On some compact embeddings of a weighted space

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Abstract

In this paper, we define an intersection spaces between weighted classical Lebesgue spaces and an amalgam space with two weight functions on a locally compact abelian group with Haar measure. We consider the basic properties of the space such as Banach algebra, translation invariant, Banach module, a generalized type of Segal algebra etc. Also, we investigate some inclusions, compact embeddings in sense to weights and further discuss multipliers of this space.

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1 Introduction

The history of potential theory begins in 17th century. Its development can be traced to such greats as Dirichlet, Gauss, Fourier, Lagrange, Laplace, Newton, Poincaré, Riemann. We refer to the book by Kellogg [22] for references to some of the old works.

Suppose that G is a locally compact abelian group with Haar measure μ. For $1 \leq p, q \leq \infty$, an amalgam space $(L^p, \ell^q)(G)$ is a Banach space of all measurable (equivalence classes of) functions on G which belong locally to L^p and globally to ℓ^q. Several authors have considered special cases of amalgams such as H. E. Krogstad [24], T. S. Liu, A. Van Rooij and J. K. Wang [26], P. Szeptycki [33] and N. Wiener [35]. Also, F. Holland [21] presented the first systematic study of amalgams on the real line. In 1979, J. Stewart [32] extended Holland's definition to locally compact abelian groups using the Structure Theorem for locally compact groups.

For $1 \leq p < \infty$, the spaces $B^p(G) = L^1(G) \cap L^p(G)$ is a Banach algebra with the norm $\|.\|_{B^p(G)}$ defined by $\|f\|_{B^p(G)} = \|f\|_1 + \|f\|_p$ and usual convolution product. The Banach algebras $B^p(G)$ have been studied by C. R. Warner [34] and L. Y. H. Yap [36] extended some of the results on $B^p(G)$ to the Segal algebras

$$B(p,q)(G) = L^1(G) \cap L(p,q)(G)$$

where $L(p,q)(G)$ is Lorentz spaces.

Amalgam spaces (L^p, ℓ^q) are based on the work of Wiener [35]. These spaces are considerable used in harmonic and time-frequency analysis. For a historical journey, we refer [15]. Beside this work, a number of authors worked on amalgam spaces or some special cases of these spaces, see [4], [5], [18]. Moreover, the weighted amalgam spaces were investigate by [19]. In [12], the author define the intersection of weighted Lebesgue and weighted Lorentz spaces and give some basic and compact embedding properties.

In this paper, we define a space $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G) = L^r_{\vartheta_1}(G) \cap \left(L^p_{\vartheta_2}, \ell^q_{\vartheta_3}\right)(G)$ and investigate the basic properties of the space. Also, we consider several inclusions and embeddings under some

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conditions. Moreover, we generalize some of the results of [1], [34] and [36] to weighted amalgam version.

2 Notation and preliminaries

In this paper, we denote by $C_c(G)$ the linear space of continuous functions on G, which have compact support. The translation operator T_y is given by $T_y f(x) = f(x-y)$ for $x \in G$. $(B, \|.\|_B)$ is called (strongly) translation invariant if one has $T_y f \in B$ (and $\|T_y f\|_B = \|f\|_B$) for all $f \in B$ and $y \in G$. A Banach function space (shortly BF-space) on G is a Banach space $(B, \|.\|_B)$ of measurable functions which is continuously embedded into $L^1_{loc}(G)$, i.e. for any compact subset $K \subset G$ there exists some constant $C_K > 0$ such that $\|f\chi_K\|_1 \leq C_K \|f\|_B$ for all $f \in B$. A BF-space is called solid if $g \in B$, $f \in L^1_{loc}(G)$ and $|f(x)| \leq |g(x)|$ locally almost every where (shortly l.a.e) implies $f \in B$ and $\|f\|_B \leq \|g\|_B$. It is easy to see that $(B, \|.\|_B)$ is solid if and only if it is a L^∞-module.

A measurable and locally integrable function $\vartheta : G \longrightarrow (0, \infty)$ is called a weight function. Moreover the weight ϑ satisfying $\vartheta(x) \geq 1$ and $\vartheta(x+y) \leq \vartheta(x)\vartheta(y)$ for all $x, y \in G$ will be called a Beurling's weight function. We say that $\vartheta_1 \prec \vartheta_2$ if and only if there exists a $C > 0$ such that $\vartheta_1(x) \leq C\vartheta_2(x)$ for all $x \in G$. Two weight functions are called equivalent and written $\vartheta_1 \approx \vartheta_2$, if $\vartheta_1 \prec \vartheta_2$ and $\vartheta_2 \prec \vartheta_1$. A weight function ϑ is said to satisfy the Beurling-Domar (shortly BD) condition, if

$$\sum_{n \geq 1} \frac{\log \vartheta(nx)}{n^2} < \infty$$

for all $x \in G$, see [10]. Moreover, it is known that every weight function is equivalent to a continuous weight, see [27, Lemma 4]. Hence, we deduce that $\vartheta(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. For example, if we choose the polynomial type weight function $\vartheta_s(x) = (1+|x|)^s$ for $s \geq 0$, then we have $\vartheta_s(x) \longrightarrow \infty$ as $x \longrightarrow \infty$, see [29].

Let $1 \leq p < \infty$. Then the weighted Lebesgue space $L^p_\vartheta(G) = \{f : f\vartheta \in L^p(G)\}$ is a Banach space with norm $\|f\|_{p,\vartheta} = \|f\vartheta\|_p$ and its dual space $L^q_{\vartheta^{-1}}(G)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, if $1 < p < \infty$, then $L^p_\vartheta(G)$ is a reflexive Banach space. Particularly, for $p = 1$, $L^1_\vartheta(G)$ is a Banach algebra under convolution, called a Beurling algebra. It is obvious that $\|.\|_1 \leq \|.\|_{1,\vartheta}$ and $L^1_\vartheta(G) \subset L^1(G)$. It is known that $L^p_{\vartheta_2}(G) \subset L^p_{\vartheta_1}(G)$ if and only if $\vartheta_1 \prec \vartheta_2$, see [14]. We denote by $L^p_{loc}(G)$ $(1 \leq p \leq \infty)$ the space of (equivalence classes of) functions on G such that f restricted to any compact subset E of G belongs to $L^p(G)$.

Let A be a Banach algebra. A Banach space B is said to be a Banach A- module if there exists a bilinear operation $\cdot : A \times B \to B$ such that

(i) $(f \cdot g) \cdot h = f \cdot (g \cdot h)$ for all $f, g \in A$, $h \in B$.

(ii) For some constant $C \geq 1$, $\|f \cdot h\|_B \leq C \|f\|_A \|h\|_B$ for all $f \in A$, $h \in B$, see [11].

Let V and W be two Banach modules over a Banach algebra A. Then a multiplier from V into W is a bounded linear operator T from V into W, which commutes with module multiplication, i.e. $T(av) = aT(v)$ for $a \in A$ and $v \in V$. We denote by $Hom_A(V, W)$ the space of all multipliers from V into W. Also we write $Hom_A(V, V) = Hom_A(V)$. It is known that

$$Hom_A(V, W^*) \cong (V \otimes_A W)^*$$

where W^* is dual of W and $V \otimes_A W$ is the $A-$module tensor product of V and W,see [30, Corollary 2.13].

We will denote by $M(G)$ the space of bounded regular Borel measures on G. We let

$$M(\vartheta) = \left\{ \mu \in M(G) : \int_G \vartheta d|\mu| < \infty \right\}.$$

It is known that the space of multipliers from $L^1_\vartheta(G)$ to from $L^1_\vartheta(G)$ is homeomorphic to $M(\vartheta)$, see [16].

In [7], Cigler revealed a generalization of Segal algebra. To define this we assume that $S_\vartheta(G) = S_\vartheta$ is a subalgebra of $L^1_\vartheta(G)$ satisfying the following conditions:

(S1) S_ϑ is dense in $L^1_\vartheta(G)$.

(S2) S_ϑ is a Banach algebra under some norm $\|.\|_{S_\vartheta}$ and invariant under translations.

(S3) $\|T_y f\|_{S_\vartheta} \leq \vartheta(y) \|f\|_{S_\vartheta}$ for all $y \in G$ and for each $f \in S_\vartheta$.

(S4) If $f \in S_\vartheta$, then for every $\varepsilon > 0$ there exists a neighborhood U of the identity element of G such that $\|T_y f - f\|_{S_\vartheta} < \varepsilon$ for all $y \in U$.

(S5) $\|f\|_{1,\vartheta} \leq \|f\|_{S_\vartheta}$ for all $f \in S_\vartheta$.

The amalgam of L^p and ℓ^q on the real line is the normed space

$$(L^p, \ell^q) = \left\{ f \in L^p_{loc}(\mathbb{R}) : \|f\|_{pq} < \infty \right\}$$

where

$$\|f\|_{pq} = \left[\sum_{n=-\infty}^{\infty} \left[\int_n^{n+1} |f(x)|^p \, dx \right]^{\frac{q}{p}} \right]^{\frac{1}{q}}. \tag{2.1}$$

We make the appropriate changes for p, q infinite. The norm $\|.\|_{pq}$ makes (L^p, ℓ^q) into a Banach space, see [21]. Note that $C_c(G)$ is included in all amalgam spaces. If $1 \leq p, q < \infty$, then the dual space of (L^p, ℓ^q) is isometrically isomorphic to $\left(L^{p'}, \ell^{q'}\right)$, where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$.

J. Stewart [32] give an alternative definition of $(L^p, \ell^q)(G)$ based on the Structure Theorem [20, Theorem 24.30]. Indeed, let $G = \mathbb{R}^a \times G_1$, where a is a nonnegative integer and G_1 is a locally compact abelian group which contains an open compact subgroup H. Also, assume that $I = [0,1)^a \times H$ and $J = \mathbb{Z}^a \times T$ where T is a transversal of H in G_1, i.e. $G_1 = \bigcup_{t \in T} (t + H)$ is a coset decomposition of G_1. For $\alpha \in J$ we define $I_\alpha = \alpha + I$, and therefore G is equal to the disjoint union of relatively compact sets I_α. We normalize μ so that $\mu(I) = \mu(I_\alpha) = 1$ for all α. Let $1 \leq p, q \leq \infty$. The amalgam space $(L^p, \ell^q)(G) = (L^p, \ell^q)$ is a Banach space

$$\left\{ f \in L^p_{loc}(G) : \|f\|_{pq} < \infty \right\},$$

where

$$\begin{aligned} \|f\|_{pq} &= \left[\sum_{\alpha\in J}\|f\|^q_{L^p(I_\alpha)}\right]^{1/q} \text{ if } 1\le p,q<\infty, \\ \|f\|_{\infty q} &= \left[\sum_{\alpha\in J}\sup_{x\in I_\alpha}|f(x)|^q\right]^{1/q} \text{ if } p=\infty,\ 1\le q<\infty, \\ \|f\|_{p\infty} &= \sup_{\alpha\in J}\|f\|_{L^p(I_\alpha)} \text{ if } 1\le p<\infty,\ q=\infty. \end{aligned} \tag{2.2}$$

If $G=\mathbb{R}$, then we have $J=\mathbb{Z}$, $I_\alpha=[\alpha,\alpha+1)$ and (2.2) becomes (2.1).

Throughout this paper, G will denote a locally compact abelian group with Haar measure and J and I_α will define as above. Moreover, we assume that $1\le p,q,r<\infty$ and every weights we used are Beurling's weight functions on G.

3 The space $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$

Definition 3.1. Denote the amalgam spaces with double weights $\left(L^p_{\vartheta_2},\ell^q_{\vartheta_3}\right)(G)=\left(L^p_{\vartheta_2},\ell^q_{\vartheta_3}\right)$ are the normed space

$$\left(L^p_{\vartheta_2},\ell^q_{\vartheta_3}\right)=\left\{f\in L^p_{loc}(G):\|f\|_{pq,\vartheta_2,\vartheta_3}<\infty\right\}$$

where

$$\|f\|_{pq,\vartheta_2,\vartheta_3}=\left[\sum_{n\in J}\|f\chi_{I_n}\|^q_{p,\vartheta_2}\,\vartheta_3^q(n)\right]^{\frac{1}{q}}=\left[\sum_{n\in J}\|f\|^q_{L^p_{\vartheta_2}(I_n)}\,\vartheta_3^q(n)\right]^{\frac{1}{q}}.$$

It is easy to see that the space $\left(L^p_{\vartheta_2},\ell^q_{\vartheta_3}\right)$ is a Banach space with respect to $\|.\|_{pq,\vartheta_2,\vartheta_3}$.

The proof the following theorem is obvious, see [1].

Theorem 3.2. Let $1\le p,q<\infty$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\left(L^p_{\vartheta_2},\ell^q_{\vartheta_3}\right)$ and $\|f_n-f\|_{pq,\vartheta_2,\vartheta_3}\longrightarrow 0$ as $n\longrightarrow\infty$, where $f\in\left(L^p_{\vartheta_2},\ell^q_{\vartheta_3}\right)$. Then $\{f_n\}_{n\in\mathbb{N}}$ has a subsequence which converges pointwise almost everywhere to f.

Now, we define the vector space $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)=L^r_{\vartheta_1}(G)\cap\left(L^p_{\vartheta_2},\ell^q_{\vartheta_3}\right)$ and equip this space with the sum norm

$$\|f\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}=\|f\|_{r,\vartheta_1}+\|f\|_{pq,\vartheta_2,\vartheta_3}$$

where $f\in B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$.

Theorem 3.3. The space $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ is a Banach space with respect to $\|.\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}$.

Proof. Assume that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$. Thus given $\varepsilon>0$, there exists an $n_1\in\mathbb{N}$ such that for all $n,m\geqslant n_1$ implies

$$\|f_n-f_m\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}=\|f_n-f_m\|_{r,\vartheta_1}+\|f_n-f_m\|_{pq,\vartheta_2,\vartheta_3}<\varepsilon.$$

Therefore, $\{f_n\}_{n\in IN}\subset L^r_{\vartheta_1}(G)$ and $\{f_n\}_{n\in IN}\subset\left(L^p_{\vartheta_2},\ell^q_{\vartheta_3}\right)$ are Cauchy sequences with respect to $\|.\|_{r,\vartheta_1}$ and $\|.\|_{pq,\vartheta_2,\vartheta_3}$, respectively. Since the spaces $\left(L^r_{\vartheta_1}(G),\|.\|_{r,\vartheta_1}\right)$ and $\left(\left(L^p_{\vartheta_2},\ell^q_{\vartheta_3}\right),\|.\|_{pq,\vartheta_2,\vartheta_3}\right)$

are two Banach spaces, there exist $f \in L^r_{\vartheta_1}(G)$ and $g \in \left(L^p_{\vartheta_2}, \ell^q_{\vartheta_3}\right)$ such that $\|f_n - f\|_{r,\vartheta_1} \longrightarrow 0$, $\|f_n - g\|_{pq,\vartheta_2,\vartheta_3} \longrightarrow 0$. Since ϑ_1 is a Beurling's weight, we have $L^r_{\vartheta_1}(G) \hookrightarrow L^r(G)$, see [3]. Thus $f_{n_k} \longrightarrow g$ in $L^r(G)$. Moreover, if we consider the fact that the convergence in L^r is necessity with convergence in measure [23] and Theorem 3.2, then there is a subsequence $\left\{f_{n_{k_l}}\right\}_{l\in IN}$ of $\{f_{n_k}\}_{k\in IN}$ such that $f_{n_{k_l}} \longrightarrow g$, *a.e.* Now, we denote

$$K_1 = \{x \in G : f_{n_k}(x) \nrightarrow f(x)\}$$

and

$$K_2 = \left\{x \in G : f_{n_{k_l}}(x) \nrightarrow g(x)\right\}.$$

It is clear that $|K_1| = 0$, $|K_2| = 0$. Moreover, there is an $n_2 \in \mathbb{N}$ such that for all $n_k \geqslant n_2$ implies

$$|f_{n_k}(x) - f(x)| < \frac{\varepsilon}{2}$$

for every $x \in \mathbb{R}^d - K_1$. Similarly, we have an $n_3 \in \mathbb{N}$ such that for all $n_{k_l} \geqslant n_3$ implies

$$\left|f_{n_{k_l}}(x) - g(x)\right| < \frac{\varepsilon}{2}$$

for every $x \in G - K_2$. Now, we set $n_0 = \max\{n_2, n_3\}$. There exists an $n_0 \in \mathbb{N}$ such that for all $n_{k_l} \geqslant n_0$ implies the inequality

$$|f(x) - g(x)| \leq \left|f(x) - f_{n_{k_l}}(x)\right| + \left|f_{n_{k_l}}(x) - g(x)\right| < \varepsilon$$

holds for every $x \in G - (K_1 \cup K_2)$. Hence, we have $f = g$, a.e. Since the elements of $L^r_{\vartheta_1}(G)$ and $\left(L^p_{\vartheta_2}, \ell^q_{\vartheta_3}\right)$ are equivalence classes, we get that $f = g$. This follows that $f \in B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$. Moreover, it is easy to see that

$$\|f_n - f\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3} = \|f_n - f\|_{r,\vartheta_1} + \|f_n - f\|_{pq,\vartheta_2,\vartheta_3} \longrightarrow 0.$$

That is the desired result. Q.E.D.

Theorem 3.4. If $1 < p < \infty$, $q \neq \infty$ and $\vartheta_1 \prec \vartheta_2$ or $\vartheta_2 \prec \vartheta_1$, then the space $\left(B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G), \|.\|^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}\right)$ is a Banach algebra with respect to convolution.

Proof. Let $\vartheta_2 \prec \vartheta_1$. Thus, there is a $c > 0$ such that $\|f\|_{1,\vartheta_2} \leq c_1 \|f\|_{1,\vartheta_1}$. It is note that the space $\left(B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G), \|.\|^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}\right)$ is a Banach space by the Theorem 3.3. Now, let $f, g \in B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ be given. Since $L^1_{\vartheta_1}(G)$ is a Banach algebra under convolution, we have $f * g \in L^1_{\vartheta_1}(G)$ and

$$\|f * g\|_{1,\vartheta_1} \leq \|f\|_{1,\vartheta_1} \|g\|_{1,\vartheta_1}. \tag{3.1}$$

Moreover, $\left(L^p_{\vartheta_2}, \ell^q_{\vartheta_3}\right)$ is a Banach $L^1_{\vartheta_2}(G)$- module for $1 < p < \infty$, $q \neq \infty$. Indeed, we have

$$\begin{aligned} \|f * g\|_{pq,\vartheta_2,\vartheta_3} &\leq \left[\sum_{n\in J} \|f\|^q_{L^1_{\vartheta_2}(I_n)} \|g\|^q_{L^p_{\vartheta_2}(I_n)} \vartheta^q_3(n)\right]^{\frac{1}{q}} \\ &\leq \|f\|_{1,\vartheta_2} \left[\sum_{n\in J} \|g\|^q_{L^p_{\vartheta_2}(I_n)} \vartheta^q_3(n)\right]^{\frac{1}{q}} = \|f\|_{1,\vartheta_2} \|g\|_{pq,\vartheta_2,\vartheta_3}. \end{aligned} \tag{3.2}$$

where $c_1 \geq 1$. By using (3.1), (3.2) and the definition of $\|.\|^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}$ we have

$$\begin{aligned}
\|f*g\|^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3} &\leq \|f\|_{1,\vartheta_1}\|g\|_{1,\vartheta_1} + \|f\|_{1,\vartheta_2}\|g\|_{pq,\vartheta_2,\vartheta_3} \\
&\leq \|f\|_{1,\vartheta_1}\|g\|_{1,\vartheta_1} + c_1\|f\|_{1,\vartheta_1}\|g\|_{pq,\vartheta_2,\vartheta_3} \\
&\leq \max\{1,c_1\}\|f\|_{1,\vartheta_1}\left(\|g\|_{1,\vartheta_1} + \|g\|_{pq,\vartheta_2,\vartheta_3}\right) \\
&\leq \max\{1,c_1\}\|f\|^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}\|g\|^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}.
\end{aligned}$$

In similar way, the desired result can be found with the case $\vartheta_1 \prec \vartheta_2$. Q.E.D.

Theorem 3.5. The space $\left(B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G), \|.\|^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}\right)$ is a solid BF-space.

Proof. Let $K \subset G$ be given a compact subset and $f \in B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$. Since ϑ_1 is a Beurling's weight function, we have

$$\int_K |f(x)|\,dx \leq \|f\|_1 \leq \|f\|^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}.$$

Let $f \in B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ and $g \in L^\infty(G)$. Since $L^1_{\vartheta_1}(G)$ and $\left(L^p_{\vartheta_2}, \ell^q_{\vartheta_3}\right)$ are solid BF-space ([2], [13]), then

$$\|fg\|^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3} \leq \|f\|_{1,\vartheta_1}\|g\|_\infty + \|f\|_{pq,\vartheta_2,\vartheta_3}\|g\|_\infty = \|f\|^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}\|g\|_\infty.$$

This completes the proof. Q.E.D.

Theorem 3.6. The following statements hold.

(i) The space $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ is translation invariant and for every $f \in B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ the inequality $\|T_y f\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3} \leq \max\{\vartheta_1(y), \vartheta_2(y)\vartheta_3(y)\}\|f\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}$ holds.

(ii) The map $y \longrightarrow T_y f$ is continuous from G into $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ for every $f \in B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$.

Proof. Let $f \in B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$. It is easy to show that $T_y f \in L^r_{\vartheta_1}(G)$ and $\|T_y f\|_{r,\vartheta_1} \leq \vartheta_1(y)\|f\|_{r,\vartheta_1}$ for all $y \in G$. If we use the similar method in [19], we get

$$\begin{aligned}
\|T_y f\|_{pq,\vartheta_2,\vartheta_3} &= \left[\sum_{n\in J}\|T_y f\|^q_{L^p_{\vartheta_2}(I_n)}\vartheta_3^q(n)\right]^{\frac{1}{q}} \\
&\leq \vartheta_2(y)\vartheta_3(y)\left[\sum_{n\in J}\|f\|^q_{p,\vartheta_2}\vartheta_3^q(n)\right]^{\frac{1}{q}} \\
&= \vartheta_2(y)\vartheta_3(y)\|f\|_{pq,\vartheta_2,\vartheta_3}.
\end{aligned}$$

This follows that

$$\begin{aligned}
\|T_y f\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3} &\leq \vartheta_1(y)\|f\|_{r,\vartheta_1} + \vartheta_2(y)\vartheta_3(y)\|f\|_{pq,\vartheta_2,\vartheta_3} \\
&\leq \max\{\vartheta_1(y), \vartheta_2(y)\vartheta_3(y)\}\|f\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}.
\end{aligned}$$

This completes *(i)*. Now, let $f \in B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$. Then $f \in L^r_{\vartheta_1}(G)$ and $f \in \left(L^p_{\vartheta_2}, \ell^q_{\vartheta_3}\right)$. It is known that the translation operator is continuous from G into $L^r_{\vartheta_1}(G)$, see [14]. Moreover, since $C_c(G)$ is dense in $\ell^q_{\vartheta_3}$ and the map $y \longrightarrow T_y f$ is continuous in $L^p_{\vartheta_2}(G)$ for all $f \in L^p_{\vartheta_2}(G)$, we get that the map $y \longrightarrow T_y f$ is continuous in $\left(L^p_{\vartheta_2}, \ell^q_{\vartheta_3}\right)$, see [28]. This follows *(ii)*. Q.E.D.

Theorem 3.7. The space $B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ is a S_{ϑ_1} algebra.

Proof. We have already proved the some conditions in Theorem 3.4 and Theorem 3.6 for S_{ϑ_1} algebra. Now, we prove that $B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ is dense in $L^1_{\vartheta_1}(G)$. Since $C_c(G) \subset B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ and $C_c(G)$ is dense in $L^1_{\vartheta_1}(G)$, then $B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ is dense in $L^1_{\vartheta_1}(G)$. Q.E.D.

Theorem 3.8. If $\vartheta_1 \prec \vartheta_0$ and $\vartheta_2 \prec \vartheta_0$, then $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ is a Banach $L^1_{\vartheta_0}(G)$- module with respect to $\|.\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}$.

Proof. Assume that $\vartheta_1 \prec \vartheta_0$ and $\vartheta_2 \prec \vartheta_0$ are hold. It is obvious that $L^1_{\vartheta_0}(G) \hookrightarrow L^1_{\vartheta_1}(G)$ and $L^1_{\vartheta_0}(G) \hookrightarrow L^1_{\vartheta_2}(G)$. Thus, we find $c_1, c_2 > 0$ such that $\|f\|_{1,\vartheta_1} \leq c_1 \|f\|_{1,\vartheta_0}$ and $\|f\|_{1,\vartheta_2} \leq c_1 \|f\|_{1,\vartheta_0}$ for any $f \in L^1_{\vartheta_0}(G)$. Since $\left(L^p_{\vartheta_2}, \ell^q_{\vartheta_3}\right)$ is a Banach $L^1_{\vartheta_2}(G) -$ module for $1 < p < \infty$, $q \neq \infty$ and $L^r_{\vartheta_1}(G)$ is a Banach $L^1_{\vartheta_1}(G)$- module for $1 \leq r < \infty$, we get

$$\begin{aligned} \|f * g\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3} &\leq \|f\|_{r,\vartheta_1} \|g\|_{1,\vartheta_1} + \|f\|_{pq,\vartheta_2,\vartheta_3} \|g\|_{1,\vartheta_2} \\ &\leq c_1 \|f\|_{r,\vartheta_1} \|g\|_{1,\vartheta_0} + c_2 \|f\|_{pq,\vartheta_2,\vartheta_3} \|g\|_{1,\vartheta_0} \\ &\leq \max\{c_1, c_2\} \|f\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3} \|g\|_{1,\vartheta_0} \end{aligned}$$

for any $f \in B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ and $g \in L^1_{\vartheta_0}(G)$. If we define a new norm $|||.|||$ on $L^1_{\vartheta_0}(G)$ such that $|||.||| = \max\{c_1, c_2\} \|.\|_{1,\vartheta_0}$, then this norm is equivalent to the norm $\|.\|_{1,\vartheta_0}$ on $L^1_{\vartheta_0}(G)$. This completes the proof. Q.E.D.

Consider the mapping Φ from $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ into $L^r_{\vartheta_1}(G) \times \left(L^p_{\vartheta_2}, \ell^q_{\vartheta_3}\right)$ defined by $\Phi(f) = (f, f)$. This is a linear isometry of $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ into $L^r_{\vartheta_1}(G) \times \left(L^p_{\vartheta_2}, \ell^q_{\vartheta_3}\right)$ with the norm

$$|||(f, f)||| = \|f\|_{r,\vartheta_1} + \|f\|_{pq,\vartheta_2,\vartheta_3}$$

for $f \in B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$. Hence it is easy to see that $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ is a closed subspace of the Banach space $L^r_{\vartheta_1}(G) \times \left(L^p_{\vartheta_2}, \ell^q_{\vartheta_3}\right)$. Let

$$H = \left\{(f, f) : f \in B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)\right\}$$

and

$$K = \left\{ \begin{array}{c} (\varphi, \psi) : (\varphi, \psi) \in L^{r'}_{\vartheta_1^{-1}}(G) \times \left(L^{p'}_{\vartheta_2^{-1}}, \ell^{q'}_{\vartheta_3^{-1}}\right), \\ \int\limits_G f(x)\varphi(x)dx + \int\limits_G f(y)\psi(y)dy = 0, \text{ for all } (f, f) \in H \end{array} \right\},$$

where $\frac{1}{r} + \frac{1}{r'} = 1$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

The following theorem is easily proved by Duality Theorem 1.7 in [25].

Theorem 3.9. The dual space $\left(B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)\right)^*$ of $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ is isomorphic to $L^{r'}_{\vartheta_1^{-1}}(G) \times \left(L^{p'}_{\vartheta_2^{-1}}, \ell^{q'}_{\vartheta_3^{-1}}\right)/K$.

If we use the similar method in [19, Theorem 11.8.3] with weighted case, then we have the following result.

Theorem 3.10. ([19], [31])Assume that $L^p_{\vartheta_2} * L^t_{\vartheta_2} \subset L^m_{\vartheta_2}$ and $\ell^q_{\vartheta_3} * \ell^s_{\vartheta_3} \subset \ell^n_{\vartheta_3}$. Then

$$\left(L^p_{\vartheta_2}, \ell^q_{\vartheta_3}\right) * \left(L^t_{\vartheta_2}, \ell^s_{\vartheta_3}\right) \subset \left(L^m_{\vartheta_2}, \ell^n_{\vartheta_3}\right)$$

,that is, there exists a $c > 0$ such that

$$\|f * g\|_{mn,\vartheta_2,\vartheta_3} \leq c \|f\|_{pq,\vartheta_2,\vartheta_3} \|g\|_{ts,\vartheta_2,\vartheta_3}$$

for all $f \in \left(L^p_{\vartheta_2}, \ell^q_{\vartheta_3}\right)$, $g \in \left(L^t_{\vartheta_2}, \ell^s_{\vartheta_3}\right)$.

Theorem 3.11. Assume that all assumptions in Theorem 3.10 hold. Then

$$B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G) * B^{1,t,s}_{\vartheta_1,\vartheta_2,\vartheta_3}(G) \subset B^{1,m,n}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$$

,that is, there exists a $c > 0$ such that

$$\|f * g\|^{1,m,n}_{\vartheta_1,\vartheta_2,\vartheta_3} \leq c \|f\|^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3} \|g\|^{1,t,s}_{\vartheta_1,\vartheta_2,\vartheta_3}$$

for all $f \in B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$, $g \in B^{1,t,s}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$.

Proof. Let $f \in B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ and $g \in B^{1,t,s}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$. By Theorem 3.10, there is a $c > 0$ such that

$$\begin{aligned} \|f * g\|^{1,m,n}_{\vartheta_1,\vartheta_2,\vartheta_3} &\leq \|f\|_{1,\vartheta_1} \|g\|_{1,\vartheta_1} + c \|f\|_{pq,\vartheta_2,\vartheta_3} \|g\|_{ts,\vartheta_2,\vartheta_3} \\ &\leq \|f\|_{1,\vartheta_1} \|g\|^{1,t,s}_{\vartheta_1,\vartheta_2,\vartheta_3} + c \|f\|_{pq,\vartheta_2,\vartheta_3} \|g\|^{1,t,s}_{\vartheta_1,\vartheta_2,\vartheta_3} \\ &\leq \max\{1, c\} \|f\|^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3} \|g\|^{1,t,s}_{\vartheta_1,\vartheta_2,\vartheta_3}. \end{aligned}$$

This follows that $B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G) * B^{1,t,s}_{\vartheta_1,\vartheta_2,\vartheta_3}(G) \subset B^{1,m,n}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$. Q.E.D.

4 Inclusions of the spaces $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$

Theorem 4.1. The inclusion $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G) \subset B^{r,p,q}_{\vartheta_4,\vartheta_5,\vartheta_6}(G)$ holds if and only if the embedding $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G) \hookrightarrow B^{r,p,q}_{\vartheta_4,\vartheta_5,\vartheta_6}(G)$ is satisfied.

Proof. The sufficient condition of the theorem is clear by definition of embedding. Now, assume that $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G) \subset B^{r,p,q}_{\vartheta_4,\vartheta_5,\vartheta_6}(G)$ holds. Moreover, we define the sum norm $|||.||| = \|.\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3} + \|.\|^{r,p,q}_{\vartheta_4,\vartheta_5,\vartheta_6}$. It is easy to see that $\left(B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G), |||.|||\right)$ is a Banach space. Now, let us define the unit function I from $\left(B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G), |||.|||\right)$ into $\left(B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G), \|.\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}\right)$. Since the inequality

$$\|I(f)\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3} = \|f\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3} \leq |||f|||$$

is satisfied, I is continuous. If we consider the Banach's theorem, then I is a homeomorphism, see [6]. That means the norms $\|\|.\|\|$ and $\|.\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}$ are equivalent. Thus, for every $f \in B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ there exists $c > 0$ such that

$$\|\|f\|\| \leq c \|f\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}. \tag{4.1}$$

Therefore, by using (4.1) and the definition of norm $\|\|.\|\|$, we obtain

$$\|f\|^{r,p,q}_{\vartheta_4,\vartheta_5,\vartheta_6} \leq \|\|f\|\| \leq c \|f\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}.$$

That is the desired result. Q.E.D.

Now, we give some continuous embeddings of $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ under some conditions.

Theorem 4.2. The following statements are true.

(i) If $q_1 \leq q_2$ and $\vartheta_4 \prec \vartheta_1$, then $B^{r,p,q_1}_{\vartheta_1,\vartheta_2,\vartheta_3}(G) \hookrightarrow B^{r,p,q_2}_{\vartheta_4,\vartheta_2,\vartheta_3}(G)$.

(ii) If $p_2 \leq p_1$ and $\vartheta_4 \prec \vartheta_1$, then $B^{r,p_1,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G) \hookrightarrow B^{r,p_2,q}_{\vartheta_4,\vartheta_2,\vartheta_3}(G)$.

(iii) If $p_2 \leq p_1$, $q_1 \leq q_2$ and $\vartheta_4 \prec \vartheta_1$, then $B^{r,p_1,q_1}_{\vartheta_1,\vartheta_2,\vartheta_3}(G) \hookrightarrow B^{r,p_2,q_2}_{\vartheta_4,\vartheta_2,\vartheta_3}(G)$.

(iv) If $\vartheta_4 \prec \vartheta_1, \vartheta_5 \prec \vartheta_2$ and $\vartheta_6 \prec \vartheta_3$, then $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G) \hookrightarrow B^{r,p,q}_{\vartheta_4,\vartheta_5,\vartheta_6}(G)$.

Proof. Let $q_1 \leq q_2$ and $\vartheta_4 \prec \vartheta_1$. Then we get $L^r_{\vartheta_1}(G) \hookrightarrow L^r_{\vartheta_4}(G)$. Moreover, since the embedding $\left(L^p_{\vartheta_2}, \ell^{q_1}_{\vartheta_3}\right) \hookrightarrow \left(L^p_{\vartheta_2}, \ell^{q_2}_{\vartheta_3}\right)$ holds under the hypotheses, there exists $c_1 > 0$ such that

$$\|f\|^{r,p,q_2}_{\vartheta_4,\vartheta_2,\vartheta_3} \leq c_1 \|f\|_{r,\vartheta_1} + \|f\|_{pq_1,\vartheta_2,\vartheta_3} \leq \max\{1, c_1\} \|f\|^{r,p,q_1}_{\vartheta_1,\vartheta_2,\vartheta_3}$$

for all $f \in B^{r,p,q_1}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$. This completes (i). Now, let $f \in B^{r,p_1,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ be given. Hence, we get $f \in L^r_{\vartheta_1}(G)$ and $f \in \left(L^{p_1}_{\vartheta_2}, \ell^q_{\vartheta_3}\right)$. Moreover, it is clear that $L^{p_1}_{\vartheta_2}(I_n) \hookrightarrow L^{p_2}_{\vartheta_2}(I_n)$ with $p_2 \leq p_1$ for $n \in J$. This yields that there exist $c_1, c_2 > 0$ such that

$$\|f\|^{r,p_2,q}_{\vartheta_4,\vartheta_2,\vartheta_3} \leq c_1 \|f\|_{r,\vartheta_1} + c_2 \|f\|_{p_1 q,\vartheta_2,\vartheta_3} \leq \max\{c_1, c_2\} \|f\|^{r,p_1,q}_{\vartheta_1,\vartheta_2,\vartheta_3}.$$

This proves (ii). If we consider (i) and (ii), then we obtain (iii). Moreover, the proof of (iv) is easy to see. Q.E.D.

Now, we consider a privatized space of $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ as $B^{r,p,1}_{\vartheta_1,\vartheta_2}(G) = L^r_{\vartheta_1}(G) \cap \left(L^p_{\vartheta_2}, \ell^1\right)$ with the norm

$$\|.\|^{r,p,1}_{\vartheta_1,\vartheta_2} = \|.\|_{r,\vartheta_1} + \|.\|_{p1,\vartheta_2}.$$

Theorem 4.3. For any $f \in B^{r,p,1}_{\vartheta_1,\vartheta_2}(G)$ and $y \in G$ there exist constants $C_1 = C_1(f) > 0$ and $C_2 = C_2(f) > 0$ such that

$$C_1 \vartheta(y) \leq \|T_y f\|^{r,p,1}_{\vartheta_1,\vartheta_2} \leq C_2 \vartheta(y)$$

where $\vartheta = \vartheta_1 + \vartheta_2$.

Proof. Let $f \in B^{r,p,1}_{\vartheta_1,\vartheta_2}(G)$. Then by Lemma 2.2 in [14], there exists a constant $a_1 = a_1(f)$ and $a_2 = a_2(f) > 0$ such that

$$a_1\vartheta_1(y) \leq \|T_y f\|_{r,\vartheta_1} \leq a_2\vartheta_1(y) \tag{4.2}$$

for all $y \in G$. By the Theorem 3.6, we have

$$\|T_y f\|_{p1,\vartheta_2} \leq \vartheta_2(y)\|f\|_{p1,\vartheta_2}. \tag{4.3}$$

Moreover, it is clear that

$$\left(L^p_{\vartheta_2},\ell^1\right) \subset \left(L^1_{\vartheta_2},\ell^1\right) = L^1_{\vartheta_2}(G)$$

is satisfied, see [19]. This follows that there exists $b > 0$ such that

$$\|T_y f\|_{1,\vartheta_2} \leq b\|T_y f\|_{p1,\vartheta_2}. \tag{4.4}$$

If we combine (4.2), (4.3) and (4.4), then there are $b_1 = b_1(f)$ and $b_2 = b_2(f) > 0$ such that

$$b_1\vartheta_2(y) \leq \|T_y f\|_{1,\vartheta_2} \leq b\|T_y f\|_{p1,\vartheta_2} \leq b_2 b\vartheta_2(y) \tag{4.5}$$

where $b_2 = \|f\|_{p1,\vartheta_2}$. From (4.2) and (4.5), we get

$$a_1\vartheta_1(y) + \frac{b_1}{b}\vartheta_2(y) \leq \|T_y f\|^{r,p,1}_{\vartheta_1,\vartheta_2} \leq a_2\vartheta_1(y) + b_2\vartheta_2(y).$$

This yields that

$$C_1(\vartheta_1(y) + \vartheta_2(y)) \leq \|T_y f\|^{r,p,1}_{\vartheta_1,\vartheta_2} \leq C_2(\vartheta_1(y) + \vartheta_2(y))$$

where $C_1 = \min\left\{a_1, \frac{b_1}{b}\right\}$, $C_2 = \max\{a_2, b_2\}$ for all $y \in G$. Q.E.D.

Theorem 4.4. Let $\vartheta_1, \vartheta_2, \vartheta_3$ and ϑ_4 be Beurling's weight functions on G and $\vartheta = \vartheta_1 + \vartheta_2$, $\vartheta' = \vartheta_3 + \vartheta_4$. If $B^{r,p,1}_{\vartheta_1,\vartheta_2}(G) \subset B^{r,p,1}_{\vartheta_3,\vartheta_4}(G)$, then $\vartheta' \prec \vartheta$.

Proof. Assume that $B^{r,p,1}_{\vartheta_1,\vartheta_2}(G) \subset B^{r,p,1}_{\vartheta_3,\vartheta_4}(G)$ holds. Hence, there is a $C > 0$ such that

$$\|T_y f\|^{r,p,1}_{\vartheta_3,\vartheta_4} \leq C\|T_y f\|^{r,p,1}_{\vartheta_1,\vartheta_2}.$$

By the Theorem 4.3, there exist $C_1 = C_1(f), C_2 = C_2(f), C_3 = C_3(f), C_4 = C_4(f) > 0$ such that

$$C_1\vartheta(y) \leq \|T_y f\|^{r,p,1}_{\vartheta_1,\vartheta_2} \leq C_2\vartheta(y)$$

and

$$C_3\vartheta'(y) \leq \|T_y f\|^{r,p,1}_{\vartheta_3,\vartheta_4} \leq C_4\vartheta'(y)$$

where $\vartheta = \vartheta_1 + \vartheta_2$, $\vartheta' = \vartheta_3 + \vartheta_4$. This follows by the Theorem 4.1 that

$$C_3\vartheta'(y) \leq \|T_y f\|^{r,p,1}_{\vartheta_3,\vartheta_4} \leq C\|T_y f\|^{r,p,1}_{\vartheta_1,\vartheta_2} \leq CC_2\vartheta(y)$$

or equivalently

$$\vartheta'(y) \leq \frac{CC_2}{C_3}\vartheta(y)$$

for all $y \in G$. This completes the proof. Q.E.D.

Remark 4.5. Note that the converse implication of the Theorem 4.4 holds if $\vartheta_3 \prec \vartheta_1$ and $\vartheta_4 \prec \vartheta_2$. Indeed, since $\vartheta_3 \prec \vartheta_1$ and $\vartheta_4 \prec \vartheta_2$, there exist $c_1, c_2 > 0$ such that $\vartheta_3(x) \leq c_1\vartheta_1(x)$ and $\vartheta_4(x) \leq c_2\vartheta_2(x)$ for all $x \in G$. Thus, we have

$$\|f\|^{r,p,1}_{\vartheta_3,\vartheta_4} \quad \leq \quad c_1 \|f\|_{r,\vartheta_1} + c_2 \|f\|_{p1,\vartheta_2} \leq \max\{c_1, c_2\} \|f\|^{r,p,1}_{\vartheta_1,\vartheta_2}$$

for all $f \in B^{r,p,1}_{\vartheta_1,\vartheta_2}(G)$. Hence, the inclusion $B^{r,p,1}_{\vartheta_1,\vartheta_2}(G) \subset B^{r,p,1}_{\vartheta_3,\vartheta_4}(G)$ holds.

The following corollary can be easily proven by using the Theorem 4.4 and Theorem 4.2.

Corollary 4.6. The following expressions are true.

(i) The equality $B^{r,p,1}_{\vartheta_1,\vartheta_2}(G) = B^{r,p,1}_{\vartheta_3,\vartheta_4}(G)$ is satisfied if $\vartheta_1 \approx \vartheta_3$ and $\vartheta_2 \approx \vartheta_4$.

(ii) If the assumptions $\vartheta_1 \approx \vartheta_4, \vartheta_2 \approx \vartheta_5$ and $\vartheta_3 \approx \vartheta_6$ are satisfied, then we have $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G) = B^{r,p,q}_{\vartheta_4,\vartheta_5,\vartheta_6}(G)$.

(iii) Let $q_1 \leq q_2$ and $p_2 \leq p_1$. If $\vartheta_4 \prec \vartheta_1, \vartheta_5 \prec \vartheta_2$ and $\vartheta_6 \prec \vartheta_3$, then the embedding $B^{r,p_1,q_1}_{\vartheta_1,\vartheta_2,\vartheta_3}(G) \hookrightarrow B^{r,p_2,q_2}_{\vartheta_4,\vartheta_5,\vartheta_6}(G)$ holds.

5 Compact embeddings of the space $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}\left(\mathbb{R}^d\right)$

Now, we investigate compact embeddings of the spaces $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ with the similar methods in [18]. Also, we will take $G = \mathbb{R}^d$ with Lebesgue measure dx for compact embeddings.

Theorem 5.1. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}\left(\mathbb{R}^d\right)$. If $\{f_n\}_{n\in\mathbb{N}}$ converges to zero in $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}\left(\mathbb{R}^d\right)$, then $\{f_n\}_{n\in\mathbb{N}}$ converges to zero in the vague topology (which means that

$$\int_{\mathbb{R}^d} f_n(x)k(x)dx \longrightarrow 0$$

for $n \longrightarrow \infty$ for all $k \in C_c(\mathbb{R}^d)$, see [8]).

Proof. Let $k \in C_c(\mathbb{R}^d)$. By the Hölder inequality, we have

$$\left|\int_{\mathbb{R}^d} f_n(x)k(x)dx\right| \leq \|k\|_{r',\vartheta_1} \|f_n\|_{r,\vartheta_1} \leq \|k\|_{r',\vartheta_1} \|f_n\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}$$

where $\frac{1}{r'} + \frac{1}{r} = 1$. This follows that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to zero in vague topology. Q.E.D.

Theorem 5.2. If $\vartheta_3 \prec \vartheta_2 \prec \vartheta_1, \vartheta \prec \vartheta_1$ and $\frac{\vartheta(x)}{(\vartheta_1(x))^2}$ doesn't tend to zero in $\mathbb{R}^d$ as $x \longrightarrow \infty$, then the embedding of the space $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}\left(\mathbb{R}^d\right)$ into $L^r_\vartheta\left(\mathbb{R}^d\right)$ is never compact.

Proof. Since $\vartheta \prec \vartheta_1$, there exists $C_1 > 0$ such that $\vartheta(x) \leq C_1\vartheta_1(x)$. This implies $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}\left(\mathbb{R}^d\right) \subset L^r_\vartheta(\mathbb{R}^d)$. Assume that $\{t_n\}_{n\in\mathbb{N}}$ is a sequence with $t_n \longrightarrow \infty$ in $\mathbb{R}^d$. Since $\frac{\vartheta(x)}{(\vartheta_1(x))^2}$ does not tend to zero as $x \longrightarrow \infty$, there exists $\delta > 0$ such that $\frac{\vartheta(x)}{(\vartheta_1(x))^2} \geq \delta > 0$ for $x \longrightarrow \infty$. To end the

proof, we take any fixed $f \in B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}\left(\mathbb{R}^d\right)$ and define a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$, where $f_n = \left(\vartheta_1\left(t_n\right)\right)^{-2} T_{t_n} f$. This sequence is bounded in $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}\left(\mathbb{R}^d\right)$. Indeed, since $\vartheta_3 \prec \vartheta_2 \prec \vartheta_1$, there exist $C_2, C_3 > 0$ such that $\vartheta_2\left(x\right) \leq C_2\vartheta_1\left(x\right)$ and $\vartheta_3\left(x\right) \leq C_3\vartheta_1\left(x\right)$ for all $\mathbb{R}^d$. Moreover, by Theorem 3.6, we get

$$\begin{aligned}
\left\|f_n\right\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3} &= \left(\vartheta_1\left(t_n\right)\right)^{-2} \left\|T_{t_n} f\right\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3} \\
&\leq \left(\vartheta_1\left(t_n\right)\right)^{-2} \max\left\{\vartheta_1\left(t_n\right), \vartheta_2\left(t_n\right)\vartheta_3\left(t_n\right)\right\} \left\|f\right\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3} \\
&\leq \left(\vartheta_1\left(t_n\right)\right)^{-2} \max\left\{\left(\vartheta_1\left(t_n\right)\right)^2, C_2\vartheta_1\left(t_n\right) C_3\vartheta_1\left(t_n\right)\right\} \left\|f\right\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3} \\
&= \max\left\{1, C_2C_3\right\} \left\|f\right\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}.
\end{aligned}$$

Now, we will prove that there would not exists norm convergence of subsequence of $\{f_n\}_{n\in\mathbb{N}}$ in $L^r_\vartheta(\mathbb{R}^d)$. The sequence obtained above certainly converges to zero in the vague topology. To prove this, for all $k \in C_c(\mathbb{R}^d)$ we obtain

$$\begin{aligned}
\left|\int_{\mathbb{R}^d} f_n(x)k(x)dx\right| &\leq \frac{1}{\vartheta_1\left(t_n\right)} \left\|k\right\|_{r',\vartheta_1} \left\|f_n\right\|_{r,\vartheta_1} \\
&\leq \frac{1}{\vartheta_1\left(t_n\right)} \left\|k\right\|_{r',\vartheta_1} \left\|f_n\right\|^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3} \qquad (5.1)
\end{aligned}$$

where $\frac{1}{r} + \frac{1}{r'} = 1$. Since the right side of (5.1) tends zero for $n \longrightarrow \infty$, we have

$$\int_{\mathbb{R}^d} f_n(x)k(x)dx \longrightarrow 0.$$

Finally by Theorem 5.1, the only possible limit of $\{f_n\}_{n\in\mathbb{N}}$ in $L^r_\vartheta(\mathbb{R}^d)$ is zero. It is known by Lemma 2.2 in [14] that there exists $C_4, C_5 > 0$ depending on f such that

$$C_4\vartheta\left(y\right) \leq \left\|T_y f\right\|_{r,\vartheta} \leq C_5\vartheta\left(y\right)$$

for all $y \in \mathbb{R}^d$. Thus we have

$$\left\|f_n\right\|_{r,\vartheta} = \left(\vartheta_1\left(t_n\right)\right)^{-2} \left\|T_{t_n} f\right\|_{r,\vartheta} \geq C_4\left(\vartheta_1\left(t_n\right)\right)^{-2} \vartheta\left(t_n\right). \qquad (5.2)$$

Since $\frac{\vartheta(t_n)}{(\vartheta_1(t_n))^2} \geq \delta > 0$ for all t_n, by using (5.2) we get

$$\left\|f_n\right\|_{r,\vartheta} \geq C_4\left(\vartheta_1\left(t_n\right)\right)^{-2} \vartheta\left(t_n\right) \geq C_4\delta.$$

Then there would not possible to find norm convergent subsequence of $\{f_n\}_{n\in\mathbb{N}}$ in $L^r_\vartheta(\mathbb{R}^d)$. This completes the proof. Q.E.D.

Theorem 5.3. If $\vartheta_4 \prec \vartheta_1$ and $\frac{\vartheta_4(y)}{(\vartheta_1(y))^2}$ doesn't tend to zero in $\mathbb{R}^d$, then the embedding $i : B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}\left(\mathbb{R}^d\right) \hookrightarrow B^{r,p,q}_{\vartheta_4,\vartheta_2,\vartheta_3}\left(\mathbb{R}^d\right)$ is never compact.

Proof. Since $\vartheta_4 \prec \vartheta_1$, then it is clear that $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}\left(\mathbb{R}^d\right) \subset B^{r,p,q}_{\vartheta_4,\vartheta_2,\vartheta_3}\left(\mathbb{R}^d\right)$. It is also known by Theorem 4.1 that the unit map from $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}\left(\mathbb{R}^d\right)$ into $B^{r,p,q}_{\vartheta_4,\vartheta_2,\vartheta_3}\left(\mathbb{R}^d\right)$ is continuous. Now take any bounded sequence of $\{f_n\}_{n\in\mathbb{N}}$ in $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}\left(\mathbb{R}^d\right)$. If there exists norm convergent subsequence of $\{f_n\}_{n\in\mathbb{N}}$ in $B^{r,p,q}_{\vartheta_4,\vartheta_2,\vartheta_3}\left(\mathbb{R}^d\right)$, this subsequence also converges in $L^r_{\vartheta_4}\left(\mathbb{R}^d\right)$. But this is not possible by Theorem 5.2, because the embedding of the space $B^{r,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}\left(\mathbb{R}^d\right)$ into $L^r_{\vartheta_4}\left(\mathbb{R}^d\right)$ is never compact. Q.E.D.

Theorem 5.4. If $\vartheta_3 \prec \vartheta_2 \prec \vartheta_1$ and $\frac{\vartheta_3(x)}{\vartheta_1(x)}$ does not tend to zero in $\mathbb{R}^d$ as $x \longrightarrow \infty$, then the embedding of the space $B^{r,p,1}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right)$ into $\left(L^p_{\vartheta_3},\ell^1\right)$ is never compact.

Proof. Since $\vartheta_3 \prec \vartheta_2 \prec \vartheta_1$, there exist $C_1, C_2 > 0$ such that $\vartheta_2(x) \leq C_1\vartheta_1(x)$ and $\vartheta_3(x) \leq C_2\vartheta_2(x)$ for all $\mathbb{R}^d$. Thus we get that $B^{r,p,1}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right) \subset \left(L^p_{\vartheta_3},\ell^1\right)$. Assume that $(t_n)_{n\in\mathbb{N}}$ is a sequence such that $t_n \longrightarrow \infty$ as $n \longrightarrow \infty$. Since $\frac{\vartheta_3(x)}{\vartheta_1(x)}$ does not tend to zero in $\mathbb{R}^d$ as $x \longrightarrow \infty$, there exists $\delta > 0$ such that $\frac{\vartheta_3(x)}{\vartheta_1(x)} \geq \delta > 0$ for $x \longrightarrow \infty$. To obtain the desired result, we take any fixed $f \in B^{r,p,1}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right)$ and define a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$, where $f_n = \left(\vartheta_1(t_n)\right)^{-1} T_{t_n} f$. This sequence is bounded in $B^{r,p,1}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right)$. Indeed, we have

$$\begin{aligned}
\|f_n\|^{r,p,1}_{\vartheta_1,\vartheta_2} &= \left(\vartheta_1(t_n)\right)^{-1} \|T_{t_n} f\|^{r,p,1}_{\vartheta_1,\vartheta_2} \\
&\leq \left(\vartheta_1(t_n)\right)^{-1} \max\left\{\vartheta_1(t_n), \vartheta_2(t_n)\right\} \|f\|^{r,p,1}_{\vartheta_1,\vartheta_2} \\
&\leq \left(\vartheta_1(t_n)\right)^{-1} \max\left\{\vartheta_1(t_n), C_1\vartheta_1(t_n)\right\} \|f\|^{r,p,1}_{\vartheta_1,\vartheta_2} \\
&= \max\{1, C_1\} \|f\|^{r,p,1}_{\vartheta_1,\vartheta_2}.
\end{aligned}$$

Now, we will prove that there would not exists norm convergence of subsequence of $\{f_n\}_{n\in\mathbb{N}}$ in $\left(L^p_{\vartheta_3},\ell^1\right)$. The sequence obtained above certainly converges to zero in the vague topology. To prove this, for all $k \in C_c\left(\mathbb{R}^d\right)$ we write

$$\left|\int_{\mathbb{R}^d} f_n(x)k(x)dx\right| \leq \frac{1}{\vartheta_1(t_n)} \|k\|_{r',\vartheta_1} \|f_n\|_{r,\vartheta_1} \leq \frac{1}{\vartheta_1(t_n)} \|k\|_{r',\vartheta_1} \|f_n\|^{r,p,1}_{\vartheta_1,\vartheta_2}.$$

Since right hand side of previous inequality tends zero for $n \longrightarrow \infty$, then we have

$$\int_{\mathbb{R}^d} f_n(x)k(x)dx \longrightarrow 0.$$

Finally, by Theorem 5.1, the only possible limit of $\{f_n\}_{n\in\mathbb{N}}$ in $\left(L^p_{\vartheta_3},\ell^1\right)$ is zero. By Theorem 4.3, there are $b_1 = b_1(f), b_2 = b_2(f) > 0$ such that

$$b_1\vartheta_3(y) \leq \|T_y f\|_{1,\vartheta_3} \leq b\|T_y f\|_{p1,\vartheta_3} \leq bb_2\vartheta_3(y)$$

or equivalently

$$\frac{b_1}{b}\vartheta_3(y) \leq \|T_y f\|_{p1,\vartheta_3} \leq b_2\vartheta_3(y)$$

for $y \in G$. This yields

$$\|f_n\|_{p1,\vartheta_3} = (\vartheta_1(t_n))^{-1} \|T_{t_n} f\|_{p1,\vartheta_3} \geq \frac{b_1}{b} (\vartheta_1(t_n))^{-1} \vartheta_3(t_n). \tag{5.3}$$

Since $\frac{\vartheta_3(t_n)}{\vartheta_1(t_n)} \geq \delta > 0$ for all t_n, by using (5.3), we get

$$\|f_n\|_{p1,\vartheta_3} \geq \frac{b_1}{b} (\vartheta_1(t_n))^{-1} \vartheta_3(t_n) \geq \frac{b_1}{b}\delta.$$

Then there would not possible to find norm convergent subsequence of $\{f_n\}_{n\in\mathbb{N}}$ in $\left(L^p_{\vartheta_3}, \ell^1\right)$. This completes the proof. Q.E.D.

Theorem 5.5. If $\vartheta_1 \prec \vartheta_2, \vartheta_3 \prec \vartheta_2$ and $\frac{\vartheta_3(x)}{\vartheta_2(x)}$ does not tend to zero in $\mathbb{R}^d$ as $x \longrightarrow \infty$, then the embedding of the space $B^{r,p,1}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right)$ into $\left(L^p_{\vartheta_3}, \ell^1\right)$ is never compact.

Proof. If we define the sequence in Theorem 5.4 as $f_n = (\vartheta_2(t_n))^{-1} T_{t_n} f$, then the proof of the following theorem is similar with the previous one. Q.E.D.

Theorem 5.6. The embedding of the space $B^{r,p,q}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right)$ into $B^{r,p,q}_{\vartheta_3,\vartheta_4}\left(\mathbb{R}^d\right)$ is never compact if

(i) $\vartheta_4 \prec \vartheta_2 \prec \vartheta_1, \vartheta_3 \prec \vartheta_1$ and $\frac{\vartheta_3(x)}{\vartheta_1(x)}$ does not tend to zero in $\mathbb{R}^d$ as $x \longrightarrow \infty$, or

(ii) $\vartheta_3 \prec \vartheta_1 \prec \vartheta_2, \vartheta_4 \prec \vartheta_2$ and $\frac{\vartheta_3(x)}{\vartheta_2(x)}$ does not tend to zero in $\mathbb{R}^d$ as $x \longrightarrow \infty$.

Proof. Let $\vartheta_4 \prec \vartheta_2 \prec \vartheta_1, \vartheta_3 \prec \vartheta_1$. Thus, there exist $C_1, C_2 > 0$ such that $\vartheta_4(x) \leq C_1\vartheta_2(x)$ and $\vartheta_3(x) \leq C_2\vartheta_1(x)$ for all $\mathbb{R}^d$. This follows that $B^{r,p,q}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right) \subset B^{r,p,q}_{\vartheta_3,\vartheta_4}\left(\mathbb{R}^d\right)$ and the unit function $I : B^{r,p,q}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right) \longrightarrow B^{r,p,q}_{\vartheta_3,\vartheta_4}\left(\mathbb{R}^d\right)$ is continuous. Now, suppose that $\frac{\vartheta_3(x)}{\vartheta_1(x)}$ does not tend to zero in $\mathbb{R}^d$ as $x \longrightarrow \infty$ and $\{f_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $B^{r,p,q}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right)$. If any subsequence of $\{f_n\}_{n\in\mathbb{N}}$ is convergent in $B^{r,p,q}_{\vartheta_3,\vartheta_4}\left(\mathbb{R}^d\right)$, then this subsequence is also convergent in $L^r_{\vartheta_3}\left(\mathbb{R}^d\right)$. However, this conflict with Theorem 5.2, because of the fact that the embedding of $B^{r,p,q}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right)$ into $L^r_{\vartheta_3}\left(\mathbb{R}^d\right)$ is never compact. This completes *(i)*. In similar way, *(ii)* can be proved. Q.E.D.

Theorem 5.7. The embedding of the space $B^{r,p,1}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right)$ into $B^{r,p,1}_{\vartheta_3,\vartheta_4}\left(\mathbb{R}^d\right)$ is never compact if

(i) $\vartheta_3 \prec \vartheta_1 \prec \vartheta_2, \vartheta_4 \prec \vartheta_2$ and $\frac{\vartheta_4(x)}{\vartheta_2(x)}$ does not tend to zero in $\mathbb{R}^d$ as $x \longrightarrow \infty$, or

(ii) $\vartheta_4 \prec \vartheta_2 \prec \vartheta_1, \vartheta_3 \prec \vartheta_1$ and $\frac{\vartheta_4(x)}{\vartheta_1(x)}$ does not tend to zero in $\mathbb{R}^d$ as $x \longrightarrow \infty$.

Proof. Let $\vartheta_3 \prec \vartheta_1 \prec \vartheta_2, \vartheta_4 \prec \vartheta_2$. Then, there are $C_1, C_2 > 0$ such that $\vartheta_4(x) \leq C_1\vartheta_2(x)$ and $\vartheta_3(x) \leq C_2\vartheta_1(x)$ for all $\mathbb{R}^d$. This follows that $B^{r,p,1}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right) \subset B^{r,p,1}_{\vartheta_3,\vartheta_4}\left(\mathbb{R}^d\right)$ and the unit function $I : B^{r,p,1}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right) \longrightarrow B^{r,p,1}_{\vartheta_3,\vartheta_4}\left(\mathbb{R}^d\right)$ is continuous. Now, suppose that $\frac{\vartheta_4(x)}{\vartheta_2(x)}$ does not tend to zero in $\mathbb{R}^d$ as $x \longrightarrow \infty$ and $\{f_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $B^{r,p,1}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right)$. If any subsequence of $\{f_n\}_{n\in\mathbb{N}}$ is convergent in $B^{r,p,1}_{\vartheta_3,\vartheta_4}\left(\mathbb{R}^d\right)$, then this subsequence is also convergent in $\left(L^p_{\vartheta_4}, \ell^1\right)$. However, this conflict with Theorem 5.4, because of the fact that the embedding of $B^{r,p,1}_{\vartheta_1,\vartheta_2}\left(\mathbb{R}^d\right)$ into $\left(L^p_{\vartheta_4}, \ell^1\right)$ is never compact. This completes *(i)*. Finally, the statement *(ii)* can be proved in similar method. Q.E.D.

6 Multipliers of $B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$

In this section, we discuss multipliers of the spaces $B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$. We define the space

$$M_{B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)} = \{\mu \in M(\vartheta_1) : \|\mu\|_M \leq C(\mu)\}$$

where

$$\|\mu\|_M = \sup\left\{ \frac{\|\mu * f\|^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}}{\|f\|_{1,\vartheta_1}} : f \in L^1_{\vartheta_1}(G),\ f \neq 0,\ \widehat{f} \in C_c(\widehat{G}) \right\}.$$

By the Proposition 2.1 in [17], we have $M_{B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)} \neq \{0\}$.

Theorem 6.1. If ϑ_1 satisfies (BD), then for a linear operator $T : L^1_{\vartheta_1}(G) \longrightarrow B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ the following are equivalent:

(i) $T \in Hom_{L^1_{\vartheta_1}(G)}\left(L^1_{\vartheta_1}(G), B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)\right)$.

(ii) There exists a unique $\mu \in M_{B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)}$ such that $Tf = \mu * f$ for every $f \in L^1_{\vartheta_1}(G)$. Moreover the correspondence between T and μ defines an isomorphism between $Hom_{L^1_{\vartheta_1}(G)}\left(L^1_{\vartheta_1}(G), B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)\right)$ and $M_{B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)}$.

Proof. It is known that $B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ is a S_{ϑ_1} space by Theorem 3.7. Thus, the proof is completed by Proposition 2.4 in [17]. Q.E.D.

Theorem 6.2. If ϑ_1 satisfies (BD) and $T \in Hom_{L^1_{\vartheta_1}(G)}\left(B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)\right)$, then there exists a unique pseudo measure $\sigma \in \left(A(\widehat{G})\right)^*$ (see [29]), such that $Tf = \sigma * f$ for all $f \in B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$.

Proof. It is known that $B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$ is a S_{ϑ_1} space by Theorem 3.7 and a Banach module over $L^1_{\vartheta_1}(G)$ by Theorem 3.8. Thus, the proof is completed by Theorem 5 in [9]. Q.E.D.

Theorem 6.3. The multiplier space $Hom_{L^1_{\vartheta_1}(G)}\left(L^1_{\vartheta_1}(G), \left(B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)\right)^*\right)$ is isomorphic to $L^\infty_{\vartheta_1^{-1}}(G) \times \left(L^{p'}_{\vartheta_2^{-1}}, \ell^{q'}_{\vartheta_3^{-1}}\right)/K$.

Proof. By Theorem 3.8, we write $L^1_{\vartheta_1}(G) * B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G) = B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)$. Hence by Corollary 2.13 in [30] and Theorem 3.9, we have

$$\begin{aligned} Hom_{L^1_{\vartheta_1}(G)}\left(L^1_{\vartheta_1}(G), \left(B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)\right)^*\right) &= \left(L^1_{\vartheta_1}(G) * B^{1,p,q}_{\vartheta_1,\vartheta_2,\vartheta_3}(G)\right)^* \\ &= L^\infty_{\vartheta_1^{-1}}(G) \times \left(L^{p'}_{\vartheta_2^{-1}}, \ell^{q'}_{\vartheta_3^{-1}}\right)/K \end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Q.E.D.

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