Coefficient estimates for a general subclass of m-fold symmetric bi-univalent functions

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Abstract

In the present paper, we introduce and investigate an interesting subclass $\mathcal{B}_{\Sigma_m}^{h,p}(\lambda,\gamma)$ of m-fold symmetric bi-univalent functions in the open unit disk $\mathbb{U}$. Furthermore, we obtain estimates on the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions belonging to this subclass. The results presented in this paper would generalize and improve some recent works of several earlier authors.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions f which are analytic in the open unit disc $\mathbb{U}=\{z\in\mathbb{C}:\ |z|<1\}$, of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n. \tag{1.1}$$

The subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$ is denoted by $\mathcal{S}$. Every function $f\in\mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z))=z \qquad (z\in\mathbb{U})$$

and

$$f(f^{-1}(w))=w \qquad \left(|w|<r_0(f),\ r_0(f)\geq\frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots. \tag{1.2}$$

A function $f\in\mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$, if both f and f^{-1} are univalent in $\mathbb{U}$. We denote $\sigma_{\mathcal{B}}$ the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). Some examples of bi-univalent functions are given as follows:

$$\frac{z}{1-z},\qquad -\log(1-z)\quad and\quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

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Lewin [14] investigated the class $\sigma_{\mathcal{B}}$ of bi-univalent functions and showed that $|a_2| < 1.51$ for the functions belonging to $\sigma_{\mathcal{B}}$. Subsequently, Brannan and Clunie [4] conjectured that $|a_2| \leq \sqrt{2}$. Netanyahu [15], on the other hand, showed that $\max_{f\in\Sigma} |a_2| = \frac{4}{3}$.
Many recent works, which are devoted to study the bi-univalent functions class $\sigma_{\mathcal{B}}$ and obtain non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. For a brief history and interesting examples of functions in the class $\sigma_{\mathcal{B}}$, one may refer to a pioneeing paper by Srivastava et al. [28]. In fact that this widely-cited work by Srivastava et al. [28] actually revived the study of analytic and bi-univalent functions in recent years and that it has led to a flood of papers on the subject by (for example) Srivastava et al.[7, 21, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 35, 37], and others authors (see, among others,[6, 9, 10, 11, 12, 19, 39, 40]). The coefficient estimate problem i.e. bound of $|a_n|$ $(n \in \mathbb{N} - \{2, 3\})$ for each $f \in \sigma_{\mathcal{B}}$, is still an open problem. In fact there is no direct way to get bound for coefficients greater than three. In special cases there are some papers in which the Faber polynomial methods were used for determining upper bounds for higher-order coefficients (for example see [31, 39, 40]).

For each function $f \in \mathcal{S}$ function

$$h(z) = \sqrt[m]{f(z^m)} \tag{1.3}$$

is univalent and maps unit disk $\mathbb{U}$ into a region with m-fold symmetry. A function f is said to be m-fold symmetric, if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \quad (z \in \mathbb{U}, m \in \mathbb{N}). \tag{1.4}$$

We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in $\mathbb{U}$, which are normalized by the series expansion (1.4). In fact, the functions in class $\mathcal{S}$ are one-fold symmetric.

In [30] Srivastava et al. defined m-fold symmetric bi-univalent functions analogues to the concept of m-fold symmetric univalent functions. They gave some important results, such as each function $f \in \sigma_{\mathcal{B}}$ generates an m-fold symmetric bi-univalent function for each $m \in \mathbb{N}$. Furthermore, for the normalized form of f given by (1.4), they obtained the series expansion for f^{-1} as follows:

$$\begin{aligned} g(w) := f^{-1}(w) &= w - a_{m+1}w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} \\ &-[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}]w^{3m+1} + \cdots . \end{aligned} \tag{1.5}$$

We denote by Σ_m the class of m-fold symmetric bi-univalent functions in $\mathbb{U}$. For $m = 1$, formula (1.5) coincides with formula (1.2) of the class $\sigma_{\mathcal{B}}$. Some examples of m-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}, \qquad \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)^{\frac{1}{m}}\right] \quad \textit{and} \qquad [-\log(1-z^m)]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left(\frac{w^m}{1+w^m}\right)^{\frac{1}{m}}, \qquad \left(\frac{e^{2w^m}-1}{e^{2w^m}+1}\right)^{\frac{1}{m}} \qquad \textit{and} \qquad \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}},$$

respectively.

Recently, many authors investigated bounds for various subclasses of m-fold bi-univalent functions (see [1, 9, 10, 12, 18, 20, 23, 24, 32] and [33]).

The goal of the this paper is to introduce new subclass of Σ_m and obtain estimates on initial coefficients $|a_{m+1}|,\ |a_{2m+1}|$ for functions in subclass and improve some recent works of many authors.

2 Coefficient estimates for the functions subclass $\mathcal{B}_{\Sigma_m}^{h,p}(\lambda,\gamma)$

In this section, we introduce the general subclass $\mathcal{B}_{\Sigma_m}^{h,p}(\lambda,\gamma)$.

Definition 2.1. Let the functions $h, p : \mathbb{U} \to \mathbb{C}$ be so constrained that

$$\min\{\Re\left(\left(h(z)\right), \Re\left(p(z)\right)\right\} > 0 \quad (z \in \mathbb{U}) \quad \textit{and} \quad h(0) = p(0) = 1.$$

A function $f \in \Sigma_m$ given by (1.4) is said to be in the subclass $\mathcal{B}_{\Sigma_m}^{h,p}(\lambda,\gamma)$, if the following conditions are satisfied:

$$\frac{1}{2}\left(\frac{z^{1-\gamma}f'(z)}{\big(f(z)\big)^{1-\gamma}} + \left(\frac{z^{1-\gamma}f'(z)}{\big(f(z)\big)^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right) \in h(\mathbb{U}) \qquad (0 < \lambda \leq 1,\ \gamma \geq 0,\ z \in \mathbb{U}) \tag{2.1}$$

and

$$\frac{1}{2}\left(\frac{w^{1-\gamma}g'(w)}{\big(g(w)\big)^{1-\gamma}} + \left(\frac{w^{1-\gamma}g'(w)}{\big(g(w)\big)^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right) \in p(\mathbb{U}) \qquad (0 < \lambda \leq 1,\ \gamma \geq 0,\ w \in \mathbb{U}), \tag{2.2}$$

where g is the extension of f^{-1} to $\mathbb{U}$ as defined in (1.5).

Remark 2.2. There are many selections of the functions $h(z)$ and $p(z)$ which would provide interesting classes of m-fold symmetric bi-univalent functions Σ_m. For example, if we take

$$h(z) = p(z) = \left(\frac{1+z^m}{1-z^m}\right)^{\alpha} = 1 + 2\alpha z^m + 2\alpha^2 z^{2m} + \cdots \quad (0 < \alpha \leq 1),$$

then the subclass $\mathcal{B}_{\Sigma_m}^{h,p}(\lambda,\gamma)$ reduces to the subclass $\mathcal{SS}_{\Sigma_m}^{*}(\lambda,\gamma;\alpha)$ which was investigated by Wanas et al.[34]. Precisely if $f \in \mathcal{SS}_{\Sigma_m}^{*}(\lambda,\gamma;\alpha)$, then

$$\left|arg\left\{\frac{1}{2}\left(\frac{z^{1-\gamma}f'(z)}{\big(f(z)\big)^{1-\gamma}} + \left(\frac{z^{1-\gamma}f'(z)}{\big(f(z)\big)^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right)\right\}\right| < \frac{\alpha\pi}{2}$$

and

$$\left|arg\left\{\frac{1}{2}\left(\frac{w^{1-\gamma}g'(w)}{\big(g(w)\big)^{1-\gamma}} + \left(\frac{w^{1-\gamma}g'(w)}{\big(g(w)\big)^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right)\right\}\right| < \frac{\alpha\pi}{2}.$$

By setting $\gamma = 0$, the subclass $\mathcal{SS}^*_{\Sigma_m}(\lambda, \gamma; \alpha)$ reduces to the subclass $\mathcal{S}_{\Sigma_m}(\alpha, \lambda)$ which was introduced by Altinkaya et al. [2].

By setting $\gamma = 0$ and $\lambda = 1$, the subclass $\mathcal{SS}^*_{\Sigma_m}(\lambda, \gamma; \alpha)$ reduces to the subclass $\mathcal{S}^{\alpha}_{\Sigma_m}$ which was introduced by Altinkaya et al. [3].

By setting $\gamma = 1$ and $\lambda = 1$, the subclass $\mathcal{SS}^*_{\Sigma_m}(\lambda, \gamma; \alpha)$ reduces to the subclass $\mathcal{H}^{\alpha}_{\Sigma,m}$ which was introduced by Srivastava et al. [30].

If we take

$$h(z) = p(z) = \frac{1 + (1 - 2\beta)z^m}{1 - z^m} = 1 + 2(1 - \beta)z^m + 2(1 - \beta)z^{2m} + \cdots \quad (0 \leq \beta < 1),$$

then the subclass $\mathcal{B}^{h,p}_{\Sigma_m}(\lambda, \gamma)$ reduces to the subclass $\mathcal{S}^*_{\Sigma_m}(\lambda, \gamma; \beta)$ which was investigated by Wanas et al.[34]. Precisely if $f \in \mathcal{S}^*_{\Sigma_m}(\lambda, \gamma; \beta)$ then

$$\Re\left\{\frac{1}{2}\left(\frac{z^{1-\gamma}f'(z)}{\big(f(z)\big)^{1-\gamma}} + \left(\frac{z^{1-\gamma}f'(z)}{\big(f(z)\big)^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right)\right\} > \beta$$

and

$$\Re\left\{\frac{1}{2}\left(\frac{w^{1-\gamma}g'(w)}{\big(g(w)\big)^{1-\gamma}} + \left(\frac{w^{1-\gamma}g'(w)}{\big(g(w)\big)^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right)\right\} > \beta.$$

By setting $\gamma = 0$, the subclass $\mathcal{S}^*_{\Sigma_m}(\lambda, \gamma; \beta)$ reduces to the subclass $\mathcal{S}_{\Sigma_m}(\beta, \lambda)$ which was introduced by Altinkaya et al. [2].

By setting $\gamma = 0$ and $\lambda = 1$, the subclass $\mathcal{S}^*_{\Sigma_m}(\lambda, \gamma; \beta)$ reduces to the subclass $\mathcal{S}^{\beta}_{\Sigma_m}$ which was introduced by Altinkaya et al. [3].

By setting $\gamma = 1$ and $\lambda = 1$, the subclass $\mathcal{S}^*_{\Sigma_m}(\lambda, \gamma; \beta)$ reduces to the subclass $\mathcal{H}_{\Sigma,m}(\beta)$ which was introduced by Srivastava et al. [30].

Remark 2.3. For one-fold symmetric bi-univalent functions, we denote the subclass $\mathcal{B}^{h,p}_{\Sigma_1}(\lambda, \gamma) = \mathcal{B}^{h,p}_{\Sigma}(\lambda, \gamma)$. Special cases of this subclass illustrated below:

- By putting $h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}$ and $\lambda = 1$, then the subclass $\mathcal{B}^{h,p}_{\Sigma}(\lambda, \gamma)$ reduces to the subclass $P_{\Sigma}(\alpha, \gamma)$ that was introduced and studied by Prema et al. [17].
- By putting $h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}$, $\lambda = 1$ and $\gamma = 0$, then the subclass $\mathcal{B}^{h,p}_{\Sigma}(\lambda, \gamma)$ reduces to the subclass $\mathcal{S}^*_{\Sigma}(\alpha)$ that was introduced and studied by Brannan et al. [5].
- By putting $h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}$ and $\lambda = \gamma = 1$, then the subclass $\mathcal{B}^{h,p}_{\Sigma}(\lambda, \gamma)$ reduces to the subclass $\mathcal{H}^{\alpha}_{\Sigma}$ that was introduced and studied by Srivastava et al. [28].
- By putting $h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z}$ and $\lambda = 1$, then the subclass $\mathcal{B}^{h,p}_{\Sigma}(\lambda, \gamma)$ reduces to the subclass $P_{\Sigma}(\beta, \gamma)$ that was introduced and studied by Prema et al. [17]

- By putting $h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z}$, $\lambda = 1$ and $\gamma = 0$, then the subclass $\mathcal{B}_{\Sigma}^{h,p}(\lambda, \gamma)$ reduces to the subclass $\mathcal{S}_{\Sigma}^{*}(\beta)$ that was introduced and studied by Brannan et al. [5].
- By putting $h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z}$ and $\lambda = \gamma = 1$, then the subclass $\mathcal{B}_{\Sigma}^{h,p}(\lambda, \gamma)$ reduces to the subclass $\mathcal{H}_{\Sigma}(\beta)$ that was introduced and studied by Srivastava et al. [28].

Theorem 2.4. *Let $f(z)$ given by (1.4) be in subclass $\mathcal{B}_{\Sigma_m}^{h,p}(\lambda, \gamma)$ $(0 < \lambda \leq 1,\ \gamma \geq 0)$. Then*

$$|a_{m+1}| \leq \min\left\{ \frac{\lambda\sqrt{2(|h_m|^2 + |p_m|^2)}}{(m+\gamma)(1+\lambda)}, \frac{2\lambda}{(m+\gamma)}\sqrt{\frac{(|h_{2m}| + |p_{2m}|)}{2\lambda(\lambda+1)(1+\frac{m}{m+\gamma}) + 2(1-\lambda)}} \right\}$$

and

$$\begin{aligned} |a_{2m+1}| \leq \min\Bigg\{ & \frac{\lambda^2(m+1)(|h_m|^2 + |p_m|^2)}{(m+\gamma)^2(\lambda+1)^2} + \frac{\lambda(|h_{2m}| + |p_{2m}|)}{(2m+\gamma)(\lambda+1)} \\ & , \frac{\lambda^2(\lambda+1)(2m+\gamma)(2m+\gamma+1) + \lambda(1-\lambda)(m+\gamma)^2}{(m+\gamma)(2m+\gamma)(\lambda+1)\big[\lambda(\lambda+1)(2m+\gamma) + (m+\gamma)(1-\lambda)\big]}|h_{2m}| \\ & + \frac{\big|\lambda^2(\lambda+1)(2m+\gamma)(1-\gamma) - \lambda(1-\lambda)(m+\gamma)^2\big|}{(m+\gamma)(2m+\gamma)(\lambda+1)\big[\lambda(\lambda+1)(2m+\gamma) + (m+\gamma)(1-\lambda)\big]}|p_{2m}| \Bigg\}. \end{aligned}$$

Proof. First of all, we write the argument inequalities in (2.1) and (2.2) in their equivalent forms as follows:

$$\frac{1}{2}\left(\frac{z^{1-\gamma}f'(z)}{\big(f(z)\big)^{1-\gamma}} + \left(\frac{z^{1-\gamma}f'(z)}{\big(f(z)\big)^{1-\gamma}} \right)^{\frac{1}{\lambda}} \right) = h(z) \quad (z \in \mathbb{U}) \tag{2.3}$$

and

$$\frac{1}{2}\left(\frac{w^{1-\gamma}g'(w)}{\big(g(w)\big)^{1-\gamma}} + \left(\frac{w^{1-\gamma}g'(w)}{\big(g(w)\big)^{1-\gamma}} \right)^{\frac{1}{\lambda}} \right) = p(w) \quad (w \in \mathbb{U}), \tag{2.4}$$

respectively, where functions $h(z)$ and $p(w)$ satisfy the conditions of Definition 2.1.

Furtheremore, the functions $h(z)$ and $p(w)$ have the forms:

$$h(z) = 1 + h_m z^m + h_{2m} z^{2m} + h_{3m} z^{3m} + \cdots \tag{2.5}$$

and

$$p(w) = 1 + p_m w^m + p_{2m} w^{2m} + p_{3m} w^{3m} + \cdots, \tag{2.6}$$

respectively.

Now, upon substituting from (2.5) and (2.6) into (2.3) and (2.4), respectively, and equating the coefficients, we get

$$\frac{(m+\gamma)(1+\lambda)}{2\lambda} a_{m+1} = h_m, \tag{2.7}$$

$$\frac{(2m+\gamma)(1+\lambda)}{4\lambda}(2a_{2m+1} + (\gamma-1)a_{m+1}^2) + \frac{(m+\gamma)^2(1-\lambda)}{4\lambda^2} a_{m+1}^2 = h_{2m}, \tag{2.8}$$

$$\frac{-(m+\gamma)(1+\lambda)}{2\lambda} a_{m+1} = p_m, \tag{2.9}$$

and

$$\frac{(2m+\gamma)(1+\lambda)}{4\lambda}[(2m+\gamma+1)a_{m+1}^2-2a_{2m+1}]+\frac{(m+\gamma)^2(1-\lambda)}{4\lambda^2}a_{m+1}^2=p_{2m}. \tag{2.10}$$

From (2.7) and (2.9), we get

$$h_m=-p_m$$

and

$$a_{m+1}^2=\frac{2\lambda^2(h_m^2+p_m^2)}{(m+\gamma)^2(1+\lambda)^2}. \tag{2.11}$$

Adding (2.8) and (2.10), we get

$$a_{m+1}^2=\frac{2\lambda^2(h_{2m}+p_{2m})}{(m+\gamma)^2\big[\lambda(\lambda+1)(1+\frac{m}{m+\gamma})+(1-\lambda)\big]}. \tag{2.12}$$

Therefore, we find from the equations (2.11) and (2.12) that

$$|a_{m+1}|\leq\frac{\lambda\sqrt{2(|h_m|^2+|p_m|^2)}}{(m+\gamma)(1+\lambda)},$$

$$|a_{m+1}|\leq\frac{2\lambda}{(m+\gamma)}\sqrt{\frac{(|h_{2m}|+|p_{2m}|)}{2\lambda(\lambda+1)(1+\frac{m}{m+\gamma})+2(1-\lambda)}}.$$

respectively. So we get the desired estimate on the coefficient $|a_{m+1}|$.

Next, in order to find the bound on the coefficient $|a_{2m+1}|$, we subtract (2.10) from (2.8), we get

$$a_{2m+1}=\frac{(m+1)}{2}a_{m+1}^2+\frac{\lambda(h_{2m}-p_{2m})}{(2m+\gamma)(\lambda+1)}. \tag{2.13}$$

Therefore, we find from (2.11) and (2.13)

$$a_{2m+1}=\frac{\lambda^2(m+1)(h_m^2+p_m^2)}{(m+\gamma)^2(\lambda+1)^2}+\frac{\lambda(h_{2m}-p_{2m})}{(2m+\gamma)(\lambda+1)}.$$

We obtain from (2.12) and (2.13)

$$a_{2m+1}=\frac{\lambda^2(m+1)(h_{2m}+p_{2m})}{(m+\gamma)\big[\lambda(\lambda+1)(2m+\gamma)+(m+\gamma)(1-\lambda)\big]}+\frac{\lambda(h_{2m}-p_{2m})}{(2m+\gamma)(\lambda+1)}.$$

So

$$a_{2m+1}=\frac{\lambda^2(\lambda+1)(2m+\gamma)(2m+\gamma+1)+\lambda(1-\lambda)(m+\gamma)^2}{(m+\gamma)(2m+\gamma)(\lambda+1)\big[\lambda(\lambda+1)(2m+\gamma)+(m+\gamma)(1-\lambda)\big]}h_{2m}$$
$$+\frac{\lambda^2(\lambda+1)(2m+\gamma)(1-\gamma)-\lambda(1-\lambda)(m+\gamma)^2}{(m+\gamma)(2m+\gamma)(\lambda+1)\big[\lambda(\lambda+1)(2m+\gamma)+(m+\gamma)(1-\lambda)\big]}p_{2m}.$$

Q.E.D.

3 Corollaries and Consequences

By putting

$$h(z) = p(z) = \left(\frac{1+z^m}{1-z^m}\right)^{\alpha} = 1 + 2\alpha z^m + 2\alpha^2 z^{2m} + \cdots \ (0 < \alpha \leq 1,\ z \in \mathbb{U})$$

in Theorem 2.4, we conclude the following result.

Corollary 3.1. *Let $f(z)$ given by (1.4) be in subclass $\mathcal{SS}^{*}_{\Sigma,m}(\lambda,\gamma;\alpha)$ $(0 < \lambda \leq 1, \gamma \geq 0,\ 0 < \alpha \leq 1,\ m \in \mathbb{N})$. Then*

$$|a_{m+1}| \leq \min\left\{\frac{4\alpha\lambda}{(m+\gamma)(1+\lambda)}, \frac{4\alpha\lambda}{(m+\gamma)\sqrt{2\lambda(\lambda+1)(1+\frac{m}{m+\gamma})+2(1-\lambda)}}\right\}$$

and

$$\begin{aligned}|a_{2m+1}| \leq \min\Bigg\{&\frac{8\lambda^2(m+1)\alpha^2}{(m+\gamma)^2(\lambda+1)^2} + \frac{4\alpha^2\lambda}{(2m+\gamma)(\lambda+1)}\\ &, \frac{2\big[\lambda^2(\lambda+1)(2m+\gamma)(2m+\gamma+1)+\lambda(1-\lambda)(m+\gamma)^2\big]\alpha^2}{(m+\gamma)(2m+\gamma)(\lambda+1)\big[\lambda(\lambda+1)(2m+\gamma)+(m+\gamma)(1-\lambda)\big]}\\ &+\frac{2\big|\lambda^2(\lambda+1)(2m+\gamma)(1-\gamma)-\lambda(1-\lambda)(m+\gamma)^2\big|\alpha^2}{(m+\gamma)(2m+\gamma)(\lambda+1)\big[\lambda(\lambda+1)(2m+\gamma)+(m+\gamma)(1-\lambda)\big]}\Bigg\}.\end{aligned}$$

Remark 3.2. The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 3.1 are better than those given in [34, Theorem 1]. Because

$$\begin{aligned}&\frac{4\alpha\lambda}{(m+\gamma)\sqrt{2\lambda(\lambda+1)(1+\frac{m}{m+\gamma})+2(1-\lambda)}} \leq\\ &\qquad\frac{4\alpha\lambda}{(m+\gamma)\sqrt{(\lambda+1)\left[2\lambda\alpha(\frac{m}{m+\gamma}+1)+(1-\alpha)(\lambda+1)\right]+2\alpha(1-\lambda)}}\end{aligned}$$

and

$$\frac{8\lambda^2(m+1)\alpha^2}{(m+\gamma)^2(\lambda+1)^2} + \frac{4\alpha^2\lambda}{(2m+\gamma)(\lambda+1)} \leq \frac{8\lambda^2(m+1)\alpha^2}{(m+\gamma)^2(\lambda+1)^2} + \frac{4\alpha\lambda}{(2m+\gamma)(\lambda+1)}.$$

By setting $\gamma = 0$ in Corollary 3.1, we conclude the following result.

Corollary 3.3. *Let $f(z)$ given by (1.4) be in subclass $\mathcal{S}_{\Sigma_m}(\alpha,\lambda)$ $(0 < \lambda \leq 1,\ 0 < \alpha \leq 1,\ m \in \mathbb{N})$. Then*

$$|a_{m+1}| \leq \min\left\{\frac{4\alpha\lambda}{m(1+\lambda)}, \frac{4\alpha\lambda}{m\sqrt{4\lambda^2+2\lambda+2}}\right\}$$

and

$$|a_{2m+1}| \leq \min\left\{\frac{8(m+1)\lambda^2\alpha^2}{m^2(\lambda+1)^2}+\frac{2\alpha^2\lambda}{m(\lambda+1)}\right.$$
$$\left.,\frac{[2\lambda^2(\lambda+1)(2m+1)+\lambda(1-\lambda)m]+\left|2\lambda^2(\lambda+1)-\lambda(1-\lambda)m\right|}{m^2(\lambda+1)\left[2\lambda^2+\lambda+1\right]}\alpha^2\right\}.$$

Remark 3.4. The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 3.3 are better than those given in [2, Theorem 1]. Because

$$\frac{4\alpha\lambda}{m\sqrt{4\lambda^2+2\lambda+2}} \leq \frac{4\alpha\lambda}{m\sqrt{(\lambda+1)\left[4\lambda\alpha+(1-\alpha)(\lambda+1)\right]+2\alpha(1-\lambda)}}$$

and

$$\frac{8(m+1)\lambda^2\alpha^2}{m^2(\lambda+1)^2}+\frac{2\alpha^2\lambda}{m(\lambda+1)} \leq \frac{8(m+1)\lambda^2\alpha^2}{m^2(\lambda+1)^2}+\frac{2\alpha\lambda}{m(\lambda+1)}.$$

By setting $\gamma=0$ and $\lambda=1$ in Corollary 3.1, we conclude the following result.

Corollary 3.5. *Let $f(z)$ given by (1.4) be in subclass $\mathcal{S}^{\alpha}_{\Sigma_m}$ $(0<\alpha\leq 1,\ m\in\mathbb{N})$. Then*

$$|a_{m+1}| \leq \min\left\{\frac{2\alpha}{m},\frac{\sqrt{2}\alpha}{m}\right\}=\frac{\sqrt{2}\alpha}{m}$$

and

$$|a_{2m+1}| \leq \min\left\{\frac{2(m+1)\alpha^2}{m^2}+\frac{\alpha^2}{m},\frac{(m+1)\alpha^2}{m^2}\right\}=\frac{(m+1)\alpha^2}{m^2}.$$

Remark 3.6. The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 3.5 are better than those given in [3, Corollary 6]. Because

$$\frac{\sqrt{2}\alpha}{m} \leq \frac{2\alpha}{m\sqrt{\alpha+1}}$$

and

$$\frac{(m+1)\alpha^2}{m^2} \leq \frac{2(m+1)\alpha^2}{m^2}+\frac{\alpha}{m}.$$

By setting $\gamma=1$ and $\lambda=1$ in Corollary 3.1, we conclude the following result.

Corollary 3.7. *Let $f(z)$ given by (1.4) be in subclass $\mathcal{H}^{\alpha}_{\Sigma,m}$ $(0<\alpha\leq 1,\ m\in\mathbb{N})$. Then*

$$|a_{m+1}| \leq \min\left\{\frac{2\alpha}{m+1},\frac{2\alpha}{\sqrt{(m+1)(2m+1)}}\right\}$$

and

$$|a_{2m+1}| \leq \min\left\{\frac{2\alpha^2}{m+1}+\frac{2\alpha^2}{2m+1},\frac{2\alpha^2}{2m+1}\right\}=\frac{2\alpha^2}{2m+1}.$$

Remark 3.8. The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 3.7 are better than those given in [30, Theorem 2]. Because

$$\frac{2\alpha}{\sqrt{(m+1)(2m+1)}} \leq \frac{2\alpha}{\sqrt{(m+1)(\alpha m+m+1)}}$$

and

$$\frac{2\alpha^2}{2m+1} \leq \frac{2\alpha\big[(2m+1)\alpha+m+1\big]}{(m+1)(2m+1)}.$$

By putting

$$h(z)=p(z)=\frac{1+(1-2\beta)z^m}{1-z^m}=1+2(1-\beta)z^m+2(1-\beta)z^{2m}+\cdots \ (0\leq\beta<1,\ z\in\mathbb{U})$$

in Theorem 2.4, we conclude the following result.

Corollary 3.9. *Let $f(z)$ given by (1.4) be in subclass $\mathcal{S}^*_{\Sigma_m}(\lambda,\gamma;\beta)$ $(0<\lambda\leq 1, \gamma\geq 0,\ 0\leq\beta<1,\ m\in\mathbb{N})$. Then*

$$|a_{m+1}|\leq \min\left\{\frac{4\lambda(1-\beta)}{(m+\gamma)(1+\lambda)}, \frac{2\lambda}{(m+\gamma)}\sqrt{\frac{2(1-\beta)}{\big(1+\frac{m}{m+\gamma}\big)\lambda^2+\frac{m}{m+\gamma}\lambda+1}}\ \right\}$$

and

$$\begin{aligned}|a_{2m+1}|\leq \min\Bigg\{&\frac{8\lambda^2(m+1)(1-\beta)^2}{(m+\gamma)^2(\lambda+1)^2}+\frac{4\lambda(1-\beta)}{(2m+\gamma)(\lambda+1)}\\ &,\frac{2\big\{\lambda^2(\lambda+1)(2m+\gamma)(2m+\gamma+1)+\lambda(1-\lambda)(m+\gamma)^2\big\}(1-\beta)}{(m+\gamma)(2m+\gamma)(\lambda+1)\big[\lambda(\lambda+1)(2m+\gamma)+(m+\gamma)(1-\lambda)\big]}\\ &+\frac{2\big|\lambda^2(\lambda+1)(2m+\gamma)(1-\gamma)-\lambda(1-\lambda)(m+\gamma)^2\big|(1-\beta)}{(m+\gamma)(2m+\gamma)(\lambda+1)\big[\lambda(\lambda+1)(2m+\gamma)+(m+\gamma)(1-\lambda)\big]}.\Bigg\}.\end{aligned}$$

Remark 3.10. The bound on $|a_{m+1}|$ given in Corollary 3.9 is better than those given in [34, Theorem 2]. Because

$$\frac{4\lambda(1-\beta)}{(m+\gamma)(1+\lambda)}\leq\frac{2\lambda}{(m+\gamma)}\sqrt{\frac{2(1-\beta)}{\big(1+\frac{m}{m+\gamma}\big)\lambda^2+\frac{m}{m+\gamma}\lambda+1}};\quad \beta\geq\frac{(3m+\gamma)\lambda^2-2\gamma\lambda+m+\gamma}{2\big[(2m+\gamma)\lambda^2+m\lambda+m+\gamma\big]}.$$

By setting $\gamma=0$ in Corollary 3.9, we conclude the following result.

Corollary 3.11. *Let $f(z)$ given by (1.4) be in subclass $\mathcal{S}_{\Sigma_m}(\alpha,\lambda)$ $(0<\lambda\leq 1,\ 0\leq\beta<1,\ m\in\mathbb{N})$. Then*

$$|a_{m+1}|\leq\min\left\{\frac{4\lambda(1-\beta)}{m(1+\lambda)},\frac{2\lambda}{m}\sqrt{\frac{2(1-\beta)}{2\lambda^2+\lambda+1}}\right\}$$

and

$$|a_{2m+1}| \leq \min\left\{ \frac{\left[2(2m+1)\lambda^2(\lambda+1)+\lambda(1-\lambda)m\right]+\left|2\lambda^2(1+\lambda)-\lambda(1-\lambda)m\right|}{m^2(\lambda+1)\left[2\lambda^2+\lambda+1\right]}(1-\beta) \right.$$
$$\left., \frac{8\lambda^2(m+1)(1-\beta)^2}{m^2(\lambda+1)^2}+\frac{2\lambda(1-\beta)}{m(\lambda+1)} \right\}.$$

Remark 3.12. The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 3.11 are better than those given in [2, Theorem 2]. Because

$$\frac{4\lambda(1-\beta)}{m(1+\lambda)} \leq \frac{2\lambda}{m}\sqrt{\frac{2(1-\beta)}{2\lambda^2+\lambda+1}}; \quad \beta \geq \frac{3\lambda^2+1}{2(2\lambda^2+\lambda+1)},$$

if $\lambda(1-\lambda)m \leq 2\lambda^2(1+\lambda)$, we have

$$\frac{4\lambda^2(m+1)(1-\beta)}{m^2[2\lambda^2+\lambda+1]} \leq \frac{8\lambda^2(m+1)(1-\beta)^2}{m^2(\lambda+1)^2}+\frac{2\lambda(1-\beta)}{m(\lambda+1)}; \quad \beta \leq \frac{3\lambda^2+1}{2(2\lambda^2+\lambda+1)}$$

and if $\lambda(1-\lambda)m \geq 2\lambda^2(1+\lambda)$, we have

$$\frac{4\lambda^2(1-\beta)}{m[2\lambda^2+\lambda+1]}+\frac{2\lambda(1-\lambda)(1-\beta)}{m(1+\lambda)[2\lambda^2+\lambda+1]} \leq \frac{8\lambda^2(m+1)(1-\beta)^2}{m^2(\lambda+1)^2}+\frac{2\lambda(1-\beta)}{m(\lambda+1)};$$
$$\beta \leq \frac{m(3\lambda^2+1)+2(2\lambda^2+\lambda+1)}{2(m+1)(2\lambda^2+\lambda+1)}.$$

By setting $\gamma = 0$ and $\lambda = 1$ in Corollary 3.9, we conclude the following result.

Corollary 3.13. *Let $f(z)$ given by (1.4) be in subclass $\mathcal{S}^{\beta}_{\Sigma_m}$ $(0 \leq \beta < 1,\ m \in \mathbb{N})$. Then*

$$|a_{m+1}| \leq \min\left\{ \frac{2(1-\beta)}{m}, \frac{\sqrt{2(1-\beta)}}{m} \right\}$$

and

$$|a_{2m+1}| \leq \min\left\{ \frac{2(m+1)(1-\beta)^2}{m^2}+\frac{1-\beta}{m}, \frac{(m+1)(1-\beta)}{m^2} \right\}.$$

Remark 3.14. The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 3.13 are better than those given in [3, Corollary 7]. Because

$$\frac{2(1-\beta)}{m} \leq \frac{\sqrt{2(1-\beta)}}{m}; \quad \beta \geq \frac{1}{2}$$

and

$$\frac{(m+1)(1-\beta)}{m^2} \leq \frac{2(m+1)(1-\beta)^2}{m^2}+\frac{1-\beta}{m}; \quad \beta \leq \frac{1}{2}.$$

By setting $\gamma = 1$ and $\lambda = 1$ in Corollary 3.9, we conclude the following result.

Corollary 3.15. *Let $f(z)$ given by (1.4) be in subclass $\mathcal{H}_{\Sigma,m}(\beta)$ $(0 \leq \beta < 1,\ m \in \mathbb{N})$. Then*

$$|a_{m+1}| \leq \min\left\{\frac{2(1-\beta)}{m+1}, 2\sqrt{\frac{1-\beta}{(m+1)(2m+1)}}\right\}$$

and

$$|a_{2m+1}| \leq \min\left\{\frac{2(1-\beta)^2}{(m+1)} + \frac{2(1-\beta)}{2m+1}, \frac{2(1-\beta)}{2m+1}\right\} = \frac{2(1-\beta)}{2m+1}.$$

Remark 3.16. The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 3.15 are better than those given in [30, Theorem 3]. Because

$$\frac{2(1-\beta)}{m+1} \leq 2\sqrt{\frac{1-\beta}{(m+1)(2m+1)}}; \quad \beta \geq \frac{m}{2m+1}$$

and

$$\frac{2(1-\beta)}{2m+1} \leq \frac{2(1-\beta)^2}{(m+1)} + \frac{2(1-\beta)}{2m+1}.$$

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