Meromorphic solutions of nonlinear ordinary differential equations

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Abstract

Meromorphic solutions of a class of nonlinear ordinary differential equations in $g(z)$ under the assumption that finite-degree polynomials with coefficients as small functions of g share one-value with the nonlinear differential operator of the equation. The concept of sharing one-value is used in the sense of counting multiplicity or ignoring multiplicity.

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1 Introduction

Let us consider differential equation which is in the form

$$L(g) = P(g), \tag{1.1}$$

where $L(g)$ is a nonlinear differential operator and $P(g)$ is an infinite power series in $g(z)$. Suggestively equation (1.1) may be written as

$$\sum_{l=1}^{k} b_l g^{(l)} = \sum_{i=0}^{\infty} a_i g^i, \tag{1.2}$$

where $g^{(l)} = \dfrac{d^l g}{dz^l}$, a_0, a_1, ... and b_1, b_2,...,b_k are small functions of g. By taking an infinite series in g on the right hand side of equation (1.1) we are encompassing transcendental functions of g, expandable as a Taylor series, within the ambit of our study.

When computation is done, however, on the right hand side of equation (1.2) we consider only finite number of terms. Now let us focus on general differential equation of the form:

$$\sum_{l=1}^{k} b_l g^{(l)} = \sum_{i=0}^{n} a_i g^i. \tag{1.3}$$

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Solving a nonlinear differential equation is, in general, difficult since there are no general methods of solving these equations. In Table 1 we document several classical differential equations which conform to the class of differential equations described by equation (1.3) and these are not handled in a unified way.

Sl.No.	order	n values	Equation
1.	1	1	Linear differential equation $b_1 g' = a_0 + a_1 g$
2.	1	2	Ricatti differential equation $b_1 g' = a_0 + a_1 g + a_2 g^2$
3.	1	3	Abel's differential equation $b_1 g' = a_0 + a_1 g + a_2 g^2 + a_3 g^3$
4.	1	n	Bernoulli differential equation $b_1 g' = a_1 g + a_n g^n$
5.	1	$2n+1$	Ginzburg-Landau differential equation $b_1 g' = a_1 g + a_3 g^3 + ... + a_{2n+1} g^{2n+1}$
6.	2	0	Kiddler differential equation $b_1 g' + b_2 g'' = 0$
7.	2	0	Langmuir-Blodgett differential equation $b_2 g'' = a_0$
8.	2	0	Poisson-Boltzmann differential equation $b_2 g'' = e^g - \alpha/z$
9.	2	1	Vandre Pol differential equation $b_2 g'' + b_1 g' = a_1 g$
10.	2	2	Painleve I transcendent differential equation $b_2 g'' = a_0 + a_2 g^2$
11.	2	3	Painleve II transcendent differential equation $b_2 g'' = a_0 + a_1 g + a_3 g^3$
12.	2	3	Jacobi elliptic equation $b_2 g'' = a_0 + a_1 g + a_2 g^2 + a_3 g^3$
13.	3	0	Blasius equation $b_3 g''' + b_2 g'' = 0$
14.	2	3	Duffing equation $b_2 g'' + b_1 g' = \gamma cos\omega z + a_1 g + a_0 g^3$

Literature survey on the application of Nevanlinna Theory in solving ordinary differential equations (see [1], [2], [4], [5] and [6]) exemplies the fact that the theory could be put to great use in handling a class of nonlinear ordinary differential equations that have the form given in equation (1.3). Further, there are ready made results on uniqueness of meromorphic functions and entire functions that could be used in ensuring that any such solution of nonlinear ordinary differential equations can be useful. As a first step towards a more unambiguous attempt at solving more difficult equations than equation (1.3), we assume that $P(g) = \sum_{i=0}^{n} a_i g^i$ in equation (1.3) shares one value with the nonlinear differential operator $L(g) = \sum_{l=1}^{k} b_l g^{(l)}$. In so doing Nevanlinna theory shows us there are 2 routes to obtaining meromorphic solutions of the class of nonlinear differential equations represented by equation (1.3):

a) sharing one value in the sense of counting multiplicities(CM),

b) sharing one value in the sense of ignoring multiplicities(IM).

In the paper, we tread both these routes and present meromorphic solutions of equation (1.3). The outline of the paper is to provide essentials of Nevanlinna theory in Appendix B and to provide three useful lemmas, with proofs wherever necessary, in Appendix A and make use of these appendices in the proof of the theorem to get meromorphic solutions of equation (1.3).

2 Main results with proofs

Theorem 2.1. Suppose g is a non-constant meromorphic function. Let $L(g)-1$ and $L(g)-g$ share the value 0 CM where b_1, b_2,...,b_k denotes small functions of g with $b_k \not\equiv 0$ and let $\overline{N}_{=n}(r,g) \neq S(r,g)$ with n be positive integer, such that g and $L(g)$ share the value 1 CM and g satisfies the following expression

$$g(z) = \frac{\int \dots \int \left(1 - ce^{-\sum_{l=1}^{k}\left(\frac{z^l}{(-1)^{l-1}b_l n\dots(n+l-1)}\right)}\right)^{n-1} dz\dots dz(l\ times)}{\left(1 - ce^{-\sum_{l=1}^{k}\left(\frac{z^l}{(-1)^{l-1}b_l n\dots(n+l-1)}\right)}\right)^{n}}, \tag{2.1}$$

where $c \neq 0$ be a constant and $k, n(\geq 1)$ are positive integers.

Proof. Since $L(g)-1$ and $L(g)-g(z)$ share 0 CM and $\beta(z)$ be entire function so that

$$\frac{L(g)-g(z)}{L(g)-1} = e^{\beta(z)}. \tag{2.2}$$

Let z_∞ be a pole of g with multiplicity $n \geq 1$ and the Laurent expansion of g about z_∞ is

$$g(z) = \frac{a_n}{(z-z_\infty)^n} + \frac{a_{n-1}}{(z-z_\infty)^{n-1}} + \dots\dots, (a_n \neq 0). \tag{2.3}$$

By differentiating equation (2.3) 'l' times, we get

$$g^{(l)}(z) = \frac{(-1)^l n(n+1)\dots(n+l-1)}{(z-z_\infty)^{n+l}}a_n + \frac{(-1)^l (n-1)n(n+1)\dots(n+l-2)}{(z-z_\infty)^{n+l-1}}a_{n-1} + \dots \tag{2.4}$$

Multiplying equation (2.4) by b_l and summing with respect to l from 1 to k, we get

$$\begin{aligned} L(g) = \sum_{l=1}^{k} b_l \Bigg[& \frac{(-1)^l n(n+1)\dots(n+l-1)}{(z-z_\infty)^{n+l}}a_n \\ & + \frac{(-1)^l (n-1)n(n+1)\dots(n+l-2)}{(z-z_\infty)^{n+l-1}}a_{n-1} + \dots \Bigg]. \end{aligned} \tag{2.5}$$

Equations (2.3) and (2.5) give us

$$\begin{aligned} L(g) - g(z) = & \sum_{l=1}^{k} b_l \frac{(-1)^l n(n+1)\dots(n+l-1)}{(z-z_\infty)^{n+l}}a_n \\ & + \sum_{l=1}^{k} b_l \frac{(-1)^l (n-1)n(n+1)\dots(n+l-2)}{(z-z_\infty)^{n+l-1}}a_{n-1} + \dots \\ & - \frac{a_n}{(z-z_\infty)^n} - \frac{a_{n-1}}{(z-z_\infty)^{n-1}} - \dots . \end{aligned} \tag{2.6}$$

It follows from equations (2.2), (2.5) and (2.6) that

$$e^{\beta(z)} = 1 - \sum_{l=1}^{k} \frac{(z - z_\infty)^l}{(-1)^l b_l n(n+1)...(n+l-1)} + ... \ . \tag{2.7}$$

We now differentiate equation (2.7) w.r.t. 'z' to get

$$\beta'(z)e^{\beta(z)} = -\sum_{l=1}^{k} \frac{l(z - z_\infty)^{l-1}}{(-1)^l b_l n(n+1)...(n+l-1)} + ... \ . \tag{2.8}$$

Eliminating $e^{\beta(z)}$ between equations (2.7) and (2.8), we get

$$\beta'(z) = -\sum_{l=1}^{k} \frac{l(z - z_\infty)^{l-1}}{(-1)^l b_l n(n+1)...(n+l-1)} + ... \ . \tag{2.9}$$

Differentiating (2.9) '$l-1$' times w.r.t. 'z', we get

$$\beta^{(l)}(z) = \sum_{l=1}^{k} \frac{l!}{(-1)^{l-1} b_l n(n+1)...(n+l-1)} + ... \ . \tag{2.10}$$

Now we have two cases.

Case 1:

$$\beta^{(l)}(z) \equiv \sum_{l=1}^{k} \frac{l!}{(-1)^{l-1} b_l n(n+1)...(n+l-1)} \quad (n \in I^+). \tag{2.11}$$

From equations (2.2) and (2.11), we have

$$\frac{L(g) - g(z)}{L(g) - 1} = \frac{1}{c} e^{\sum_{l=1}^{k} \left(\frac{z^l}{(-1)^{l-1} b_l n(n+1)...(n+l-1)} \right)}, \tag{2.12}$$

where c is a non-zero constant. With some algebra, equation (2.12) may be written in compact form as:

$$\begin{aligned} &\frac{d^l}{dz^l} \left[\left(1 - ce^{-\sum_{l=1}^{k} \left(\frac{z^l}{(-1)^{l-1} b_l n...(n+l-1)} \right)} \right)^n g \right] \\ &= \left(1 - ce^{-\sum_{l=1}^{k} \left(\frac{z^l}{(-1)^{l-1} b_l n...(n+l-1)} \right)} \right)^{n-1} . \end{aligned} \tag{2.13}$$

By integrating equation (2.13) 'l' times, we get

$$g(z)=\frac{\int\ldots\int\left(1-ce^{-\sum_{l=1}^{k}\left(\frac{z^l}{(-1)^{l-1}b_l n\ldots(n+l-1)}\right)}\right)^{n-1}dz\ldots dz(l\ times)}{\left(1-ce^{-\sum_{l=1}^{k}\left(\frac{z^l}{(-1)^{l-1}b_l n\ldots(n+l-1)}\right)}\right)^{n}}. \tag{2.14}$$

We can easily see that equation (2.14) is essentially the identity (2.1). From equation (2.14), we find that g and $\sum_{l=1}^{k}b_l g^{(l)}$ share 1 CM.

Case 2: $\beta^{(l)}(z)\not\equiv\sum_{l=1}^{k}\frac{l!}{(-1)^{l-1}b_l n(n+1)\ldots(n+l-1)}$, for all $(n\in I^{+})$.
In this case we may write

$$\begin{aligned}\overline{N}_{=n}(r,g)&\leq N\left(r,\frac{1}{\beta^{(l)}(z)-\sum_{l=1}^{k}\frac{l!}{(-1)^{l-1}b_l n(n+1)\ldots(n+l-1)}}\right),\\&\leq T(r,\beta^{(l)}(z))+O(1),\\&\leq o(T(r,e^{\beta(z)})),\\&\leq S(r,e^{\beta(z)}).\end{aligned} \tag{2.15}$$

From equation (2.2), we find that

$$\begin{aligned}T(r,e^{\beta})&\leq T(r,L(g)-g)+T(r,L(g)-1)+O(1),\\&\leq T(r,g)+2T(r,L(g))+O(1),\\&\leq(2k+3)T(r,g)+S(r,g).\end{aligned} \tag{2.16}$$

Since $T(r,g^{(k)})\leq(k+1)T(r,g)+S(r,g)$, equations (2.15) and (2.16) yield the result $\overline{N}_{=n}(r,g)=S(r,g)$, for all positive integers n, which contradicts our assumption made earlier. Q.E.D.

Using Theorem 2.1 we can now prove Theorem 2.2 .

Theorem 2.2. Suppose g is a non-constant meromorphic function satisfying a general differential equation (1.3), where a_0, a_1,...,a_n and b_1, b_2,...,b_k denotes small functions of g with a_n, $b_k\not\equiv 0$. Let $L(g)-1$ and $L(g)-g$ share the value 0 CM such that g and $L(g)$ share the value 1 CM and g satisfying the equation (2.1) taking $n=1$, we have

$$g(z)=\frac{\frac{z^l}{l!}+A}{1-ce^{-\sum_{l=1}^{k}\left(\frac{z^l}{(-1)^{l-1}b_l l!}\right)}}, \tag{2.17}$$

where A and $c\neq 0$ are constants.

Proof. From the general differential equation (1.3), it is easy to conclude that $N_{(n}(r,g)+m(r,g)=S(r,g)$. This implies $T(r,g)=N_{n-1)}(r,g)+S(r,g)$. Thus $N_{n-1)}(r,g)\neq S(r,g)$, i.e., $N_{=n-1}(r,g)\neq S(r,g)$. In Theorem 2.1, we have the identity (2.1) from which $n=1$ gives us equation (2.17). Substituting equation (2.17) into equation (1.3), we arrive at the following equation

$$\begin{aligned}\sum_{l=1}^{k} b_l \left[\frac{\frac{z^l}{l!}+A}{1-ce^{-\left(\frac{z^l}{(-1)^{l-1}l!b_l}\right)}}\right]^{(l)} &= \sum_{l=1}^{k}\left[a_0+a_1\left(\frac{\frac{z^l}{l!}+A}{1-ce^{-\left(\frac{z^l}{(-1)^{l-1}l!b_l}\right)}}\right)+.....\right.\\ &\left.+a_n\left(\frac{\frac{z^l}{l!}+A}{1-ce^{-\left(\frac{z^l}{(-1)^{l-1}j!b_l}\right)}}\right)^n\right].\end{aligned} \tag{2.18}$$

Suppose $a_0\not\equiv 0$, now using equation (2.18) we can get $T(r,e^{-z})=S(r,e^{-z})$ this is not possible. Therefore, we have $a_0\equiv 0$. Substituting $a_0\equiv 0$ in equation (2.18) and fixing the values of k,n and equating the coefficients of like powers of ce^{-z}, we can obtain $a_1(z),a_2(z),...,a_n(z)$. Q.E.D.

Having proved a theorem on a meromorphic solution of equation (1.3) sharing 1 CM, in what follows we prove an analogous theorem for one IM.

Theorem 2.3. Suppose g is a non-constant meromorphic function satisfying the general differential equation (1.3), where a_0, a_1,...,a_n and b_1, b_2,...,b_k denotes small functions of g with $a_n,\ b_k\not\equiv 0$. Let g and $L(g)$ share the value 1 IM, now we have either

$$L(g)-1=a_n(g-1)\left[-\left(1+\frac{a_1}{a_n}+...+\frac{a_{n-1}}{a_n}\right)g^{n-1}-...-\left(1+\frac{a_1}{a_n}\right)g-1\right], \tag{2.19}$$

or

$$\begin{aligned}L(g)-1=a_n(g-1)\left[\left(A-\frac{z^k}{k!}-\frac{a_1}{a_n}-...-\frac{a_{n-1}}{a_n}+1\right)g^{n-1}+...\right.\\ \left.+\left(A-\frac{z^k}{k!}-\frac{a_1}{a_n}+1\right)g+A-\frac{z^k}{k!}+1\right],\end{aligned} \tag{2.20}$$

where A is a constant.

Proof. Suppose z_n is a zero of $L(g)-1$ and $a_j(z_n)\not\equiv 0,\infty(j=1,2,...,n)$. As g and $L(g)$ share the value 1 IM, we know that z_n is a zero of $g-1$ of order $n-1$. From this and equation (1.3), we have

$$(a_0+a_1+...+a_n)(z_n)\equiv 1. \tag{2.21}$$

If $a_0 + a_1 + ... + a_n \not\equiv 1$, then

$$\begin{aligned}\overline{N}\left(r, \frac{1}{L(g)-1}\right) = N\left(r, \frac{1}{g-1}\right) &\leq N\left(r, \frac{1}{a_0+a_1+...+a_n-1}\right) + S(r,g),\\ &\leq T(r, a_0+a_1+...+a_n) + S(r,g),\\ &\leq T(r,a_0) + T(r,a_1) + ... + T(r,a_n) + S(r,g),\\ &= S(r,g).\end{aligned} \tag{2.22}$$

From the general differential equation (1.3), it is easy to conclude that

$$N_{(n}(r,g) + m(r,g) = S(r,g). \tag{2.23}$$

From equations (2.22) and (2.23), and using Theorem 1.23 in [7], we obtain

$$\begin{aligned}T(r,g) &\leq N\left(r, \frac{1}{g-1}\right) + \overline{N}\left(r, \frac{1}{L(g)-1}\right) + \overline{N}(r,g) - N_0\left(r, \frac{1}{L'(g)}\right)\\ &+ S(r,g),\\ &= \overline{N}_{(n}(r,g) + N_{n-1)}(r,g) - N_0\left(r, \frac{1}{L'(g)}\right) + S(r,g).\end{aligned}$$

Consequently,

$$N_0\left(r, \frac{1}{L'(g)}\right) = S(r,g). \tag{2.24}$$

Applying Lemmas A.1 and A.2, with $\lambda = 1$, then either

$$\begin{aligned}kN_{n-1)}(r,g) \leq \overline{N}_{(n}(r,g) + N_{n-1)}\left(r, \frac{1}{L(g)-1}\right) + \overline{N}\left(r, \frac{1}{L'(g)}\right)\\ + S(r,g),\end{aligned} \tag{2.25}$$

or

$$g(z) = \frac{(-1)^{n-1}(n-2)!(k+n-1)^{k+n-1}}{cb_k(k+n-2)![z+(k+n-1)A]^{n-1}} + \frac{z^k}{k!} + I_{k-1}(z) + J_{k-1}(g), \tag{2.26}$$

where A and $c \neq 0$ are constants. If we now combine equations (2.25), (2.23) and (2.22), we can easily find that

$$T(r,g) \leq \frac{1}{k}\overline{N}\left(r, \frac{1}{L'(g)}\right) + S(r,g). \tag{2.27}$$

Equation (2.27) together with equation (2.24) yields the following result:

$$T(r,g) \leq \overline{N}_{(n}\left(r, \frac{1}{L(g)-1}\right) + S(r,g). \tag{2.28}$$

Equations (2.22) and (2.28) gives $T(r,g) = S(r,g)$, which is not possible. Thus, we infer that equation (2.25) does not hold. From equation (2.26), we find that

$$\begin{aligned}g(z) - 1 = \frac{cb_k(k+n-2)![z+(k+n-1)A]^{n-1}[z^k + k!(I_{k-1}(z) + J_{k-1}(g) - 1)]}{cb_k k!(k+n-2)![z+(k+n-1)A]^{n-1}}\\ + \frac{(-1)^{n-1}k!(n-2)!(k+n-1)^{k+n-1}}{cb_k k!(k+n-2)![z+(k+n-1)A]^{n-1}},\end{aligned} \tag{2.29}$$

and

$$L(g)-1=\sum_{l=1}^{k}\left[\frac{b_l(-1)^{l+n-1}(l+n-1)^{l+n-1}(n+l-2)!}{cb_k[z+(l+n-1)A]^{l+n-1}}+\frac{z^{k-l}}{(k-l)!}+I_{k-(l+1)}(z)\right.$$
$$\left.+J_{k-(l+1)}(g)\right]-1. \tag{2.30}$$

Now g and $L(g)$ cannot share 1 IM and this contradicts the condition of Theorem 2.3. Hence

$$a_0+a_1+...+a_n\equiv 1. \tag{2.31}$$

Substituting equation (2.31) into the equation (1.3), we can easily obtain that

$$L(g)-1=a_n(g-1)\left[\left(\frac{a_1}{a_n}+\frac{a_2}{a_n}+...+1\right)+\left(\frac{a_2}{a_n}+\frac{a_3}{a_n}+...+1\right)g+...+g^{n-1}\right]. \tag{2.32}$$

If $1+\frac{a_1}{a_n}+\frac{a_2}{a_n}+...+\frac{a_{n-1}}{a_n}=-1$ or $1+\left(\frac{a_1}{a_n}\right)^{(k)}+\left(\frac{a_2}{a_n}\right)^{(k)}+...+\left(\frac{a_{n-1}}{a_n}\right)^{(k)}=0$, then we respectively arrive at the conclusion (2.19) or (2.20). Otherwise, we conclude that

$$N\left(r,\frac{1}{\left(\frac{a_1}{a_n}+\frac{a_2}{a_n}+...+1\right)+\left(\frac{a_2}{a_n}+\frac{a_3}{a_n}+...+1\right)g+...+g^{n-1}}\right)=S(r,g).$$

Indeed, if $\left(\frac{a_1}{a_n}+\frac{a_2}{a_n}+...+1\right)+\left(\frac{a_2}{a_n}+\frac{a_3}{a_n}+...+1\right)g+...+g^{n-1}$ has a zero of multiplicity p at z_0, say, such that $\frac{1}{a_n}\frac{L(g)-1}{g-1}$ has a zero of multiplicity p at z_0 as well, then we must have

(i) $g(z_0)=\infty$ or

(ii) $g'(z_0)=g''(z_0)=...=g^{(k)}(z_0)=1=g(z_0)=...=g^{n-1}(z_0)=1$ or

(iii) $a_n(z_0)=\infty$.

If $g(z_0)=\infty$, then a_n having a pole of multiplicity $p+1$ at z_0, likewise suppose $g^{(k)}(z_0)=g^{(k-1)}(z_0)=...=g'(z_0)=g^{n-1}(z_0)=...=g(z_0)=n$, then $\frac{a_1}{a_n}+2\frac{a_2}{a_n}+...+n$ will have a zero of multiplicity $\frac{n(n-1)}{2}$ at z_0 and $1+\left(\frac{a_1}{a_n}\right)^{(k)}+...+\left(\frac{a_{n-1}}{a_n}\right)^{(k)}$ with a zero of multiplicity $\min\{p-k,p+k-s\}$ at z_0, where s denotes the possible multiplicity of the pole of a_n at z_0. In the case (iii), a_n will be having a pole of multiplicity p at z_0. Thus,

$$N\left(r,\frac{1}{\left(\frac{a_1}{a_n}+\frac{a_2}{a_n}+...+1\right)+\left(\frac{a_2}{a_n}+\frac{a_3}{a_n}+...+1\right)g+...+g^{n-1}}\right)$$

$$\leq N(r,a_n)+N\left(r,\frac{1}{\frac{a_1}{a_n}+2\frac{a_2}{a_n}+...+n}\right)+N\left(r,\frac{1}{1+\left(\frac{a_1}{a_n}\right)^{(k)}+...+\left(\frac{a_{n-1}}{a_n}\right)^{(k)}}\right),$$

$$=S(r,g). \tag{2.33}$$

Equation (2.32) may now be written as

$$\frac{\left[\left(\frac{a_1}{a_n}+...+1\right)+\left(\frac{a_2}{a_n}+...+1\right)g+...+g^{n-1}\right]^{(k)}}{\left[\left(\frac{a_1}{a_n}+...+1\right)+\left(\frac{a_2}{a_n}+...+1\right)g+...+g^{n-1}\right]}$$
$$-\frac{\left[1+\left(\frac{a_1}{a_n}\right)^{(k)}+...+\left(\frac{a_{n-1}}{a_n}\right)^{(k)}\right]}{\left[\left(\frac{a_1}{a_n}+...+1\right)+\left(\frac{a_2}{a_n}+...+1\right)g+...+g^{n-1}\right]}=a_n(g-1). \tag{2.34}$$

From equations (2.34) and (2.23) it follows that, if $1+\left(\frac{a_1}{a_n}\right)^{(k)}+...+\left(\frac{a_{n-1}}{a_n}\right)^{(k)}\not\equiv 0$, we have

$$m\left(r,\frac{1}{\left(\frac{a_1}{a_n}+\frac{a_2}{a_n}+...+1\right)+\left(\frac{a_2}{a_n}+\frac{a_3}{a_n}+...+1\right)g+...+g^{n-1}}\right)=S(r,g). \tag{2.35}$$

From equations (2.33) and (2.35) we can deduce that $T(r,g)=S(r,g)$, this is not possible. Hence

$$1+\left(\frac{a_1}{a_n}\right)^{(k)}+...+\left(\frac{a_{n-1}}{a_n}\right)^{(k)}\equiv 0. \tag{2.36}$$

By integrating equation (2.36) k times w.r.t. 'z', we get

$$\frac{a_1}{a_n}+\frac{a_2}{a_n}+...+\frac{a_{n-1}}{a_n}=A-\frac{z^k}{k!}. \tag{2.37}$$

From equations (2.32) and (2.37), we arrive at equation (2.20). Q.E.D.

Appendix

A

Appendix A. Lemmas

To prove our results, we use the following three lemmas.

Lemma A.1. Suppose k and $n(\geq 1)$ are positive integers and g be a meromorphic function so that $L'(g)$ is not constant. Now we have either

$$(L'(g))^{k+n-1} = c\,(L(g)-\lambda)^{k+n}\,, \tag{A.1}$$

where c is some nonzero constant c, or

$$\begin{aligned} kN_{n-1)}(r,g) \leq \overline{N}_{(n}(r,g) + N_{n-1)}\left(r,\frac{1}{L(g)-\lambda}\right) + \overline{N}\left(r,\frac{1}{L'(g)}\right) \\ + S(r,g), \end{aligned} \tag{A.2}$$

where λ is a constant.

Proof. Let us take

$$\psi = (k+1)\frac{L''(g)}{L'(g)} - (k+2)\frac{L'(g)}{L'(g)-\lambda}. \tag{A.3}$$

Let z_∞ be a pole of g of order $(n-1)$ and from equation (A.3) we arrive at the following expression

$$\psi(z) = O\left((z-z_\infty)^{k+n-1}\right),$$

which proves that z_∞ is a zero of ψ of multiplicity $k+n-1$. Thus, if equation (A.1) is not true, i.e., $\psi \not\equiv 0$, then

$$\begin{aligned} kN_{n-1)}(r,g) &\leq N\left(r,\frac{1}{\psi}\right) \\ &\leq T(r,\psi) + O(1). \end{aligned} \tag{A.4}$$

Note that ψ can have poles of order $(n-1)$ at zeros of $L'(g)$ or $L(g)-\lambda$ or multiple poles of g. Hence from equation (A.3) we can obtain

$$N(r,\psi) \leq \overline{N}_{(n}(r,g) + N_{n-1)}\left(r,\frac{1}{L(g)-\lambda}\right) + \overline{N}\left(r,\frac{1}{L'(g)}\right). \tag{A.5}$$

Again from equation (A.3) we find that

$$m(r,\psi) = S(r,g). \tag{A.6}$$

Adding equations (A.5) and (A.6) then on substituting the resulting equation in equation (A.4), we get equation (A.2).

Lemma A.2. Let k and $n(\geq 1)$ are positive integers and also g be a non-constant meromorphic function. Then either equation (A.2) holds, or

$$g(z) = \frac{(-1)^{n-1}(n-2)!(k+n-1)^{k+n-1}}{cb_k(k+n-2)![z+(k+n-1)A]^{n-1}} + \lambda\frac{z^k}{k!} + I_{k-1}(z) + J_{k-1}(g), \tag{A.7}$$

where $c \neq 0$, A, λ are constants and $J_{k-1}(g)$ is a Integral of g and $I_{k-1}(z)$ is a polynomial in z of degree at most $k-1$.

Proof. Let $L(g)$ be a constant such that g is a polynomial of degree at most k and then $N_{n-1)}(r,g) = S(r,g)$. From this case equation (A.2) holds. Now let assume that $L(g)$ is not constant. From Lemma A.1 if equation (A.2) is not true then we have equation (A.1). We now write equation (A.1) as

$$\left(\frac{L'(g)}{L(g)-\lambda}\right)^{k+n-1} = c(L(g)-\lambda). \tag{A.8}$$

Differentiating equation (A.8) we obtain

$$(k+n-1)\left(\frac{L'(g)}{L(g)-\lambda}\right)^{k+n-2}\left(\frac{L(g)}{L(g)-\lambda}\right)' = cL'(g). \tag{A.9}$$

Combining equations (A.8) and (A.9), we get

$$\left(\frac{L'(g)}{L(g)-\lambda}\right)^{-2}\left(\frac{L'(g)}{L(g)-\lambda}\right)' = \frac{1}{k+n-1}\ .$$

Integrating this once and then using equation (A.8) we have

$$L(g)-\lambda = \frac{1}{c}\left[\frac{-(k+n-1)}{z+(k+n-1)A}\right]^{k+n-1}\ . \tag{A.10}$$

By integrating equation (A.10) 'k' times rearranging the resulting equation, we arrive at equation (A.7).

Appendix B. Basics of Nevanlinna Theory

Proximity function: $m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} log^+ \left|f(re^{i\theta})\right| d\theta$,
where $log^+ x = max(log\ x, 0)$ for all $x \geq 0$ and f is a meromorphic function.

Counting function: $N(r,f) = \int_0^r \frac{n(t,f)-n(0,f)}{t}dt + n(0,f)log\ r$,
where $n(t,f)$ denotes the number of poles of $f(z)$ in the disc $|z| \leq t$ where multiple poles are counted according to their multiplicities. The term $n(0,f)$ denotes the multiplicity of poles of $f(z)$ at the origin. At this juncture note that if $f(0) \neq \infty$, then $n(0,f) = 0$.

Characteristic function: $T(r,f) = m(r,f) + N(r,f)$.

Reduced Counting function: $\overline{N}(r,f) = \int_0^r \frac{\overline{n}(t,f)-\overline{n}(0,f)}{t}dt + \overline{n}(0,f)log\ r$,
where $\overline{n}(t,f)$ denotes the number of poles of $f(z)$ in the disc $|z| \leq t$ where multiple poles are counted only once. The term $\overline{n}(0,f)$ denotes the multiplicity of poles of $f(z)$ at the origin.

$\overline{\mathbf{N}}_{=\mathbf{n}}(\mathbf{r},\mathbf{f})$: It is the reduced counting function of all the poles of f having multiplicity n.

$\mathbf{N}_{\mathbf{n-1})}(\mathbf{r},\mathbf{f})$: It is the counting function for the poles of f having multiplicity $\leq n-1$.

$\overline{\mathbf{N}}_{(\mathbf{n}}(\mathbf{r},\mathbf{f})$: It is the reduced counting function for the poles of f with multiplicity $\geq n$.

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