted by a class $H \in H^{n+2}(X, \mathbb{Z}_p)$. Here $E_n^{hS\mathbb{G}_n}$ are the homotopy fixed point spectra under the action of the subgroup $S\mathbb{G}_n$ of the Morava stabilizer group where $S\mathbb{G}_n$ is the kernel of the determinant homomorphism $\det : \mathbb{G}_n \to \mathbb{Z}_p^\times$. These spectra were utilized in [8] by C. Westerland to study higher chromatic analogues of the J-homomorphism. We investigate some of the properties of these new twisted theories and discuss why we consider them as a generalization of twisted K-theory to higher chromatic levels.

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## 1 Introduction

Let $R_n$ denote the homotopy fixed point spectra

$$R_n = E_n^{hS\mathbb{G}_n}$$

where $E_n$ is the $n$th Morava E-theory, $\mathbb{G}_n$ is the extended Morava stabilizer group, and $S\mathbb{G}_n$ is the kernel of the determinant homomorphism $\det : \mathbb{G}_n \to \mathbb{Z}_p^\times$. The spectra $R_n$ were introduced by Craig Westerland in [8] to study the higher chromatic analogues of the image of the $J$-homomorphism. In this paper, we use these spectra to introduce new twisted cohomology theories analogous to twisted K-theory.

Throughout this paper, $L_{K(n)}$ denotes the usual Bousfield localization functor with respect to Morava K-theory $K(n)$. As we will see, there is a natural map $L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1)_+ \to R_n$. Composition with the localization map

$$\Sigma^\infty K(\mathbb{Z}_p, n+1) \to L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1)_+$$

gives rise to a ring map $\Sigma^\infty K(\mathbb{Z}_p, n+1) \to R_n$, which equips $R_n$ with a $\Sigma^\infty K(\mathbb{Z}_p, n+1)$-module structure. We define twisted $R_n$-theory as follows. Let $X$ be a space equipped with a cohomology class $H \in H^{n+2}(X, \mathbb{Z}_p)$ and consider the induced bundle $P \to X$. Then $\Sigma^\infty K(\mathbb{Z}_p, n+1)$ acts on $\Sigma^\infty P_+$ by virtue of the fact that $P$ is a $K(\mathbb{Z}_p, n+1)$-bundle. The *generalized Thom spectra* arising from this set up is the smash product

$$X^H := \Sigma^\infty P_+ \wedge_{\Sigma^\infty K(\mathbb{Z}_p, n+1)_+} R_n$$

Note that the smash product here (and throughout this paper) is the derived smash product, so $\Sigma^\infty P_+$ is replaced with a cofibrant $\Sigma^\infty K(\mathbb{Z}_p, n+1)_+$-module spectrum. See [1] for more information. We define the $H$-twisted $R_n$-homology of $X$ to be

$$R_{n*}(X, H) = \pi_n(X^H)$$

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We refer to these theories as **higher chromatic twisted K-theories**.

In this paper we seek to justify this definition as the right generalization of twisted K-theory into the chromatic world. We'll start by defining twisted p-adic K-theory and later we recover it as a particular case of $R_{n*}(X,H)$ for $n=1$. However, we fail to prove what we initially set out to prove- that a universal coefficient type isomorphism (discussed below) holds for these higher twisted theories. Such isomorphism holds for twisted K-theory- see theorem 2.1 below. The goal of this paper is introduce these new twisted theories to the community and establish their basic properties. We hope that subsequent research will shed more light on some of the questions that arise in this paper, particularly the question regarding a universal coefficient type isomorphism.

The above definitions are made in the context of *generalized Thom spectra*. We briefly recall the general description of twisted cohomology theories. Following [1], let $R$ be an $A_\infty$ ring spectrum. The units of $R$, $GL_1(R)$, is defined as the fiber product

$$\begin{array}{ccc} GL_1(R) & \longrightarrow & \Omega^\infty R \\ \downarrow & & \downarrow \\ \pi_0(R)^\times & \longrightarrow & \pi_0(R) \end{array}$$

where the bottom map is the inclusion of the units $\pi_0(R)^\times$ in $\pi_0(R)$. The $A_\infty$ structure of $R$ equips $GL_1(R)$ with a group-like structure, allowing one to form the classifying space $BGL_1(R)$. The construction of $GL_1$ is functorial and is adjoint of the suspension spectrum functor. This means that if $Y$ is an $A_\infty$-space, then there is an isomorphism (see [1])

$$\pi_0 \operatorname{Map}_{A_\infty}(\Sigma^\infty Y_+, R) \cong \pi_0 \operatorname{Map}_{A_\infty}(Y_+, GL_1(R)).$$

For a space $X$ equipped with a map $H: X \to BGL_1(R)$, one defines the **generalized Thom spectrum** as follows. The map $H$ induces a principal $GL_1(R)$ bundle $P_H \to X$ over $X$. Now $\Sigma^\infty(P_H)_+$ and $R$ are module spectra over $\Sigma^\infty GL_1(R)_+$. The generalized Thom spectrum $X^H$ is defined to be

$$X^H = \Sigma^\infty(P_H)_+ \wedge_{\Sigma^\infty GL_1(R)_+} R.$$

The $H$-twisted $R$-(co)homology groups of $X$ are defined to be the homotopy groups

$$R_n(X,H) := \pi_n(X^H) \qquad \text{and} \qquad R^n(X,H) := \text{R-mod}[X^H, \Sigma^{-n}R]$$

where R-mod$[X^H, \Sigma^{-n}R]$ denotes $R$-module maps from $X^H$ to $\Sigma^{-n}R$. Twisted cohomology theories are honest cohomology theories on the category of spaces $X$ equipped with a map $H: X \to BGL_1(R)$. Note that when $H$ is trivial, one recovers the usual untwisted theory.

## 2 Twisted k-theory

Twisted K-theory arises from the action of line bundles on K-theory. This action is given by a map $\Sigma^\infty \mathbb{C}P^\infty_+ \wedge K \to K$ which we now describe. Let $\mathbb{L}$ be the universal line bundle on $\mathbb{C}P^\infty \simeq BU(1)$ and let $\xi_n$ be the universal bundle on $BU(n)$. The tensor bundle $\mathbb{L} \otimes \xi_n$ is an n-dimensional bundle over $\mathbb{C}P^\infty \times BU(n)$ and is classified by a map

$$\mathbb{C}P^\infty \times BU(n) \longrightarrow BU(n)$$

By taking direct limit over $n$ and stabilizing, we get a map $\Sigma^\infty \mathbb{C}P^\infty_+ \wedge K \to K$.

There is an alternative description of this map. The action of line bundles is characterized by the inclusion $BU(1) \times \{0\} \simeq \mathbb{C}P^\infty \to BU \times \mathbb{Z}$. This inclusion stabilizes to a ring map of ring spectra $i : \Sigma^\infty \mathbb{C}P^\infty_+ \to K$. We use this map to define a $\Sigma^\infty \mathbb{C}P^\infty_+$- module structure on $K$.

$$\Sigma^\infty \mathbb{C}P^\infty_+ \wedge K \to K \wedge K \to K.$$

Here the second map is given by the ring structure of $K$.

Let $X$ be a space equipped with a three dimensional integral cohomology class $H \in H^3(X, \mathbb{Z})$. Such class is given by a map $H : X \to K(\mathbb{Z}, 3)$. We consider the path-loop fibration on $K(\mathbb{Z}, 3)$, and the principal $\mathbb{C}P^\infty$-bundle induced by $H$.

$$\begin{array}{ccc}
\mathbb{C}P^\infty & \longrightarrow & \Omega K(\mathbb{Z},3) \simeq K(\mathbb{Z},2) \\
\downarrow & & \downarrow \\
P & \longrightarrow & PK(\mathbb{Z},3) \simeq \mathrm{pt} \\
\downarrow & & \downarrow \\
X & \xrightarrow{H} & K(\mathbb{Z},3)
\end{array}$$

The action of $\mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$ on $P$ defines a $\Sigma^\infty \mathbb{C}P^\infty_+$-module structure on $\Sigma^\infty P_+$ and we can form the generalized Thom spectrum as

$$X^H = \Sigma^\infty P_+ \wedge_{\Sigma^\infty \mathbb{C}P^\infty_+} K.$$

The $H$-twisted $K$-homology of $X$ is defined to be the homotopy of this spectrum:

$$K_*(X, H) := \pi_*(X^H).$$

In order to make sense of the above constructions, one must choose a model for the K-theory spectrum and model the action of line bundles on the point-set level. This has been done before; see [2] and [3], for example. In [6], we showed that twisted K-theory of $X$ can be calculated from the untwisted K-theory of $P$ via a *universal coefficient isomorphism*. To be more precise, note that the pullback of $P$ along its own projection $P \to X$ is tautologically trivial, so in the category of spaces equipped with an $\mathbb{C}P^\infty$-bundle there is a map

$$(P \times \mathbb{C}P^\infty, P) \to (P, X)$$

which induces a map $K_*(P) \to K_*(X, H)$. In [6], we showed that this map factors through a map

$$K_*(P) \otimes_{K_*(\mathbb{C}P^\infty)} K_* \to K_*(X, H)$$

where the action of $K_*(\mathbb{C}P^\infty)$ on $K_*$ in this tensor product is given by the ring homomorphism $\varepsilon : K_*(\mathbb{C}P^\infty) \to K_*$ induced by the composition $\Sigma^\infty \mathbb{C}P^\infty_+ \wedge K \to K \wedge K \to K$.

The main result of [6] is the following universal coefficient isomorphism for twisted K-theory. It shows that the twisted K-theory of $X$ can be computed from the usual untwisted K-theory of $P$ via a tensor product.

**Theorem 2.1** (M. Khorami)**.** The map

$$K_*(P) \otimes_{K_*(\mathbb{C}P^\infty)} K_* \to K_*(X, H)$$

is an isomorphism of $K_*$-modules.

A key property of K-theory that makes this isomorphism possible is the following theorem due to Snaith.

**Theorem 2.2** (Snaith)**.** Let $\beta : S^2 \to \Sigma^\infty \mathbb{C}P^\infty_+$ be the Bott map. Then the inclusion $\mathbb{C}P^\infty \cong BU(1) \to BU \times \mathbb{Z}$ localizes to an equivalence

$$\Sigma^\infty \mathbb{C}P^\infty_+ \ [\beta^{-1}] \simeq K.$$

Here the Bott map $\beta : S^2 \to \Sigma^\infty \mathbb{C}P^\infty_+$ is the stabilization of the inclusion $\mathbb{C}P^1 \to \mathbb{C}P^\infty_+$. An immediate consequence of this theorem is the isomorphism $K_*(\mathbb{C}P^\infty)[\beta^{-1}] \cong K_*K$, where $\beta$ is regarded as an element of $K_*(\mathbb{C}P^\infty)$ by smashing with the unit map of $K$.

This isomorphism is crucial in the proof of theorem 2.1. It shows that the algebra $K_*K$ is a localization of algebra $K_*(\mathbb{C}P^\infty)$, and so the passage from $K_*(\mathbb{C}P^\infty)$ to $K_*K$ is "flat". It is preferable to work with $K_*K$, since it is the algebra of co-operations of K-theory, and as such is highly structured. For more information, see the proof of theorem 2.1 in [6].

There are many cohomology theories in algebraic topology that admit natural twists. Examples include Morava K-theory and E-theory. In [4], for prime $p = 2$, the authors define the twisted Morava K-theory $K(n)_*(X, H)$ for spaces $X$ equipped with an integral cohomology class $H \in H^{n+2}(X, \mathbb{Z})$. They show, among other things, that for primes $p > 2$, there are in fact no nontrivial twists, though nontrivial twists do exist for the 2-periodic form of Morava K-theory, as well as Morava E-theory.

## 3 Twisted p-adic K-theory

Working with mod $p$ cohomology, we can construct twisted p-adic K-theory. Recall that $K(1)$-localization of a space $X$ is given by p-completion of $K$-localization of $X$, that is,

$$L_{K(1)}X = (L_K X)^c.$$

It follows immediately that p-adic K-theory $\hat{K}$ is the $K(1)$-localization of $K$.

To begin, we construct a map $j : \Sigma^\infty K(\mathbb{Z}_p, 2)_+ \to \hat{K}$. Let

$$\hat{i} : L_{K(1)}\Sigma^\infty K(\mathbb{Z}, 2)_+ \to L_{K(1)}K \simeq \hat{K}$$

be the $K(1)$-localization of the map $i : \Sigma^\infty K(\mathbb{Z}, 2)_+ \to K$ introduced in section 2. We define the map $j$ to be the composition

$$\Sigma^\infty K(\mathbb{Z}_p, 2)_+ \to L_{K(1)}\Sigma^\infty K(\mathbb{Z}_p, 2)_+ \cong L_{K(1)}\Sigma^\infty K(\mathbb{Z}, 2)_+ \xrightarrow{\hat{i}} \hat{K}.$$

where the first map is $K(1)$-localization of $\Sigma^\infty K(\mathbb{Z}_p, 2)_+$.

Using this map we can define a $\Sigma^\infty K(\mathbb{Z}_p, 2)_+$-module structure on $\hat{K}$ in the obvious way:

$$\Sigma^\infty K(\mathbb{Z}_p, 2)_+ \wedge \hat{K} \to \hat{K} \wedge \hat{K} \to \hat{K}.$$

Here the second map is given by the ring structure of the ring spectrum $\hat{K}$.

Let $X$ be a space equipped with a cohomology class $H \in H^3(X, \mathbb{Z}_p)$. Let $P$ be the bundle induced by $H$ as before. Then $\Sigma^\infty P_+$ is a module over $\Sigma^\infty K(\mathbb{Z}_p, 2)_+$ and we define:

**Definition 3.1.** The twisted $\hat{K}$-theory of $X$ is

$$\hat{K}_*(X, H) := \pi_*(\Sigma^\infty P_+ \wedge_{\Sigma^\infty K(\mathbb{Z}_p,2)_+} \hat{K}).$$

We consider this construction as the $K(1)$-localization of twisted K-theory at prime $p$. In the following sections, we show that this construction generalizes to all chromatic levels using the spectra $R_n$.

## 4 The spectra $R_n$

Let $\mathbb{S}_n$ be the Morava stabilizer group; the group of isomorphisms of the height $n$ Honda formal group law $\Gamma_n$. Let $\mathbb{G}_n$ be the extended Morava stabilizer group:

$$\mathbb{G}_n = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \ltimes \mathbb{S}_n$$

The group $\mathbb{S}_n$ is the units in the endomorphism ring $\mathbb{O}_n$ of $\Gamma_n$. The ring $\mathbb{O}_n$ has a faithful matrix representation and the determinant defines a multiplicative map $\det : \mathbb{O}_n \to \mathbb{W}$. It's easy to check that the image of $\mathbb{S}_n$ under this map lands in the group of units of $\mathbb{Z}_p$. We extend this map to $\mathbb{G}_n$ via the sign character; by sending the Frobenius generator of $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ to $(-1)^{n-1}$. This gives rise to a homomorphism $\det : \mathbb{G}_n \to \mathbb{Z}_p^\times$. Following [8], let $S\mathbb{G}_n$ denote the kernel of this map.

Let $E_n$ denote the $n$th Morava E-theory. The work of Goerss-Hopkins-Miller equips the spectra $E_n$ with a $\mathbb{G}_n$ action. The works of Devinatz-Hopkins [5] allow one to define the homotopy fixed point spectra with respect to closed subgroups of $\mathbb{G}_n$. The homotopy fixed point spectrum with respect to the full group is the $K(n)$-local sphere $L_{K(n)}S^0$.

The $K(n)$-local spectra that we consider in this paper are the homotopy fixed point spectra of the subgroup $S\mathbb{G}_n$. Let

$$R_n = E_n^{hS\mathbb{G}_n}.$$

For all $n$, $R_n$ is an $E_\infty$ ring spectrum. These spectra were used by Craig Westerland in [8] to study the higher chromatic analogues of the image of the $J$-homomorphism. Note that when $n = 1$, $R_1$ is the $p$-adic K-theory $\hat{K}$. In [8] Westerland proves, among other things, that:

**Theorem 4.1** (C. Westerland)**.** There exist a map $\rho_n : S{<}\det{>} \to L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1)_+$ and an equivalence of $E_\infty$-ring spectra

$$L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1)_+[\rho_n^{-1}] \to R_n.$$

The spectra $S{<}\det{>}$ are invertible $K(n)$-local spectra introduced in [7].

This theorem is the main source of comparison between K-theory and $R_n$. It is to be considered as the chromatic version of the Snaith's theorem mentioned above. In fact, when $n = 1$, $R_1$ is the $p$-adic K-theory, and this result is the $K(1)$-localization of the Snaith's theorem. In addition, $\rho_1$ is $K(1)$-localization of the Bott map $\beta$.

## 5 Higher twisted chromatic theories

In this section we define what we consider as higher chromatic analogues of twisted K-theory. To simplify notation, let $Q_n$ denote $\Sigma^\infty K(\mathbb{Z}_p, n+1)_+$ and let $T_n$ denote the localized spectrum $L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1)_+$. We show that for all $n$ there is a natural map $\gamma : T_n \to R_n$. Existence of such map is quite clear in the work of C. Westerland, but for completeness we include a proof here.

**Lemma 5.1.** There exist a natural ring map of ring spectra $\gamma_n : T_n \to R_n$.

*Proof.* Consider the mod $p$ Eilenberg-MacLane space $K(\mathbb{Z}/p, n)$. The localized spectra $L_{K(n)}\Sigma^\infty K(\mathbb{Z}/p, n)_+$ split as a wedge $Z \vee Z^\perp$, where $Z$ is a certain invertible $K(n)$-local spectrum whose image in $K_*K(\mathbb{Z}/p, n)$ is a $K_*$-module of rank one (See [8]). This yields a natural map $i : Z \to \Sigma^\infty K(\mathbb{Z}/p, n)_+$.

Let $\beta$ be the Bockstein homomorphism

$$\beta : L_{K(n)}\Sigma^\infty K(\mathbb{Z}/p, n)_+ \to L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1)_+ = T_n$$

Define $\alpha = \beta i$. The main result of [8] is that

$$R_n \simeq L_{K(n)}(T_n[\alpha^{-1}])$$

Here the map $\alpha$ is inverted via a telescope construction, similar to the usual homotopy element of a ring spectrum. To be precise, note that since $Z$ is invertible, for a spectrum $A$, $L_{K(n)}(A \wedge Z) \simeq L_{K(n)}S$. Then $T_n[\alpha^{-1}]$ is the homotopy colimit of the diagram

$$T_n \to L_{K(n)}(A \wedge Z) \to L_{K(n)}(A \wedge A \wedge Z) \to \cdots$$

Finally, we define $\gamma_n$ to be the composition

$$\gamma_n : T_n \to T_n[\alpha^{-1}] \to L_{K(n)}T_n[\alpha^{-1}] \simeq R_n$$

Q.E.D.

We compose $\gamma_n$ with $K(n)$-localization map $Q_n \to T_n$ to get a map $\alpha_n : Q_n \to R_n$. We define a $Q_n$-module structure on $R_n$ by smashing this map with $R_n$ and using the ring structure of $R_n$:

$$Q_n \wedge R_n \to R_n \wedge R_n \to R_n.$$

Let $X$ be a space equipped with a cohomology class $H \in H^{n+2}(X, \mathbb{Z}_p)$. Let $P \to X$ be the $K(\mathbb{Z}_p, n+1)$-bundle on $X$ induced by $H$. Then $Q_n$ acts on $\Sigma^\infty P_+$ by virtue of the fact that $P$ is a $K(\mathbb{Z}_p, n+1)$-bundle. The *generalized Thom spectra* is

$$X^H := \Sigma^\infty P_+ \wedge_{Q_n} R_n$$

**Definition 5.2.** The $H$-twisted $R_n$-(co)homology of $X$ is

$$R_{n*}(X, H) = \pi_*(X^H) \qquad \text{and} \qquad R_n^*(X, H) = \pi_{-*}F_{R_n}(X^H, R_n)$$

where $F_{R_n}(X^H, R_n)$ denotes the spectrum of $R_n$-module maps $X^H \to R_n$.

We refer to these theories as **higher chromatic twisted** theories.

In an earlier version of this paper we attempted to work with $L_{K(n)}K(\mathbb{Z}_p, n+1)$-bundles instead of honest $K(\mathbb{Z}_p, n+1)$-bundles . As it was correctly pointed out by the referee, there are issues that arise from localizing in this way and then working with the suspension spectra.

We like to point out that the spectra $\Sigma^\infty P_+ \wedge_{Q_n} R_n$ are not necessarily $K(n)$-local. One could localize first and study $L_{K(n)}(\Sigma^\infty P_+ \wedge_{Q_n} R_n)$. The resulting "completed" theory would be of interest as well, similar to the untwisted setting.

## 6 Properties

Analogous to twisted K-theory, twisted chromatic theories have the basic properties that general twisted cohomology theories have.

**Theorem 6.1.** Let $R_{n*}(X, H)$ be twisted chromatic theory.

1. If $H = 0$, then $R_{n*}(X, H) = R_{n*}(X)$. Same holds for cohomology.
2. Let $H \in H^{n+1}(Y, \mathbb{Z}_p)$. Then for a continuous map $f : X \to Y$, there is an induced homomorphism
$$f_* : R_{n*}(X, f^*H) \to R_{n*}(Y, H)$$
where $f^* : H^{n+1}(Y, \mathbb{Z}_p) \to H^{n+1}(X, \mathbb{Z}_p)$ is the induced map on cohomology.
3. For twists $H_1, H_2 \in H^{n+1}(X, \mathbb{Z}_p)$ of $X$, there is a cup product homomorphism
$$R_n^*(X, H_1) \otimes R_n^*(X, H_2) \to R_n^*(X, H_1 + H_2)$$
4. For a space $X$, $R_n^*(X, H)$ is a module over the ring $R_n^*(X)$.

The proof is standard. For completeness, we include a proof here.

*Proof.* When $H = 0$, then the induced bundle $P \to X$ is trivial and $P = Q_n \times X$. Therefore
$$\Sigma^\infty P_+ \wedge_{Q_n} R_n \simeq \Sigma^\infty (Q_n \times X)_+ \wedge_{Q_n} R_n \simeq \Sigma^\infty X_+ \wedge R_n.$$
This shows that $R_{n*}(X, 0) \cong R_{n*}(X)$.

For property 2, let $P_H$ denote the bundle induced on $Y$ by $H$. If $P_{f^*H}$ denotes the induced bundle $P_{f^*H}$ on $X$, then there is a map of bundles $P_{f^*H} \to P_H$. This map induces a map $X^{f^*H} \to Y^H$ on the generalized Thom spectra. The effect of this map on homotopy gives the desired map.

For property 3, let $m : R_n \wedge R_n \to R_n$ denote the ring map of $R_n$. On the function spectra level, $m$ induces a map
$$F_{R_n}(X^{H_1}, R_n) \wedge_{R_n} F_{R_n}(X^{H_2}, R_n) \xrightarrow{m^*} F_{R_n}(X^{H_1} \wedge_{R_n} X^{H_2}, R_n)$$
Consider the diagonal map $\Delta : X \to X \times X$, and let $\mathrm{pr}_1$ and $\mathrm{pr}_2$ denote the two projections onto $X$. We have
$$\Delta^*(\mathrm{pr}_1^*(H_1) + \mathrm{pr}_2^*(H_2)) = H_1 + H_2$$
It follows that $\Delta$ induces a map on the Thom spectra
$$X^{H_1+H_2} \to (X \times X)^{\mathrm{pr}_1^*(H_1)+\mathrm{pr}_2^*(H_2)} \simeq X^{H_1} \wedge_{R_n} X^{H_2}$$

and on the function spectra level, there is an induced map

$$F_{R_n}(X^{H_1} \wedge_{R_n} X^{H_2}, R_n) \to F_{R_n}(X^{H_1+H_2}, R_n)$$

Composition of this map with $m^*$ will give a map whose effect on homotopy induces the desired product.

Property 4 follows from properties 1 and 3.

Q.E.D.

We now show that when $n = 1$, this construction is compatible with that of twisted p-adic $K$-theory defined in section 3.

**Theorem 6.2.** When $n = 1$, $R_{1*}(X, H)$ is isomorphic to twisted $p$-adic K-theory $\hat{K}_*(X, H)$ defined in section 3.

*Proof.* When $n = 1$, $R_1$ is the p-adic K-theory $\hat{K}$, and $T_1 = L_{K(1)}\Sigma^\infty K(\mathbb{Z}_p, 2)_+$. It suffices to show that the map $Q_1 \to T_1 \xrightarrow{\gamma_1} R_1$ coincides with the map $j$ used to define twisted p-adic K-theory in section 3. To see this, note that in theorem 4.1, $\rho_1$ is $K(1)$-localization of the Bott map $\beta$. When $n = 1$, the map $\gamma_1$ is the same as the map induced by inversion of $\rho_1$:

$$L_{K(1)}\Sigma^\infty K(\mathbb{Z}_p, 2)_+ \to L_{K(1)}\Sigma^\infty K(\mathbb{Z}_p, 2)_+[\rho_1^{-1}] = \hat{K}.$$

This map is essentially the composition of the map $\Sigma^\infty K(\mathbb{Z}_p, 2)_+ \to L_{K(1)}\Sigma^\infty K(\mathbb{Z}, 2)_+$ with $K(1)$-localization of the map $i : \Sigma^\infty K(\mathbb{Z}, 2)_+ \to K$. The result follows. Q.E.D.

As for computation of the twisted chromatic theories, there is a standard bar spectral sequence as in theorem below.

**Theorem 6.3.** There is a bar spectral sequence

$$\mathrm{Tor}^{s,t}_{R_{n*}(Q_n)}(R_{n*}(P), R_{n*}) \Rightarrow R_{n*}(X, H)$$

*Proof.* According to the definition,

$$\begin{aligned} R_{n*}(X, H) &= \pi_*(\Sigma^\infty P_+ \wedge_{Q_n} R_n) \\ &= \pi_*((R_n \wedge \Sigma^\infty P_+) \wedge_{(Q_n \wedge R_n)} R_n). \end{aligned}$$

From this description, it follows immediately that there is a bar spectral sequence

$$\mathrm{Tor}^{s,t}_{R_{n*}(Q_n)}(R_{n*}(P), R_{n*}) \Rightarrow R_{n*}(X, H).$$

Q.E.D.

The Tor groups in this spectral sequence are rather mysterious. In the case of twisted K-theory, all higher Tor groups vanish giving rise to the universal coefficient isomorphism described in theorem 2.1 above. The question is whether the same holds for twisted chromatic theories, that is, whether there is an isomorphism

$$R_{n*}(X, H) \cong R_{n*}(P) \otimes_{R_{n*}Q_n} R_{n*}$$

In order to approach this question, one would like to invert the map

$$\rho_n : S< \det > \to L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1)_+$$

and use the equivalence in theorem 4.1 to establish an isomorphism $R_{n*}Q_n\ [\rho_n^{-1}] \cong R_{n*}R_n$. This would allow one to work with $R_{n*}R_n$ rather than $R_{n*}Q_n$ (see the paragraphs after theorem 2.2).

Unfortunately, such isomorphism does not hold. In fact the left hand side is not even meaningful since for $n \geq 1$, $S < \det >$ is not a sphere, and does not have a Hurewicz image in $R_{n*}T_n$. The spectra $S < \det >$ are only invertible $K(n)$-locally. There is no spectrum $A$ such that $A \wedge S < \det > \simeq S$. Rather, there is a spectrum $A$ such that $L_{K(n)}(A \wedge S < \det >) \simeq L_{K(n)}S$.

Therefore, in order to make sense of $R_{n*}(Q_n)\ [\rho_n^{-1}]$, it seems necessary to modify the definition of $R_{n*}(X)$ to include maps from more invertible $K(n)$-local spectra. This leads us to consider the Picard grading. Let $\mathrm{Pic}_n$ denote the $n$th Picard group.

Following [8], they define for a space $X$ and $A \in \mathrm{Pic}_n$, the group $R^A_{n*}(X)$ as

$$R^A_{n*}(X) = [A, R_n \wedge X]$$

For $A = S$, this yields $R_{n*}(X)$. We define the associated graded ring by taking direct sum over all $\mathrm{Pic}_n$.

$$R_{n\bigstar}(X) = \bigoplus_{A \in \mathrm{Pic}_n} R^A_{n*}(X)$$

By allowing maps from $\mathrm{Pic}_n$ into the Thom spectra $X^H$, one can extend the definition of $R_{n*}(X, H)$ in a similar fashion to obtain $R_{n\bigstar}(X, H)$. The question then becomes whether there is an isomorphism

$$R_{n\bigstar}(X, H) \cong R_{n\bigstar}(P) \otimes_{R_{n\bigstar}Q_n} R_{n\bigstar}.$$

Unfortunately, grading over the Picard group results in major complications in the subsequent algebra, and the author is unable to prove this result at this point.

We believe that a version of the universal coefficient theorem must hold for $R_{n*}(X, H)$, and hope that future research can establish one such isomorphism. In addition, it would be interesting to investigate geometric models for these new twisted theories, similar to geometric models for twisted K-theory.

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