On pointwise approximation properties of certain nonlinear Bernstein operators

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Abstract

The present study is concerned with the nonlinear Bernstein type operators NB_nf, acting on bounded functions, where the kernel function of the operators provide some convenient assumptions. Especially, some pointwise convergence results for these type operators are achieved at a generalized Lebesgue point of the function f.

2010 Mathematics Subject Classification. **41A35**. 41A25, 47G10
Keywords. nonlinear Bernstein operators, pointwise convergence, generalized Lebesgue point.

1 Introduction

We take in consideration the problem of approximating a given real-valued function f, described on $[0,1]$, by favour of a sequence of nonlinear Bernstein operators (NB_nf). Operators like positive, linear, convolution, moment and sampling perform a significant role in several branches of Mathematics, for instance in reconstruction of signals and images, in Fourier analysis, operator theory, probability theory and approximation theory.

Let f be a function described on the interval $[0,1]$ and let $\mathbb{N} := \{1,2,...\}$. The classical Bernstein operators B_nf described as

$$(B_nf)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x) \ , \ 0 \le x \le 1 \ , \quad n \in \mathbb{N}, \tag{1}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ is the well-known Bernstein basis. These polynomials were introduced by Bernstein [12] in 1912 to give the first useful proof of the Weierstrass approximation theorem. Some features of the polynomials (1) can be found in [3].

A singular integral generally written in the form

$$(T_nf)(x) = \int_a^b f(t) K_n(x,t)\, dt \tag{2}$$

where $K_n(x,t)$ is the kernel, described for $a \le x \le b$, $a \le t \le b$, which has the speciality that for functions $f(x)$ of a certain class and in a certain sense, $(T_nf)(x)$ converges to $f(x)$ as $n \to \infty$.

The Bernstein polynomials (1) is a finite sum of a type corresponding to the integral (2). Both (1) and (2) are specific cases of singular Stieltjes integrals. (1) may be written in the form of a

Tbilisi Mathematical Journal 12(2) (2019), pp. 47–58.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 10 August 2018.
Accepted for publication: 25 February 2019.

Stieltjes integral in the variable t,

$$(B_n f)(x) = \int_0^1 f(t)\, d_t K_n(x,t)$$

with the kernel

$$\begin{aligned} K_n(x,t) &= \sum_{k \le nt} \binom{n}{k} x^k (1-x)^{n-k} \quad , \quad 0 < t \le 1 \\ K_n(x,0) &= 0 \end{aligned}$$

which is constant in any interval $k/n \le t < (k+1)/n$, $k = 0, 1, ..., n-1$.

An earlier and significant study on pointwise convergence of linear singular integrals depending on two parameters is that of Taberski [11]. In this paper Taberski has studied that the pointwise convergence at the points which are Lebesgue points of integrable functions in $L_1(-\pi, \pi)$, by the family of convolution type singular integral operators of the form

$$U(f; x, \lambda) = \int_{-\pi}^{\pi} f(t)\, K(t - x, \lambda)\, dt \quad , \quad x \in (-\pi, \pi), \tag{3}$$

where $K(t, \lambda)$ is a kernel which provide some convenient assumptions. From that time in several papers some extensions and generalizations of the pointwise convergence theorems of the operators (3) have been given. Specifically, Karsli [4], [5], [6] and [7]obtained the pointwise convergence of some nonlinear integral type operators at Lebesgue points, generalized Lebesgue points and μ-generalize Lebesgue points of integrable functions in $L_1(a, b)$, where (a, b) is an arbitrary interval in $\mathbb{R}$.

We now express a brief and theoretical statement of the connection between approximation by linear and nonlinear operators. Approximation with nonlinear integral operators of convolution type was established by Musielak in [10] and substantially improved in [2]. In [10], the presumption of linearity of the singular integral operators was replaced by a presumption of a Lipschitz condition for the kernel function $K_\lambda(t, u)$ with respect to the second variable. After this approachment, several mathematicians have undertaken the program of extending approximation by nonlinear operators in many ways, involving pointwise and uniform convergence, Korovkin type theorems in abstract function spaces, sampling series and so on.

Lately, by virtue of the techniques of Musielak [10], Karsli, Tiryaki and Altin [9] established the following type nonlinear counterpart of the well-known Bernstein operators;

$$(NB_n f)(x) = \sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) \quad , \quad 0 \le x \le 1 \quad , \quad n \in \mathbb{N}, \tag{4}$$

acting on bounded functions f on an interval $[0, 1]$, where $P_{n,k}$ provide some convenient assumptions. In this work, we deal with a obvious nonlinear counterpart of the Bernstein operators, established in [9] and improved in [8]. Especially, some pointwise convergence results for these type operators are achieved at a generalized Lebesgue point of the function f.

A structure of the study is as follows: The following section involves foundation definitions and notations. In Section 3, we give some particular results which are essential to prove the main result. The final section, that is Section 4, the major approximation results of this study are given.

2 Preliminaries

In this section, we put together the foundation definitions and notations which will be used during the study.

Let X be the set of all bounded Lebesgue measurable functions $f : [0,1] \to \mathbb{R}$.

Let Ψ be the class of all functions $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that the function ψ is continuous and concave with $\psi(0) = 0$, $\psi(u) > 0$ for $u > 0$.

We now determine a sequence of functions. Let $\{P_{n,k}\}_{n\in\mathbb{N}}$ be a sequence functions $P_{n,k} : [0,1]\,\mathrm{x}\,\mathbb{R} \to \mathbb{R}$ defined by

$$P_{n,k}(t,u) = p_{n,k}(t)H_n(u) \tag{5}$$

for every $t \in [0,1], u \in \mathbb{R}$, where $H_n : \mathbb{R} \to \mathbb{R}$ is such that $H_n(0) = 0$ and $p_{n,k}(t)$ is the Bernstein basis.

Throughout the paper we suppose that $\mu : \mathbb{N} \to \mathbb{R}^+$ is an increasing and continuous function such that $\lim_{n\to\infty} \mu(n) = \infty$.

First of all we suppose that the following assumptions hold:

a) $H_n : \mathbb{R} \to \mathbb{R}$ is such that

$$|H_n(u) - H_n(v)| \le \psi(|u - v|), \quad \psi \in \Psi,$$

holds for every $u, v \in \mathbb{R}$, for every $n \in \mathbb{N}$. That is, H_n *provides* a $(L - \psi)$ Lipschitz condition.

b) We now set

$$K_n(x,u) := \begin{cases} \sum\limits_{k\le nu} p_{n,k}(x) & , \quad 0 < u \le 1 \\ \\ 0 & , \quad u = 0 \end{cases} \tag{6}$$

and

$$A_n(x) := \int\limits_{x-x/n^{\gamma/\beta}}^{x+(1-x)/n^{\gamma/\beta}} d_t\left(K_n(x,t)\right) \quad \text{for any fixed } x \in (0,1)$$

where $\beta > 0$, $\gamma \ge 1$ and

$$\lambda_n(x,t) := \int\limits_0^t d_u K_n(x,u). \tag{7}$$

c) Denoting by $r_n(u) := H_n(u) - u$, $u \in \mathbb{R}$ and $n \in \mathbb{N}$. Suppose that for n adequately large

$$\sup_u |r_n(u)| \le \frac{1}{\mu(n)},$$

holds.

For any function f for which the one-sided limits $f(x+), f(x-)$ exist at every point $x \in (0,1)$, we let

$$f_x(t) = \begin{cases} f(t) - f(x+) & , \ x < t \leq 1 \\ 0 & , \ t = x \\ f(t) - f(x-) & , \ 0 \leq t < x \end{cases} \tag{8}$$

and $\bigvee_0^1 \psi(|f_x|)$ is the total variation of $\psi(|f_x|)$ on $[0,1]$. The symbol $[a]$ will denote the greatest integer.

3 Auxiliary result

In this section we give specific results, which are essential to prove our theorems.

Definition 1. A point $x_0 \in \mathbb{R}$ is called a generalized Lebesgue point of the function f, if

$$\lim_{h \to 0^+} \frac{1}{h^{\alpha+1}} \int_0^h |f(x_0+t) - f(x_0)| \, dt = 0 \quad , \quad 0 \leq \alpha < 1, \tag{9}$$

holds.

Lemma 1. ([9], Lemma 2)*For all $x \in (0,1)$ and for each $n \in \mathbb{N}$, let*

$$NB_n((t-x)^\beta; x) := \int_0^1 |u-x|^\beta \, d_u(K_n(x,u)) \leq \frac{A_n(x)}{n^{\gamma/\beta}}, \ (\beta > 0) \tag{10}$$

holds, where $A_n(x)$ is as defined in Section 2. Then one has

$$\lambda_n(x,t) =: \int_0^t d_u(K_n(x,u)) \leq \frac{A_n(x)}{(x-t)^\beta n^{\gamma/\beta}} \ , \quad 0 \leq t < x, \tag{11}$$

and

$$1 - \lambda_n(x,t) = \int_t^1 d_u(K_n(x,u)) \leq \frac{A_n(x)}{(t-x)^\beta n^{\gamma/\beta}} \ , \quad x < t < 1. \tag{12}$$

The following lemma is the slight variance of the Lemma 1 in [1].

Lemma 2 . Let $\psi \in \Psi$. Then, if $x_0 \in \mathbb{R}$ is a generalized Lebesgue point of the function f, we have

$$\left| \int_0^h \psi(|f(x_0+t) - f(x_0)|) \, dt \right| = o\left(|h|^{\alpha+1}\right) \ as \ h \to 0. \tag{13}$$

Proof . In order to prove our lemma we will show the following two statements:

$$\begin{aligned}\left|\int_0^h \psi\left(\left|f\left(x_0+t\right)-f\left(x_0\right)\right|\right)dt\right| &= o\left(h^{\alpha+1}\right) \ as\ h\to 0^+,\\ \left|\int_h^0 \psi\left(\left|f\left(x_0+t\right)-f\left(x_0\right)\right|\right)dt\right| &= o\left(-h^{\alpha+1}\right) \ as\ h\to 0^-.\end{aligned}$$

Since ψ is concave, one has for $h<0$ and $h>0$, respectively,

$$\frac{1}{-h^{\alpha+1}}\int_h^0 \psi\left(\left|f\left(x_0+t\right)-f\left(x_0\right)\right|\right)dt \le \psi\left(\frac{1}{-h^{\alpha+1}}\int_h^0 \left|f\left(x_0+t\right)-f\left(x_0\right)\right|dt\right)$$

and

$$\frac{1}{h^{\alpha+1}}\int_0^h \psi\left(\left|f\left(x_0+t\right)-f\left(x_0\right)\right|\right)dt \le \psi\left(\frac{1}{h^{\alpha+1}}\int_0^h \left|f\left(x_0+t\right)-f\left(x_0\right)\right|dt\right).$$

Hence, by continuity of ψ and $\psi(0)=0$, we reach the desired result.

4 Convergence results

We pay regard to the following type nonlinear Bernstein operators,

$$\left(NB_nf\right)(x)=\sum_{k=0}^{n}P_{n,k}\left(x,f\left(\frac{k}{n}\right)\right)$$

defined for every $f\in X$ for which NB_nf is well-defined, where

$$P_{n,k}(x,u)=p_{n,k}(x)H_n(u)$$

for every $x\in[0,1]$ and $u\in\mathbb{R}$.

We are now ready to constitute the major consequence of this study:

Theorem 1. Let $\psi\in\Psi$ and $f\in L_1([0,1])$ be such that $\psi\circ|f|\in BV([0,1])$. Suppose that $P_{n,k}(x,u)$ satisfies conditions (a), (b) and (c). Then at each point $x\in(0,1)$ for which (9) holds we have for each $\epsilon>0$ and for adequately large $n\in\mathbb{N}$,

$$\begin{aligned}\left|\left(NB_nf\right)(x)-f(x)\right| \le\ & \epsilon(\alpha+1)\\ &+\frac{A_n^*(x)}{n^{\frac{\gamma}{\beta}}}\left[\bigvee_0^1\psi\left(|f_x|\right)+\sum_{k=1}^{[n^\gamma]}\bigvee_{x-x/k^{1/\beta}}\psi\left(|f_x|\right)\right]\\ &+\frac{1}{\mu(n)}\end{aligned}$$

where $A_n^*(x) = A_n(x) \max\left\{x^{-\beta}, (1-x)^{-\beta}\right\}$, $(\beta > 0)$.

Proof. Assume that

$$x + \delta < 1\ ,\ x - \delta > 0 \tag{14}$$

for any sufficiently small $\delta > 0$.

Let

$$\begin{aligned} |I_n(x)| &= |(NB_n f)(x) - f(x)| \\ &= \left|\sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) - f(x)\right|. \end{aligned}$$

¿From (4) and using triangle inequality, we can rewrite $|I_n(x)|$ as follows:

$$\begin{aligned} |I_n(x)| &\leq \left|\sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) - \sum_{k=0}^{n} P_{n,k}(x, f(x))\right| \\ &\quad + \left|\sum_{k=0}^{n} P_{n,k}(x, f(x)) - f(x)\right| \\ &= I_{n,1}(x) + I_{n,2}(x) \end{aligned}$$

¿From (c) it is easy to see that the second term of the right-hand-side of the above inequality is less than or equal to $\frac{1}{\mu(n)}$. Indeed;

$$\begin{aligned} I_{n,2}(x) &= \left|\sum_{k=0}^{n} P_{n,k}(x, f(x)) - f(x)\right| \\ &= \left|\sum_{k=0}^{n} p_{n,k}(x) H_n(f(x)) - \sum_{k=0}^{n} p_{n,k}(x) f(x)\right| \\ &= |H_n(f(x)) - f(x)| \sum_{k=0}^{n} p_{k,n}(x) \\ &\leq \frac{1}{\mu(n)} \end{aligned}$$

holds for n adequately large.

Using (a) and Lebesgue Stieltjes integral representation of Bernstein polynomial, we have the following inequality for $I_{n,1}(x)$,

$$\begin{aligned} I_{n,1}(x) &= \left|\sum_{k=0}^{n} P_{n,k}\left(x, f\left(\frac{k}{n}\right)\right) - \sum_{k=0}^{n} P_{n,k}(x, f(x))\right| \\ &\leq \sum_{k=0}^{n} \psi\left(\left|f\left(\frac{k}{n}\right) - f(x)\right|\right) p_{n,k}(x) \\ &= \int_0^1 \psi(|f(t) - f(x)|)\, d_t K_n(x,t). \end{aligned}$$

According to (b), we can split the last integral in three terms as follows:

$$\begin{aligned} I_{n,1}(x) &\leq \left(\int_0^{x-x/n^{\frac{\gamma}{\beta}}} + \int_{x-x/n^{\frac{\gamma}{\beta}}}^{x+(1-x)/n^{\frac{\gamma}{\beta}}} + \int_{x+(1-x)/n^{\frac{\gamma}{\beta}}}^{1} \right) \psi\left(|f(t)-f(x)|\right) d_t K_n(x,t) \\ &= I_{n,1,1}(x) + I_{n,1,2}(x) + I_{n,1,3}(x). \end{aligned}$$

First, we estimate $I_{n,1,2}(x)$. We have for $t \in \left[x - x/n^{\frac{\gamma}{\beta}}, x + (1-x)/n^{\frac{\gamma}{\beta}}\right]$

$$\begin{aligned} I_{n,1,2}(x) &= \int_{x-x/n^{\frac{\gamma}{\beta}}}^{x+(1-x)/n^{\frac{\gamma}{\beta}}} \psi\left(|f(t)-f(x)|\right) d_t K_n(x,t) \\ &\leq \int_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \psi\left(|f(t)-f(x)|\right) d_t K_n(x,t) \\ &\quad + \int_{x}^{x+(1-x)/n^{\frac{\gamma}{\beta}}} \psi\left(|f(t)-f(x)|\right) d_t K_n(x,t) \\ &= I_{n,1,2,1}(x) + I_{n,1,2,2}(x). \end{aligned}$$

Setting

$$F(t) := \int_t^x \psi\left(|f(y)-f(x)|\right) dy,$$

then, according to Lemma (2), for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$F(t) \leq \epsilon (x-t)^{\alpha+1} \tag{15}$$

for all $0 < x - t \leq \delta$. Now fix this δ and estimate $I_{n,1,2,1}(x)$ and $I_{n,1,2,2}(x)$ respectively. Recall the Lebesgue-Stieltjes integral representation and using (7), we can write $I_{n,1,2,1}(x)$ as follows:

$$\begin{aligned} I_{n,1,2,1}(x) &= \int_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \psi\left(|f(t)-f(x)|\right) d_t K_n(x,t) \\ &= \int_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \psi\left(|f(t)-f(x)|\right) \frac{\partial}{\partial t} \lambda_n(x,t)\, dt \\ &= \int_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \frac{\partial}{\partial t} \lambda_n(x,t)\, d\left(-F(t)\right). \end{aligned}$$

Applying partial Lebesgue-Stieltjes integration and using (15), we attain,

$$\begin{aligned} I_{n,1,2,1}(x) &= \frac{\partial}{\partial t}\left(\lambda_n\left(x, x-x/n^{\frac{\gamma}{\beta}}\right)\right) F\left(x-x/n^{\frac{\gamma}{\beta}}\right) \\ &\quad + \int_{x-x/n^{\frac{\gamma}{\beta}}}^{x} F(t)\,\frac{\partial^2}{\partial t^2}\lambda_n(x,t)\,dt \\ &\leq \frac{\partial}{\partial t}\left(\lambda_n\left(x, x-x/n^{\frac{\gamma}{\beta}}\right)\right)\epsilon\,\left(\frac{x}{n^{\gamma/\beta}}\right)^{\alpha+1} \\ &\quad +\epsilon \int_{x-x/n^{\frac{\gamma}{\beta}}}^{x} (x-t)^{\alpha+1}\,\frac{\partial^2}{\partial t^2}\lambda_n(x,t)\,dt. \end{aligned}$$

Integration by parts again gives

$$\begin{aligned} I_{n,1,2,1}(x) &= \epsilon\,\left(\frac{x}{n^{\gamma/\beta}}\right)^{\alpha+1}\frac{\partial}{\partial t}\left(\lambda_n\left(x, x-x/n^{\frac{\gamma}{\beta}}\right)\right) \\ &\quad +\epsilon\left\{-\left(\frac{x}{n^{\gamma/\beta}}\right)^{\alpha+1}\frac{\partial}{\partial t}\left(\lambda_n\left(x, x-x/n^{\frac{\gamma}{\beta}}\right)\right) + \int_{x-x/n^{\frac{\gamma}{\beta}}}^{x}(\alpha+1)(x-t)^{\alpha}\,\frac{\partial}{\partial t}\lambda_n(x,t)\,dt\right\} \\ &= \epsilon(\alpha+1)\int_{x-x/n^{\frac{\gamma}{\beta}}}^{x}(x-t)^{\alpha}\,\frac{\partial}{\partial t}\lambda_n(x,t)\,dt \\ &= \epsilon(\alpha+1)\int_{x-x/n^{\frac{\gamma}{\beta}}}^{x}(x-t)^{\alpha}\,d_tK_n(x,t) \qquad (16) \end{aligned}$$

We can use a similar method for $I_{n,1,2,2}(x)$. Then, we find the following inequality,

$$I_{n,1,2,2}(x) \leq \epsilon(\alpha+1)\int_{x}^{x+(1-x)/n^{\frac{\gamma}{\beta}}}(t-x)^{\alpha}\,d_tK_n(x,t). \qquad (17)$$

Combaning (16) and (17), we attain

$$I_{n,1,2}(x) \leq \epsilon(\alpha+1)\int_{x-x/n^{\frac{\gamma}{\beta}}}^{x+(1-x)/n^{\frac{\gamma}{\beta}}}|t-x|^{\alpha}\,d_tK_n(x,t).$$

Since $0 \leq \alpha < 1$ and in view of $\int_0^1 d_tK_n(x,t) = 1$, we have

$$I_{n,1,2}(x) \leq \epsilon(\alpha+1).$$

Next, we estimate $I_{n,1,1}(x)$. Using partial Lebesgue-Stieltjes integration, we attain

$$\begin{aligned} I_{n,1,1}(x) &= \int_0^{x-x/n^{\frac{\gamma}{\beta}}} \psi\left(|f(t)-f(x)|\right) d_t K_n(x,t) \\ &= \int_0^{x-x/n^{\frac{\gamma}{\beta}}} \psi\left(|f_x(t)|\right) \frac{\partial}{\partial t}\lambda_n(x,t)\, dt \\ &= \psi\left(\left|f_x\left(x-\frac{x}{n^{\frac{\gamma}{\beta}}}\right)\right|\right)\lambda_n\left(x,x-\frac{x}{n^{\frac{\gamma}{\beta}}}\right) \\ &\quad - \int_0^{x-x/n^{\frac{\gamma}{\beta}}} \lambda_n(x,t)\, d_t\left(\psi\left(|f_x(t)|\right)\right). \end{aligned}$$

By Lemma (1), it is obvious that

$$\lambda_n\left(x,x-\frac{x}{n^{\frac{\gamma}{\beta}}}\right) \leq A_n(x)\, x^{-\beta}\left(n^{\frac{\gamma}{\beta}}\right)^{\beta-1} \tag{18}$$

and also note that

$$\begin{aligned} \psi\left(\left|f_x\left(x-\frac{x}{n^{\frac{\gamma}{\beta}}}\right)\right|\right) &= \left|\psi\left(\left|f_x\left(x-\frac{x}{n^{\frac{\gamma}{\beta}}}\right)\right|\right)-\psi\left(|f_x(x)|\right)\right| \\ &\leq \bigvee_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \psi\left(|f_x|\right). \end{aligned} \tag{19}$$

Using (18) and (19) we can write

$$\begin{aligned} I_{n,1,1}(x) &\leq \bigvee_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \psi\left(|f_x|\right) A_n(x)\, x^{-\beta}\left(n^{\frac{\gamma}{\beta}}\right)^{\beta-1} \\ &\quad + \int_0^{x-x/n^{\frac{\gamma}{\beta}}} \lambda_n(x,t)\, d_t\left(-\bigvee_t^{x} \psi\left(|f_x|\right)\right) \\ &\leq \bigvee_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \psi\left(|f_x|\right) A_n(x)\, x^{-\beta}\left(n^{\frac{\gamma}{\beta}}\right)^{\beta-1} \\ &\quad + \frac{A_n(x)}{n^{\frac{\gamma}{\beta}}} \int_0^{x-x/n^{\frac{\gamma}{\beta}}} (x-t)^{-\beta}\, d_t\left(-\bigvee_t^{x} \psi\left(|f_x|\right)\right) \end{aligned}$$

then, applying partial integration, we attain

$$\begin{aligned} I_{n,1,1}(x) &\leq \bigvee_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \psi(|f_x|) A_n(x) x^{-\beta} \left(n^{\frac{\gamma}{\beta}}\right)^{\beta-1} \\ &\quad + \frac{A_n(x)}{n^{\frac{\gamma}{\beta}}} \left[-x^{-\beta} n^{\gamma} \bigvee_{x-x/n^{\frac{\gamma}{\beta}}}^{x} \psi(|f_x|) + x^{-\beta} \bigvee_{0}^{x} \psi(|f_x|) \right. \\ &\quad \left. + \int_{0}^{x-x/n^{\frac{\gamma}{\beta}}} \bigvee_{t}^{x} \psi(|f_x|) \frac{\beta}{(x-t)^{\beta+1}} dt \right] \\ &= \frac{A_n(x)}{n^{\frac{\gamma}{\beta}}} \left[x^{-\beta} \bigvee_{0}^{x} \psi(|f_x|) + \int_{0}^{x-x/n^{\frac{\gamma}{\beta}}} \bigvee_{t}^{x} \psi(|f_x|) \frac{\beta}{(x-t)^{\beta+1}} dt \right]. \end{aligned}$$

Changing the variable t by $x - x/u^{1/\beta}$ in the last integral, we have

$$\begin{aligned} \int_{0}^{x-x/n^{\frac{\gamma}{\beta}}} \bigvee_{t}^{x} \psi(|f_x|) \frac{\beta}{(x-t)^{\beta+1}} dt &= \frac{1}{x^{\beta}} \int_{1}^{n^{\gamma}} \bigvee_{x-x/u^{1/\beta}}^{x} \psi(|f_x|) du \\ &\leq \frac{1}{x^{\beta}} \sum_{k=1}^{[n^{\gamma}]} \bigvee_{x-x/k^{1/\beta}}^{x} \psi(|f_x|). \end{aligned}$$

Eventually, we attain

$$I_{n,1,1}(x) \leq \frac{A_n(x)}{n^{\frac{\gamma}{\beta}}} x^{-\beta} \left[\bigvee_{0}^{x} \psi(|f_x|) + \sum_{k=1}^{[n^{\gamma}]} \bigvee_{x-x/k^{1/\beta}}^{x} \psi(|f_x|) \right].$$

Using a similar method for $I_{n,1,3}(x)$, we can find

$$I_{n,1,3}(x) \leq \frac{A_n(x)}{n^{\frac{\gamma}{\beta}}} (1-x)^{-\beta} \left[\bigvee_{x}^{1} \psi(|f_x|) + \sum_{k=1}^{[n^{\gamma}]} \bigvee_{x}^{x+(1-x)/k^{1/\beta}} \psi(|f_x|) \right].$$

Collecting the above calculations, we get

$$\begin{aligned} I_{n,1}(x) &\leq \epsilon(\alpha+1) \\ &\quad + \frac{A_n^*(x)}{n^{\frac{\gamma}{\beta}}} \left[\bigvee_{0}^{1} \psi(|f_x|) + \sum_{k=1}^{[n^{\gamma}]} \bigvee_{x-x/k^{1/\beta}}^{x+(1-x)/k^{1/\beta}} \psi(|f_x|) \right] \end{aligned}$$

where $A_n^*(x) = A_n(x) \max\left\{x^{-\beta}, (1-x)^{-\beta}\right\}$, $(\beta > 0)$. This completes the proof of the theorem.

Theorem 2. Let $\psi \in \Psi$ and $f \in L_1([0,1])$ be such that $\psi \circ |f| \in BV([0,1])$. Suppose that $P_{n,k}(x,u)$ satisfies conditions (a), (b) and (c). Then at each point $x \in (0,1)$ for which (9) holds we have

$$\lim_{n\to\infty} |(NB_n f)(x) - f(x)| = 0.$$

Proof . From Theorem 1 and the definition of ψ function we reach the result, by the arbitrariness of $\epsilon > 0$.

5 Acknowledgements

The author is extremely grateful to the referees for their helpful remarks and valuable suggestions.

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