Inequalities for complex rational functions

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Abstract

In this paper, we consider a class of rational functions $r(s(z))$ of degree at most mn, where $s(z)$ is a polynomial of degree m and obtain a certain sharp compact generalization of well-known inequalities for rational functions.

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1 Introduction

Let $p(z)$ be a polynomial of degree at most n and $T := \{z : |z| = 1\}$. We denote by D_- the region inside T and by D_+ the region outside T.
According to a well-known result as Bernstein's inequality on the derivative of $p(z)$, we have

$$\max_{z\in T} |p^{'}(z)| \leq n \max_{z\in T} |p(z)|. \tag{1.1}$$

The result is best possible and equality holds for a polynomial having all its zeros at the origin (see [3], [11]). The inequality (1.1) can be sharpened by considering the class of polynomials having no zeros in D_-. In fact, P. Erdös conjectured and later Lax [8] proved that if $p(z) \neq 0$ in D_-, then(1.1) can be replaced by

$$\max_{z\in T} |p^{'}(z)| \leq \frac{n}{2} \max_{z\in T} |p(z)|. \tag{1.2}$$

Turán [12] reversed the hypothesis of the result proved by Erdös-Lax and showed that if $p(z) \neq 0$ in D_+, then

$$\max_{z\in T} |p^{'}(z)| \geq \frac{n}{2} \max_{z\in T} |p(z)|. \tag{1.3}$$

Aziz and Dawood [1] obtained the following refinement of the inequality (1.3) and proved that if $p(z)$ has all its zeros in $T \cup D_-$, then

$$\max_{z\in T} |p^{'}(z)| \geq \frac{n}{2}\{\max_{z\in T} |p(z)| + \min_{z\in T} |p(z)|\}. \tag{1.4}$$

In 1930, Bernstein [4] revisited his inequality and established the following comparative result by assuming that $p(z)$ and $q(z)$ are polynomials such as $p(z)$ has at most of degree n and $q(z)$ has exactly n zeros in $T \cup D_-$ and for $z \in T$

$$|p(z)| \leq |q(z)|,$$

then for $z \in T$

$$|p'(z)| \leq |q'(z)|. \tag{1.5}$$

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Let α be a complex number. For a polynomial $p(z)$ of degree n, $D_\alpha p(z)$, the polar derivative of $p(z)$ is defined as

$$D_\alpha p(z) = np(z) + (\alpha - z)p^{'}(z).$$

The polynomial $D_\alpha p(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}[\frac{D_\alpha p(z)}{\alpha}] = p^{'}(z). \tag{1.6}$$

For the polar derivative $D_\alpha p(z)$, Aziz and Rather [2] generalized the inequality (1.3) to the polar derivative of a polynomial. In fact, they proved that if all zeros of $p(z)$ lie in $T \cup D_-$, then for every real or complex number α with $|\alpha| \geq 1$, we get

$$\max_{z\in T}|D_\alpha p(z)| \geq \frac{n}{2}(|\alpha| - 1)\max_{z\in T}|p(z)|. \tag{1.7}$$

2 Rational functions

Let $P_{n'}$ is a class of all polynomials of degree at most n' and $a_1, a_2, ..., a_n$ be n given points in D_+. Consider the following space of rational functions with prescribed poles and with finite limit at infinity.

$$R_{n\prime,n} = R_{n\prime,n}(a_1, ..., a_n) = \{\frac{p(z)}{w(z)} : p \in \mathrm{P}_{n\prime}\},$$

where

$$w(z) = \Pi_{j=1}^{n}(z - a_j).$$

The inequalities of Bernstein and Erdös-Lax have been extended to the rational functions ([5], [10]) by replacing the polynomial $p(z)$ by rational functions $r(z)$ and z^n by Blaschke product $B(z)$ defined by

$$B(z) = \frac{w^*(z)}{w(z)} = \frac{z^n\overline{w(\frac{1}{\bar{z}})}}{w(z)} = \Pi_{j=1}^{n}(\frac{1 - \bar{a}_j z}{z - a_j}).$$

Li et al.([10], [9]) obtained Bernstein-type inequalities for rational function $r(z)$.They proved that if the rational function $r(z) \in R_{n,n}$ having all its zeros in $T \cup D_-$, then for $z \in T$

$$|r^{'}(z)| \geq \frac{1}{2}|B^{'}(z)||r(z)|. \tag{2.1}$$

The inequality is sharp and equality holds if $r(z) = \alpha B(z) + \beta$ with α,$\beta \in T$.
Also Xin Li [9] extended the inequality (1.5) for rational functions by showing that, if $r(z)$, $s(z) \in R_{n,n}$ such that $s(z)$ has all its n zeros in $T \cup D_-$ and for $z \in T$

$$|r(z)| \leq |s(z)|,$$

then for $z \in T$

$$|r'(z)| \leq |s'(z)|. \tag{2.2}$$

Recently, Hans and Tripathi [6] proved that, if $r(z)$, $s(z) \in R_{n,n}$ such that $s(z)$ has all its n zeros in $T \cup D_-$ and $|r(z)| \leq |s(z)|$ for $z \in T$, then for every real or complex number β with $|\beta| \leq 1$ and $z \in T$

$$|\ zr'(z) + \frac{\beta}{2}|B'(z)|r(z)\ | \leq |\ zs'(z) + \frac{\beta}{2}|B'(z)|s(z)\ |. \tag{2.3}$$

Also, Hans and Tripathi [6] obtained that, if $r(z) \in R_{n,n}$, then for every real or complex number β with $|\beta| \leq 1$ and $z \in T$

$$| \ zr'(z) + \frac{\beta}{2}|B'(z)|r(z) \ | \leq |1 + \frac{\beta}{2}||B'(z)| \ \max_{z\in T}|r(z)|. \tag{2.4}$$

In this paper, we consider a more general class of rational functions $r(s(z))$, defined by

$$ros(z) = r(s(z)) = \frac{p(s(z))}{w(s(z))},$$

where $s(z)$ is a polynomial of degree m, $m \geq 1$ and $r(z) \in R_{n',n}$ $(n' \leq n)$, so that $ros(z) \in R_{mn',mn}$ and

$$w(s(z)) = \Pi_{j=1}^{mn}(z - a_j).$$

Now, Balachke product is given by

$$B(z) = \frac{w^*(s(z))}{w(s(z))} = \frac{z^{mn}\overline{w(s(\frac{1}{\bar{z}}))}}{w(s(z))} = \Pi_{j=1}^{mn}(\frac{1 - \bar{a}_j z}{z - a_j}),$$

thereby proving the following results which in turn generalize the above inequalities with suitable modification.

3 Main results

Theorem 3.1. If $r(s(z)) \in R_{mn\prime,mn}$ having all its zeros in $T \cup D_-$ with t-fold zeros at the origin, then for $z \in T$

$$\begin{aligned} |(r(s(z)))'| \geq \frac{1}{2}\{&(|B'(z)| - m(n - n') + t)|r(s(z))| + \\ &(|B'(z)| - m(n - n') - t)\min_{z\in T}|r(s(z))|\}. \end{aligned} \tag{3.1}$$

Remark 3.1. In particle case if we consider $p(s(z))$ as a polynomial of degree mn, then for rational function $r(s(z)) = \dfrac{p(s(z))}{(z-\alpha)^{mn}}$, we have

$$B'(z) = mn\frac{|\alpha|^2 - 1}{(z - \alpha)^2}(\frac{1 - \overline{\alpha}z}{z - \alpha})^{mn-1}.$$

Hence for $z \in T$

$$|B'(z)| = \frac{mn(|\alpha|^2 - 1)}{|z - \alpha|^2}.$$

Now for $\min_{z\in T}|r(s(z))| = r(s(z_0))$, we get

$$\frac{|D_\alpha p(s(z))|}{|z - \alpha|^{mn+1}} \geq \frac{1}{2}\{(\frac{mn(|\alpha|^2 - 1)}{|z - \alpha|^2} + t)\frac{|p(s(z))|}{|z - \alpha|^{mn}} +$$

$$(\frac{mn(|\alpha|^2-1)}{|z-\alpha|^2}-t)\frac{|p(s(z_0))|}{|z_0-\alpha|^{mn}}\},$$

or

$$|D_\alpha p(s(z))| \geq \frac{1}{2}\{(\frac{mn(|\alpha|^2-1)}{|z-\alpha|}+t|z-\alpha|)|p(s(z))|+$$
$$(\frac{mn(|\alpha|^2-1)}{|z-\alpha|}-t|z-\alpha|)\frac{|z-\alpha|^{mn}}{|z_0-\alpha|^{mn}}|p(s(z_0))|\},$$

or

$$|D_\alpha p(s(z))| \geqslant \frac{1}{2}\{(mn+t)(|\alpha|-1)|p(s(z))|+$$
$$[(mn-t)|\alpha|-(mn+t)](\frac{|\alpha|-1}{|\alpha|+1})^{mn}|p(s(z_0))|\}.$$

Therefore, we obtain the following refinement and generalization of inequality (1.7) for the polar derivatives of a polynomial.

Corollary 3.1. Let $p(z)$ and $s(z)$ be polynomials of degrees n and m respectively, also $p(s(z))$ has all its zeros in $T \cup D_-$, with t-fold zeros at the origin. Then we have for every real or complex number α with $|\alpha| \geq 1$ and $z \in T$

$$\begin{aligned}|D_\alpha p(s(z))| &\geq \frac{1}{2}\{(mn+t)(|\alpha|-1)|p(s(z))|+\\ &[(mn-t)|\alpha|-(mn+t)](\frac{|\alpha|-1}{|\alpha|+1})^{mn}\min_{z\in T}|p(s(z))|\}.\end{aligned} \tag{3.2}$$

Dividing two sides of inequality (3.2) by $|\alpha|$ and letting $|\alpha| \to \infty$, we have the following generalization of the inequality (1.4).

Corollary 3.2. Let $p(z)$ and $s(z)$ be polynomials of degrees n and m respectivevely, also $p(s(z))$ has all its zeros in $T \cup D_-$, with t-fold zeros at the origin. Then we have for $z \in T$

$$|(p(s(z))'| \geq \frac{mn+t}{2}|p(s(z))| + \frac{mn-t}{2}\min_{z\in T}|p(s(z))|.$$

Next, we use $\min_{z\in T}|r(s(z))|$ to obtain the following refinement and generalization of inequality (2.1) and also provide a refinement of the result due to Idrees Qusim and Liman [7], more precisely:

Theorem 3.2. If $r(s(z)) \in R_{mn',mn}$ has all its zeros in $T \cup D_-$ with t-fold zeros at the origin, then for $z \in T$ and

$$M' = \max_{z\in T}|s(z)|,$$

$$\begin{aligned}|r'(s(z))| \geq \frac{1}{2mM'}\{&(|B'(z)|-m(n-n')+t)|r(s(z))|+\\ &(|B'(z)|-m(n-n')-t)\min_{z\in T}|r(s(z))|\}.\end{aligned} \tag{3.3}$$

If $n = n'$, then we have the following result which is an extension and improvement of inequality (2.1).

Corollary 3.3. If $r(s(z)) \in R_{mn,mn}$ has all its zeros in $T \cup D_-$ with t-fold zeros at the origin, then for $z \in T$ and $M^{'} = \max_{z \in T} |s(z)|$,

$$|r^{'}(s(z))| \geq \frac{1}{2mM^{'}} \{(|B^{'}(z)| + t)|r(s(z))| + \tag{3.4}$$

$$(|B^{'}(z)| - t) \min_{z \in T} |r(s(z))|\}.$$

Finally, we will prove the following result, which is an extension of inequality (2.3) for rational functions.

Theorem 3.3. Let $r(s(z)), u(s(z)) \in R_{mn',mn}$ and all the zeros of $u(s(z))$ lie in $T \cup D_-$. If both of them having t-fold of zeros at origin and we have for $z \in T$

$$|r(s(z))| \leq |u(s(z))|,$$

then for every real or complex number ρ with $|\rho| \leq \frac{1}{2}$

$$|z(r(s(z)))' + \rho(|B'(z)| - m(n - n') + t)r(s(z))| \tag{3.5}$$

$$\leq |z(u(s(z)))' + \rho(|B'(z)| - m(n - n') + t)u(s(z))|.$$

If we take

$$u(s(z)) = B(z) \max_{z \in T} |r(s(z))|$$

in above inequality, then $t = 0$ and we have the following result which is an extension of inequality (2.4) for rational functions.

Corollary 3.4. If $r(s(z)) \in R_{mn',mn}$, then for every real or complex ρ with $|\rho| \leq \frac{1}{2}$ and for $z \in T$

$$|z(r(s(z)))' + \rho(|B'(z)| - m(n - n'))r(s(z))| \tag{3.6}$$

$$\leq \{|1 + \rho||B'(z)| + |\rho|\ m(n - n')\} \max_{z \in T} |r(s(z))|.$$

4 Lemmas

For the proofs of these theorems, we need the following lemmas.

Lemma 4.1. If $r(s(z)) \in R_{mn',mn}$ having all its zeros in $T \cup D_-$ with t-fold zeros at origin, then for $r(s(z)) \neq 0$ and $z \in T$, we have

$$Re(\frac{z(r(s(z)))^{'}}{r(s(z))}) \geq \frac{1}{2}\{t + |B^{'}(z)| - m(n - n^{'})\}.$$

Proof of Lemma 4.1. Let $r(s(z)) = \dfrac{p(s(z))}{w(s(z))}$ where $p(z)$ has $n^{'}$ zeros and $s(z)$ has m ones; therefore, $p(s(z))$ has $mn^{'}$ zeros. If $r(s(z))$ has t-fold zeros at the origin, then

$$r(s(z)) = z^t \frac{\prod_{i=1}^{mn^{'}-t}(z-b_i)}{\prod_{j=1}^{mn}(z-a_j)},$$

where b_i, $|b_i| \leq 1; i = 1, ..., mn^{'} - t$, are the zeros of $p(s(z))$ and $n^{'} \leq n$.
Now it gives

$$z(\frac{(r(s(z)))^{'}}{r(s(z))}) = t + \sum_{i=1}^{mn^{'}-t} \frac{z}{z-b_i} - \sum_{j=1}^{mn} \frac{z}{z-a_j}. \tag{4.1}$$

Since all zeros $p(s(z))$ lie in $T \cup D_-$, for $z \in T$ with $z \neq b_i$; $i = 1, ..., mn^{'} - t$, we have

$$|\frac{z}{z-b_i}| \geq |\frac{z}{z-b_i} - 1|. \tag{4.2}$$

Using the fact that $Re(z) \geq \frac{1}{2}$ if and only if $|z| \geq |z-1|$, we get from inequality (4.2),

$$Re(\frac{z}{z-b_i}) \geq \frac{1}{2}$$

for $i = 1, ..., mn^{'} - t$.
Hence from equation (4.1), we get

$$\begin{aligned} Re(z\frac{(r(s(z)))^{'}}{r(s(z))}) &\geq t + \sum_{i=1}^{mn^{'}-t} \frac{1}{2} - \sum_{j=1}^{mn} Re(\frac{z}{z-a_j}) \\ &= t + \sum_{j=1}^{mn-t} Re(\frac{1}{2} - \frac{z}{z-a_j}) - \frac{1}{2}(mn - mn^{'} + t) \\ &= t + \sum_{j=1}^{mn-t} \frac{|a_j|^2 - 1}{2|z-a_j|^2} - \frac{1}{2}(mn - mn^{'} + t) \\ &= \frac{1}{2}\{t + |B^{'}(z)| - m(n-n^{'})\}. \end{aligned}$$

We also need the following lemma due to Li [9].

Lemma 4.2. Let A and B be any two complex numbers.Then
(i) If $|A| \geq |B|$ and $B \neq 0$, then $A \neq \delta B$ for all complex numbers δ satisfying $|\delta| < 1$.
(ii) Conversely, if $A \neq \delta B$ for all complex numbers δ satisfying $|\delta| < 1$, then $|A| \geq |B|$.

5 Proof of theorems

Proof of Theorem 3.1. If $\min_{z \in T} |r(s(z))| = 0$, then the result follows from Lemma 4.1. So we suppose that $\min_{z \in T} |r(s(z))| > 0$, then for $z \in T$

$$\min_{z \in T} |r(s(z))| \leq |r(s(z))|.$$

If $|\alpha| < 1$, then it follows by Rouche's theorem that

$$R(s(z)) = r(s(z)) + \alpha z^t \min_{z\in T} |r(s(z))|$$

has all its zeros in $T \cup D_-$, with t-fold zeros at the origin. Now by applying Lemma 4.1 for rational function $R(s(z))$, we get for $z \in T$

$$|\frac{z(R(s(z)))^{'}}{R(s(z))}| \geq \frac{1}{2}\{t + |B^{'}(z)| - m(n - n^{'})\}$$

or

$$|z[r(s(z)) + \alpha z^t \min_{z\in T} |r(s(z))|]^{'}| \geq$$

$$\frac{1}{2}\{t + |B^{'}(z)| - m(n - n^{'})\}|r(s(z)) + \alpha z^t \min_{z\in T} |r(s(z))||.$$

By using a suitable argument of α, we have

$$|r(s(z)) + \alpha z^t \min_{z\in T} |r(s(z))|| = |r(s(z))| + |\alpha z^t \min_{z\in T} |r(s(z))||.$$

Now by applying this equality for right hand side the above inequality, we get for $z \in T$

$$|(r(s(z)))^{'}| + |\alpha| t \min_{z\in T} |r(s(z))| \geq$$

$$\frac{1}{2}\{t + |B^{'}(z)| - m(n - n^{'})\}\{|r(s(z))| + |\alpha| \min_{z\in T} |r(s(z))|\}.$$

Hence making $|\alpha| \to 1$, we get

$$|(r(s(z)))'| \geq \frac{1}{2}\{(|B'(z)| - m(n - n') + t)|r(s(z))| +$$

$$(|(B'(z)| - m(n - n') - t) \min_{z\in T} |r(s(z))|\}.$$

This completes the proof of Theorem 3.1.

Proof of Theorem 3.2. Since $r(s(z)) \in R_{mn\prime,mn}$ and $s(z)$ is a polynomial of degree m, so by the use of inequality (1.1), we get

$$|(r(s(z)))^{'}| \leq mM^{'}|r^{'}(s(z))|.$$

Therefore, by applying this inequality for left hand side of inequality (3.1), we have

$$|r^{'}(s(z))| \geq \frac{1}{2mM'}\{(|B^{'}(z)| - m(n - n^{'}) + t)|r(s(z))| +$$

$$(|B^{'}(z)| - m(n - n^{'}) - t) \min_{z\in T} |r(s(z))|\},$$

which completes the proof of Theorem 3.2.

Proof of Theorem 3.3. First supposing that $u(s(z)) \neq 0$ for $z \in T$, then for every complex number δ with $|\delta| < 1$, it follows by Rouche's theorem that $\delta r(s(z)) + u(s(z))$ has all zeros in D_- with t-fold zeros at the origin. Now $\delta r(s(z)) + u(s(z)) \neq 0$ in $T \cup D_+$, hence by Lemma 4.1, we have for $z \in T$

$$|z(\delta r(s(z)) + u(s(z)))'|$$
$$\geq \frac{||B'(z)| - m(n-n') + t|}{2} \, |\delta r(s(z)) + u(s(z))|.$$

Since $|B'(z)| \neq 0$ (see formula 14 in [9]), it follows by using (i) of Lemma 4.2 for every real or complex β with $|\beta| < 1$

$$z(\delta r(s(z)) + u(s(z)))' + \beta \frac{|B'(z)| - m(n-n') + t}{2}(\delta r(s(z)) + u(s(z))) \neq 0$$

in $T \cup D_+$. This implies that

$$\delta\{z(r(s(z)))' + \beta \frac{|B'(z)| - m(n-n') + t}{2} r(s(z))\} \tag{5.1}$$

$$\neq -\{z(u(s(z)))' + \beta \frac{|B'(z)| - m(n-n') + t}{2} u(s(z))\}.$$

Now using (ii) of lemma 4.2 in (5.1), we get for δ with $|\delta| < 1$ and $z \in T$

$$|z(u(s(z)))' + \beta \frac{|B'(z)| - m(n-n') + t}{2} u(s(z))|$$

$$\geq |z(r(s(z)))' + \beta \frac{|B'(z)| - m(n-n') + t}{2} r(s(z))|.$$

Taking $\rho := \frac{\beta}{2}$ gives us the desired inequality when $|\rho| < \frac{1}{2}$.
Finally, by continuity, the same must behold for those zeros of $u(s(z))$ lie on T and for $|\rho| \leq \frac{1}{2}$. This completes the proof of Theorem 3.3.

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