Almost automorphic solutions for neutral type CNNs with time-varying delays and D operator on time scales*

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Abstract

In this paper, we propose a new concept of almost automorphic functions on a new type of almost periodic time scales. Based on inequality analysis techniques on time scales, the exponential dichotomy of linear dynamic equations on time scales and the Banach's fixed point theorem, we establish the existence and global exponential stability of almost automorphic solutions for a class of neutral type cellular neural networks with time-varying delays and D operator on time scales. We give a numerical example to illustrate the feasibility of our results.

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1 Introduction

Because the dynamical properties of cellular neural networks (CNNs) play important roles in the design, implementation and application of neural networks. In the past few decades, various dynamic behaviors of CNN have been extensively studied.

On the one hand, almost periodic phenomenon is more common than periodic one, and almost automorphicity is an extension of almost periodicity and plays an important role to understand the almost periodicity [1]. But the research on almost automorphicity of CNNs is still relatively few ([2-4]). Especially for neutral type cellular neural networks with delays. As we know that neutral type cellular neural networks with D operator as special types of neutral type CNNs with delays have more realistic significance than non-operator-based ones in many practical applications of neural networks dynamics [5, 6].

On the other hand, in fact, continuous and discrete systems are very important in implementation and applications. It is well known that to study systems on time scales can unify the continuous and discrete cases ([3, 7, 8]). Recently, in order to study the almost periodic dynamic equations on time scales, a concept of almost periodic time scales was proposed in [9]. Based on this concept, almost automorphic functions and weighted pseudo almost automorphic functions on time scales were defined in [10, 11], respectively. The concept of almost periodic time scales in [9] can unify the continuous and discrete situations effectively, but it is very restrictive because it requires the time scale having some global additivity. This may exclude some interesting time scales. Therefore,

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it is a challenging and important problem in theories and applications to study almost periodic problems on the time scale that does not require such a global additivity.

Motivated by the above statement, in this paper, we consider the following neutral type CNNs with time-varying delays and D operator on time scales:

$$\begin{aligned}&[x_i(t)-c_i(t)x_i(t-\sigma_i(t))]^{\Delta}\\ =\;& -a_i(t)x_i(t)+\sum_{j=1}^{n}b_{ij}(t)f_j\big(x_j(t)\big)+\sum_{j=1}^{n}c_{ij}(t)g_j\big(x_j(t-\tau_{ij}(t))\big)+I_i(t),\ t\in\mathbb{T},\end{aligned}\tag{1.1}$$

where $i=1,2,\ldots,n$, $\mathbb{T}$ is a time scale, n corresponds to the number of units in a neural network, $x_i(t)$ corresponds to the state vector of the ith unit at the time t, $a_i(t)$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs at the time t; $b_{ij}(t)$ and $c_{ij}(t)$ denote the strengths of connectivity between the cells j and i at time t and $t-\tau_{ij}(t)$, respectively; $f_i(\cdot)$ and $g_i(\cdot)$ denote the nonlinear continuous activation functions; $I_i(t)$ denotes the external inputs at time t; $\sigma_i(t)$ and $\tau_{ij}(t)$ are transmission delays at time t and satisfy $t-\sigma_i(t)\in\mathbb{T}$ and $t-\tau_{ij}(t)\in\mathbb{T}$ for $t\in\mathbb{T}$.

The initial conditions of (1.1) is of the following form:

$$x_i(s)=\varphi_i(s),\quad s\in[t_0-\tau,t_0]_{\mathbb{T}},\ \ t_0\in\mathbb{T},\tag{1.2}$$

where $\tau=\sup\limits_{t\in\mathbb{T}}\{\max\{|\sigma_i(t)|,|\{\tau_{ij}(t)|,i,j=1,2,\ldots,n\}\},\varphi_i\in C_{rd}([t_0-\tau,t_0]_{\mathbb{T}},\mathbb{R}^+),i=1,2,\ldots,n.$

Throughout this paper, we denote $[a,b]_{\mathbb{T}}=\{t|t\in[a,b]\cap\mathbb{T}\}$. For convenience, for an almost automorphic function $f:\mathbb{T}\to\mathbb{R}$, we denote $f^-=\inf\limits_{t\in\mathbb{T}}|f(t)|$ and $f^+=\sup\limits_{t\in\mathbb{T}}|f(t)|$.

To the best of our knowledge, there is no published paper considering the almost automorphic solutions for neutral type CNNs with time-varying delays and D operator on time scales. Our main purpose of this paper is to propose a new definition of automorphic functions on a new type of almost periodic time scales that is introduced in [12] and does not require the global additivity and to study the existence and stability of automorphic solutions of (1.1).

2 Preliminaries

Definition 2.1. ([9]) A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\Pi:=\{\tau\in\mathbb{R}:t\pm\tau\in\mathbb{T},\forall t\in\mathbb{T}\}\neq\{0\}.$$

Definition 2.2. ([9]) Let $x\in\mathbb{R}^n$ and $A(t)$ be an $n\times n$ rd-continuous matrix on $\mathbb{T}$, the linear system

$$x^{\Delta}(t)=A(t)x(t),\ t\in\mathbb{T}\tag{2.1}$$

is said to admit an exponential dichotomy on $\mathbb{T}$ if there exist positive constant k,α, projection P and the fundamental solution matrix $X(t)$ of (2.1), satisfying

$$\begin{aligned}|X(t)PX^{-1}(\sigma(s))|&\le ke_{\ominus\alpha}(t,\sigma(s)),\ s,t\in\mathbb{T},t\ge\sigma(s),\\ |X(t)(I-P)X^{-1}(\sigma(s))|&\le ke_{\ominus\alpha}(\sigma(s),t),\ s,t\in\mathbb{T},t\le\sigma(s),\end{aligned}$$

where $|\cdot|$ is a matrix norm on $\mathbb{T}$.

Definition 2.3. ([12, 13]) A time scale $\mathbb{T}$ is called an almost periodic time scale if the set

$$\Lambda_0 := \{\tau \in \mathbb{R} : \mathbb{T}_{\pm\tau} \neq \varnothing\} \neq \{0\},$$

where $\mathbb{T}_{\pm\tau} = \mathbb{T} \cap \{\mathbb{T} \mp \tau\} = \mathbb{T} \cap \{t \mp \tau : t \in \mathbb{T}\}$, and there exists a set Λ satisfying

(*a*) $0 \in \Lambda \subseteq \Lambda_0$,

(*b*) $\Pi(\Lambda) \setminus \{0\} \neq \varnothing$,

(*c*) $\widetilde{\mathbb{T}} := \mathbb{T}(\Pi) = \bigcap\limits_{\tau\in\Pi} \mathbb{T}_\tau \neq \varnothing$,

where $\Pi := \Pi(\Lambda) = \{\tau \in \Lambda \subseteq \Lambda_0 : \sigma \pm \tau \in \Lambda, \forall \sigma \in \Lambda\}$.

Obviously, if $t \in \mathbb{T}_\tau$, then $t+\tau \in \mathbb{T}$. If $t \in \widetilde{\mathbb{T}}$, then $t \pm \tau \in \mathbb{T}$ for $\tau \in \Pi$. If $\mathbb{T}$ is an almost periodic time scales under Definition 2.3, then $\widetilde{\mathbb{T}}$ is an almost periodic time scale under Definition 2.1.

Throughout the rest of this paper, we restrict our discussion on an almost periodic time scale $\mathbb{T}$ that is defined by Definition 2.3.

Let $BC_{rd}(\mathbb{T} \times D, \mathbb{X})$ denote the collection of all bounded rd-continuous functions from $\mathbb{T} \times S$ to $\mathbb{X}$, where $\mathbb{X}$ denotes $\mathbb{R}^n, \mathbb{C}^n$ or some Banach space, $S \subset D$ is any compact set. We call a time scale $\mathbb{T}$ is a symmetric time scale if $t \in \mathbb{T}$, then $-t \in \mathbb{T}$. From now on, we will restrict our discussion on symmetric time scales. We introduce the following definition of almost automorphic functions on time scales as follows.

Definition 2.4. Let $\mathbb{T}$ be an almost periodic time scale.

(*i*) A function $f \in BC_{rd}(\mathbb{T}, \mathbb{X})$ is said to be almost automorphic, if for any sequence $\{s_n\}_{n=1}^{\infty} \subset \Pi$, there is a subsequence $\{\tau_n\}_{n=1}^{\infty} \subset \{s_n\}_{n=1}^{\infty}$ such that

$$g(t) = \lim_{n\to\infty} f(t+\tau_n)$$

is well defined for each $t \in \widetilde{\mathbb{T}}$ and

$$\lim_{n\to\infty} g(t-\tau_n) = f(t)$$

for each $t \in \widetilde{\mathbb{T}}$. Denote by $AA(\mathbb{T}, \mathbb{X})$ the set of all such functions;

(*ii*) A function $f \in BC_{rd}(\mathbb{T} \times \mathbb{X}, \mathbb{X})$ is said to be almost automorphic, if $f(t, x)$ is almost automorphic in $t \in \mathbb{T}$ uniformly in $x \in \mathbb{B}$, where $\mathbb{B}$ is any bounded subset of $\mathbb{X}$. Denote by $AA(\mathbb{T} \times \mathbb{X}, \mathbb{X})$ the set of all such functions.

Similar to the proofs of Theorems 3.19, 3.21 and 3.22 in [11], one can easily get that

Lemma 2.5. Let $f, g \in AA(\mathbb{T}, \mathbb{X})$. Then we have the following

(*i*) $f + g \in AA(\mathbb{T}, \mathbb{X})$;

(*ii*) $cf \in AA(\mathbb{T}, \mathbb{X})$ for any constant $c \in \mathbb{R}$;

(*iii*) if $h : \mathbb{X} \to \mathbb{Y}$ is a continuous function, then the composite function $h \circ f : \mathbb{T} \to \mathbb{Y}$ is almost automorphic.

Lemma 2.6. Let $f \in AA(\mathbb{T} \times \mathbb{X}, \mathbb{X})$ and f satisfy the Lipschitz condition in $x \in \mathbb{X}$ uniformly in $t \in \mathbb{T}$. If $\varphi \in AA(\mathbb{T}, \mathbb{X})$, then $f(t, \varphi(t))$ is almost automorphic.

Consider the following linear system

$$x^{\Delta}(t) = A(t)x(t) + f(t),\ t \in \mathbb{T}. \tag{2.2}$$

Similar to the proofs of Lemma 8 in [12] and Theorem 5.6 in [11], we have

Lemma 2.7. Suppose that $\mathbb{T}$ be an almost periodic time scale under Definition 2.4, $A(t) \in AA(\mathbb{T}, \mathbb{R}^{m\times m})$ such that $\{A^{-1}(t)\}_{t\in\mathbb{T}}$ and $\{(I + \mu(t)A(t))^{-1}\}_{t\in\mathbb{T}}$ are bounded. Moreover, suppose that $f \in AA(\mathbb{T}, \mathbb{R}^m)$ and (2.1) admits an exponential dichotomy, then system (2.2) has a solution $x(t) \in AA(\mathbb{T}, \mathbb{R}^m)$ and $x(t)$ is expressed as follows:

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))f(s)\Delta s - \int_{t}^{+\infty} X(t)(I - P)X^{-1}(\sigma(s))f(s)\Delta s,$$

where $X(t)$ is the fundamental solution matrix of (2.1).

3 Main results

In this section, we will state and prove the sufficient conditions for the existence and global exponential stability of almost automorphic solutions of (1.1).

Set $\mathbb{X} = \{\varphi | \varphi \in AA(\mathbb{T}, \mathbb{R})\}$ with the norm $\|\varphi\|_{\mathbb{X}} = \sup_{t\in\mathbb{T}} |\varphi(t)|$, then $\mathbb{X}$ is a real Banach space. Let $\varphi^0(t) = (\varphi_1^0(t), \varphi_2^0(t), \ldots, \varphi_n^0(t))^T$, where $\varphi_i^0(t) = \int_{-\infty}^{t} e_{-a_i}(t, \sigma(s))I_i(s)\Delta s$, $i = 1, 2, \ldots, n$ and r be a constant satisfying $r \geq \|\varphi^0\|_{\mathbb{X}}$.

Throughout this paper, we assume that the following conditions hold:

(H_1) $a_i \in C(\mathbb{T}, \mathbb{R}^+)$ with $-a_i \in \mathcal{R}^+, c_i, b_{ij}, c_{ij} \in C(\mathbb{T}, \mathbb{R})$ and $\sigma_i, \tau_{ij} \in C(\mathbb{T}, \mathbb{R}^+)$ are almost automorphic, where $i, j = 1, 2, \ldots, n$.

(H_2) $f_i, g_i \in C(\mathbb{R}, \mathbb{R})$ and there exist nonnegative constants L_i^f and L_i^g such that $|f_i(x) - f_i(y)| \leq L_i^f|x - y|$ and $|g_i(x) - g_i(y)| \leq L_i^g|x - y|$ for all $x, y \in \mathbb{R}$ and $f_i(0) = g_i(0) = 0$, $i = 1, 2, \ldots, n$;

(H_3) $\alpha = \max\limits_{1\leq i\leq n} \left\{ c_i^+ + \frac{1}{a_i^-}\left(a_i^+ c_i^+ + \sum\limits_{j=1}^{n} \left(b_{ij}^+ L_j^f + c_{ij}^+ L_j^g \right) \right) \right\} < 1$, $i = 1, 2, \ldots, n$.

Theorem 3.1. Let (H_1)-(H_3) hold. Then, system (1.1) has a unique almost automorphic solution in $\mathbb{X}^* = \{\varphi \in \mathbb{X} | \|\varphi - \varphi^0\|_{\mathbb{X}} \leq r\}$, which is globally exponentially stable.

Proof. Let

$$Y_i(t) = x_i(t) - c_i(t)x_i(t - \sigma_i(t)).$$

Then, it follows from (1.1) that

$$\begin{aligned} Y_i^{\Delta}(t) &= -a_i(t)Y(t) - a_i(t)c_i(t)x_i(t-\sigma_i(t)) \\ &\quad + \sum_{j=1}^{n} b_{ij}(t)f_j\big(x_j(t)\big) + \sum_{j=1}^{n} c_{ij}(t)g_j\big(x_j(t-\tau_{ij}(t))\big) + I_i(t),\ \ i=1,2,\ldots,n. \end{aligned}$$

For any $\varphi \in \mathbb{X}^*$, we consider the following almost automorphic linear nonhomogeneous system:

$$Y_i^{\Delta}(t) = -a_i(t)Y_i(t) + F_i(t,\varphi) + I_i(t),\ \ i=1,2,\ldots,n, \tag{3.1}$$

where

$$F_i(t,\varphi) = -a_i(t)c_i(t)\varphi_i(t-\sigma_i(t)) + \sum_{j=1}^{n} b_{ij}(t)f_j\big(\varphi_j(t)\big) + \sum_{j=1}^{n} c_{ij}(t)g_j\big(\varphi_j(t-\tau_{ij}(t))\big).$$

Since $\inf\limits_{t\in\mathbb{T}} a_i(t) > 0$ and $-a_i \in \mathcal{R}^+$, it follows from Lemma 2.15 in [14] that the linear homogeneous system

$$Y_i^{\Delta}(t) = -a_i(t)Y_i(t),\ \ i=1,2,\ldots,n$$

admits an exponential dichotomy on $\mathbb{T}$. Thus, by Lemma 2.7, we obtain that system (3.1) has an almost automorphic solution as follows:

$$Y^{\varphi}(t) = \big(Y_1^{\varphi}(t), Y_2^{\varphi}(t), \ldots, Y_n^{\varphi}(t)\big)^T,$$

where

$$Y_i^{\varphi}(t) = \int_{-\infty}^{t} e_{-a_i}(t,\sigma(s))(F_i(s,\varphi) + I_i(s))\Delta s,\ \ i=1,2,\ldots,n.$$

For $\varphi \in \mathbb{X}^*$, we have $\|\varphi\|_{\mathbb{X}} \le \|\varphi - \varphi^0\|_{\mathbb{X}} + \|\varphi^0\|_{\mathbb{X}} \le 2r$. Define an operator as follows:

$$\Phi : \mathbb{X}^* \to \mathbb{X}^*,\ \ (\varphi_1, \varphi_2, \ldots, \varphi_n)^T \to \big((\Phi\varphi_1), (\Phi\varphi)_2, \ldots, (\Phi\varphi)_n\big)^T,$$

where

$$(\Phi\varphi)_i(t) = c_i(t)\varphi_i(t-\sigma_i(t)) + Y_i^{\varphi}(t),\ \ \forall \varphi \in \mathbb{X}^*. \tag{3.2}$$

First we show that for any $\varphi \in \mathbb{X}^*$, $\Phi\varphi \in \mathbb{X}^*$. For $i = 1, 2, \ldots, n$, we have

$$\begin{aligned} |(\Phi\varphi)_i(t) - \varphi_i^0(t)| &\le c_i^+|\varphi_i(t-\sigma_i(t))| + \int_{-\infty}^{t} e_{-a_i}(t,\sigma(s))\big(a_i^+c_i^+|\varphi(s-\sigma_i(s))| \\ &\quad + \sum_{j=1}^{n} b_{ij}^+L_j^f|\varphi_j(s)| + \sum_{j=1}^{n} c_{ij}^+L_j^g|\varphi_j(s-\tau_{ij}(s))| \\ &\le c_i^+\|\varphi\|_{\mathbb{X}} + \int_{-\infty}^{t} e_{-a_i}(t,\sigma(s))\Delta s\left(a_i^+c_i^+ + \sum_{j=1}^{n} b_{ij}^+L_j^f + \sum_{j=1}^{n} c_{ij}^+L_j^g\right)\|\varphi\|_{\mathbb{X}} \\ &\le c_i^+\|\varphi\|_{\mathbb{X}} + \frac{1}{a_i^-}\left(a_i^+c_i^+ + \sum_{j=1}^{n}\big(b_{ij}^+L_j^f + c_{ij}^+L_j^g\big)\right)\|\varphi\|_{\mathbb{X}} < r, \end{aligned}$$

therefore, we have that $\|\Phi\varphi\|_{\mathbb{X}} \le r$. This implies that the mapping Φ is a self-mapping from $\mathbb{X}^*$ to $\mathbb{X}^*$. Next, we prove that the mapping Φ is a contraction one on $\mathbb{X}^*$. In fact, for any $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)^T, \psi = (\psi_1, \psi_2, \dots, \psi_n)^T \in \mathbb{X}^*$, we find

$$\begin{aligned}
|(\Phi\varphi)_i(t) - (\Phi\psi)_i(t)| &\le c_i^+\big|\varphi_i(t-\sigma_i(t)) - \psi_i(t-\sigma_i(t))\big| + \int_{-\infty}^{t} e_{-a_i}(t,\sigma(s)\\
&\quad\times\bigg(a_i^+c_i^+\big|\varphi_i(s-\sigma_i(s)) - \psi_i(s-\sigma_i(s))\big|\\
&\quad+\sum_{j=1}^{n} b_{ij}^+L_j^f\big|\varphi_j(s) - \psi_j(s)\big|\\
&\quad+\sum_{j=1}^{n} c_{ij}^+L_j^g\big|\varphi_j(s-\tau_{ij}(s)) - \psi_j(s-\tau_{ij}(s))\big|\bigg)\Delta s\\
&\le \bigg(c_i^+ + \frac{1}{a_i^-}\bigg(a_i^+c_i^+ + \sum_{j=1}^{n}\big(b_{ij}^+L_j^f + c_{ij}^+L_j^g\big)\bigg)\bigg)\|\varphi-\psi\|_{\mathbb{X}}\\
&= \alpha\|\varphi-\psi\|_{\mathbb{X}},\ i = 1, 2, \dots.
\end{aligned}$$

Because $\alpha < 1$, so Φ is a contraction mapping. By the fixed point theorem of Banach space, it follows that Φ has a unique fixed point $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T \in \mathbb{X}^*$ such that

$$x_i^*(t) = (\Phi x^*)_i(t) = c_i(t)x_i^*(t-\sigma_i(t)) + Y_i^{x_i^*}(t)$$

and

$$\begin{aligned}
x_i^*(t) &= c_i(t)x_i^*(t-\sigma_i(t)) + \int_{-\infty}^{t} e_{-a_i}(t,\sigma(s))\bigg(-a_i(s)c_i(s)x_i^*(s-\sigma_i(s))\\
&\quad+\sum_{j=1}^{n} b_{ij}(s)f_j\big(x_j^*(s)\big) + \sum_{j=1}^{n} c_{ij}(s)g_j\big(x_j^*(s-\tau_{ij}(s))\big) + I_i(s)\bigg)\Delta s,\ \ i = 1, 2, \dots, n.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&[x_i^*(t) - c_i(t)x_i^*(t-\sigma_i(t))]^{\Delta}\\
=\ &-a_i(t)Y_i^{x_i^*}(t) - a_i(t)c_i(t)x_i^*(t-\sigma_i(t))\\
&+\sum_{j=1}^{n} b_{ij}(t)f_j\big(x_j^*(t)\big) + \sum_{j=1}^{n} c_{ij}(t)g_j\big(x_j^*(t-\tau_{ij}(t))\big) + I_i(t)\\
=\ &-a_i(t)x_i^*(t) + \sum_{j=1}^{n} b_{ij}(t)f_j\big(x_j^*(t)\big) + \sum_{j=1}^{n} c_{ij}(t)g_j\big(x_j^*(t-\tau_{ij}(t))\big) + I_i(t),\ \ i = 1, 2, \dots, n,
\end{aligned}$$

hence, $x^*(t)$ is an almost automorphic solution of system (1.1). The proof of Theorem 3.1 is completed. ∎

Theorem 3.2. Assume that (H_1)-(H_3) hold, then system (1.1) has a unique almost automorphic solution that is globally exponentially stable.

Proof. From Theorem 3.1, we see that system (1.1) has an almost automorphic solution $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T$ with initial value $\psi^*(s) = (\psi_1^*(s), \psi_2^*(s), \ldots, \psi_n^*(s))^T$. Suppose that $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ is an arbitrary solution of system (1.1) with initial value $\psi(s) = (\psi_1(s), \psi_2(s), \ldots, \psi_n(s))^T$ and

$$z_i(t) = x_i(t) - x_i^*(t), \;\; Z_i(t) = z_i(t) - c_i(t)z_i(t - \sigma_i(t)), \;\; i = 1, 2, \ldots, n.$$

From (1.1) it follows that

$$\begin{aligned} Z_i^{\Delta}(t) \;\; = \;\; & -a_i(t)Z_i(t) - a_i(t)c_i(t)z_i(t - \sigma_i(t)) + \sum_{j=1}^{n} b_{ij}(t)\Big(f_j\big(x_j(t)\big) - f_j\big(x_j^*(t)\big)\Big) \\ & + \sum_{j=1}^{n} c_{ij}(t)\Big(g_j\big(x_j(t - \tau_{ij}(t))\big) - g_j\big(x_j^*(t - \tau_{ij}(t))\big)\Big). \end{aligned} \tag{3.3}$$

According to (H_3), there exists a constant $\lambda \in (0, \min\limits_{1\le i\le n}\{a_i^-\})$ such that $1 - c_i^+ \exp(\lambda\sigma_i^+) > 0, i = 1, 2, \ldots, n$, and

$$\max_{1\le i\le n}\left\{\frac{1}{(1 - c_i^+\exp(\lambda\sigma_i^+))(a_i^- - \lambda)}\Big(\exp(\lambda\sigma_i^+)a_i^+c_i^+ + \sum_{j=1}^{n} b_{ij}^+L_j^f + \sum_{j=1}^{n} c_{ij}^+L_j^g\exp(\lambda\tau_{ij}^+)\Big)\right\} < 1.$$

Let $\|\varphi\|_0 = \sup\limits_{t\in[t_0-\tau,t_0]_{\mathbb{T}}}\max\limits_{1\le i\le n}|(\varphi_i(t) - x_i^*(t)) - c_i(t)(\varphi_i(t-\sigma_i(t)) - x_i^*(t-\sigma_i(t)))|$, $\|Z(t)\| = \max\limits_{1\le i\le n}\{|Z_i(t)|\}$. Let $\varepsilon > 0$ and $M > 1$. It is obvious that

$$\|Z(t_0)\| < (\|\varphi\|_0 + \varepsilon) \tag{3.4}$$

and

$$\|Z(t)\| < (\|\varphi\|_0 + \varepsilon)e_{\ominus\lambda}(t, t_0) < M(\|\varphi\|_0 + \varepsilon)e_{\ominus\lambda}(t, t_0), \;\; \forall t \in [t_0 - \tau, t_0]_{\mathbb{T}}. \tag{3.5}$$

We claim that

$$\|Z(t)\| < M(\|\varphi\|_0 + \varepsilon)e_{\ominus\lambda}(t, t_0), \;\; \forall t \in (t_0, +\infty)_{\mathbb{T}}. \tag{3.6}$$

If (3.6) does not hold. Then there must exist some $t_1 \in (t_0, +\infty)_{\mathbb{T}}$ and some $i \in \{1, 2, \ldots, n\}$ such that

$$\begin{cases} |Z_i(t_1)| = \|Z(t_1)\| \ge M(\|\varphi\|_0 + \varepsilon)e_{\ominus\lambda}(t_1, t_0), \\ \|Z(t)\| < M(\|\varphi\|_0 + \varepsilon)e_{\ominus\lambda}(t, t_0), \;\; t < t_1. \end{cases}$$

Therefore, there exists a $\varrho \ge 1$ such that

$$\begin{cases} |Z_i(t_1)| = \varrho M(\|\varphi\|_0 + \varepsilon)e_{\ominus\lambda}(t_1, t_0), \\ \|Z(t)\| < \varrho M(\|\varphi\|_0 + \varepsilon)e_{\ominus\lambda}(t, t_0), \;\; t < t_1. \end{cases} \tag{3.7}$$

In addition,

$$\begin{aligned} & e_\lambda(t',t_0)|z_i(t')| \\ \le\ & e_\lambda(t',t_0)|z_i(t')-c_i(t')z_i(t'-\sigma_i(t'))|+e_\lambda(t',t_0)|c_i(t')z_i(t'-\sigma_i(t'))| \\ \le\ & e_\lambda(t',t_0)|Z_i(t')|+c_i^+\exp(\lambda\sigma_i^+)e_\lambda(t'-\sigma_i(t'),t_0)|z_i(t'-\sigma_i(t'))| \\ <\ & \varrho M(\|\varphi\|_0+\varepsilon)+c_i^+\exp(\lambda\sigma_i^+)\sup_{s\le t}e_\lambda(s,t_0)|z_i(s)| \end{aligned}$$

for all $t'\le t, t<t_1$, which implies that

$$e_\lambda(t,t_0)|z_i(t)|\le\sup_{s\le t}e_\lambda(s,t_0)|z_i(s)|<\frac{\varrho M(\|\varphi\|_0+\varepsilon)}{1-c_i^+\exp(\lambda\sigma_i^+)}. \tag{3.8}$$

Moreover, from (3.3), we have

$$\begin{aligned} & e_{-a_i}(t_0,\sigma(s))\big(Z_i^\Delta(s)+a_i(s)Z_i(s)\big) \\ =\ & e_{-a_i}(t_0,\sigma(s))\bigg\{-a_i(s)c_i(s)z_i(s-\sigma_i(s))+\sum_{j=1}^n b_{ij}(s)\Big(f_j\big(x_j(s)\big)-f_j\big(x_j^*(s)\big)\Big) \\ & +\sum_{j=1}^n c_{ij}(s)\Big(g_j\big(x_j(s-\tau_{ij}(s))\big)-g_j\big(x_j^*(s-\tau_{ij}(s))\big)\Big)\bigg\},\ \ s\in[t_0,t]_{\mathbb{T}}, \end{aligned}$$

which yields to

$$\begin{aligned} Z_i(t) \ =\ & Z_i(t_0)e_{-a_i}(t,t_0)+\int_{t_0}^t e_{-a_i}(t,\sigma(s))\bigg\{-a_i(s)c_i(s)z_i(s-\sigma_i(s)) \\ & +\sum_{j=1}^n b_{ij}(s)\Big(f_j\big(x_j(s)\big)-f_j\big(x_j^*(s)\big)\Big) \\ & +\sum_{j=1}^n c_{ij}(s)\Big(g_j\big(x_j(s-\tau_{ij}(s))\big)-g_j\big(x_j^*(s-\tau_{ij}(s))\big)\Big)\bigg\}\Delta s,\ \ t\in[t_0,t_1]_{\mathbb{T}}. \end{aligned}$$

Hence, $\varrho M > 1$, (3.4), (3.5), (3.7) and (3.8) imply that

$$\begin{aligned}
& |Z_i(t_1)| \\
\leq\ & |Z_i(t_0)|e_{-a_i}(t_1,t_0) + \int_{t_0}^{t_1} e_{-a_i}(t_1,\sigma(s))\Big\{a_i^+c_i^+|z_i(s-\sigma_i(s))| \\
& + \sum_{j=1}^{n} b_{ij}^+L_j^f\big|x_j(s)-x_j^*(s)\big| \\
& + \sum_{j=1}^{n} c_{ij}^+L_j^g\big|x_j(s-\tau_{ij}(s))-x_j^*(s-\tau_{ij}(s))\big|\Big\}\Delta s \\
<\ & (\|\varphi\|_0+\varepsilon)e_{\ominus\lambda}(t_1,t_0)e_{-a_i\oplus\lambda}(t_1,t_0) + \int_{t_0}^{t_1} e_{-a_i\oplus\lambda}(t_1,\sigma(s))\bigg(\frac{\exp(\lambda\sigma_i^+)}{1-c_i^+\exp(\lambda\sigma_i^+)}a_i^+c_i^+ \\
& \sum_{j=1}^{n} b_{ij}^+L_j^f\frac{1}{1-c_i^+\exp(\lambda\sigma_i^+)} + \sum_{j=1}^{n} c_{ij}^+L_j^g\frac{\exp(\lambda\tau_{ij}^+)}{1-c_i^+\exp(\lambda\sigma_i^+)}\bigg)\Delta s\varrho M(\|\varphi\|_0+\varepsilon)e_{\ominus\lambda}(t_1,t_0) \\
\leq\ & (\|\varphi\|_0+\varepsilon)e_{\ominus\lambda}(t_1,t_0)e_{-a_i\oplus\lambda}(t_1,t_0) + \frac{1-e_{-a_i^-\oplus\lambda}(t_1,t_0)}{a_i^--\lambda}\bigg(\frac{\exp(\lambda\sigma_i^+)}{1-c_i^+\exp(\lambda\sigma_i^+)}a_i^+c_i^+ \\
& + \sum_{j=1}^{n} b_{ij}^+L_j^f\frac{1}{1-c_i^+\exp(\lambda\sigma_i^+)} + \sum_{j=1}^{n} c_{ij}^+L_j^g\frac{\exp(\lambda\tau_{ij}^+)}{1-c_i^+\exp(\lambda\sigma_i^+)}\bigg)\varrho M(\|\varphi\|_0+\varepsilon)e_{\ominus\lambda}(t_1,t_0) \\
\leq\ & \varrho M(\|\varphi\|_0+\varepsilon)e_{\ominus\lambda}(t_1,t_0)\big((\frac{1}{\varrho M}-1)e_{-a_i^-\oplus\lambda}(t_1,t_0)+1\big) \\
<\ & \varrho M(\|\varphi\|_0+\varepsilon)e_{\ominus\lambda}(t_1,t_0),
\end{aligned}$$

which contradicts the first equation (3.7). Therefore, (3.6) holds. Let $\varepsilon \to 0^+$ leads to

$$\|Z(t)\| \leq M\|\varphi\|_0 e_{\ominus\lambda}(t,t_0) \quad \forall t \in (t_0,+\infty)_{\mathbb{T}}.$$

Hence, the almost automorphic solution of system (1.1) is globally exponentially stable. The proof of Theorem 3.2 is completed. ∎

4 Examples

In this section, we give an example to illustrate the feasibility and effectiveness of our results obtained in Section 3.

Example 4.1. In system (1.1), let $n = 2$, $f_1(x) = g_1(x) = \frac{1}{12}\sin^2 x$, $f_2(x) = g_2(x) = \frac{1}{6}\cos x$ and the coefficients are taken as follows:

$$c_1(t) = 0.1\sin\sqrt{2}t,\ \ c_2(t) = \frac{1}{15}\cos\sqrt{5}t,\ \ a_1(t) = 0.8 + 0.1|\sin t|,$$

$$a_2(t) = 0.7 + 0.1|\cos t|,\ \ b_{11}(t) = 0.4 + 0.1\sin t,\ \ b_{12}(t) = 0.3 - 0.1\sin t,$$

$$b_{21}(t) = 0.4 + 0.1\cos\sqrt{7}t,\ \ b_{22}(t) = 0.2 + 0.3\sin\sqrt{3}t,\ \ c_{11}(t) = 0.1 + 0.2\cos^2 t,$$

$$c_{12}(t) = 0.3 + 0.1\cos t,\ \ c_{21}(t) = 0.5 + 0.1\cos\sqrt{11}t,\ \ c_{22}(t) = 0.2 + 0.1\sin\sqrt{2}t,$$

$$\sigma_1(t) = |\cos 2\pi t|,\ \ \sigma_2(t) = |\sin 2\pi t|,\ \ \tau_{11}(t) = 2|\cos(4\pi t)|,\ \ \tau_{12}(t) = |\cos(4\pi t)|,$$

$$\tau_{21}(t) = 3|\sin(3\pi t)|,\ \ \tau_{22}(t) = 5|\cos(\pi t)|,\ \ I_1(t) = \sin\sqrt{2}t - \cos\sqrt{7}t,\ \ I_2(t) = 17\sin\sqrt{3}t.$$

By calculating, we have

$$L_1^f = L_2^f = L_1^g = L_2^g = \frac{1}{6},\ c_1^+ = \frac{1}{10},\ c_2^+ = \frac{1}{15},\ a_1^+ = 0.9,\ a_2^+ = 0.8,$$

$$a_1^- = 0.8,\ a_2^- = 0.7,\ b_{11}^+ = 0.5,\ b_{12}^+ = 0.4,\ b_{21}^+ = 0.6,\ b_{22}^+ = 0.3.$$

For $0 \le \mu(t) \le 1$, we have $\inf_{t\in\mathbb{T}}\{1-\mu(t)a_i(t)\} > 0$, therefore, whether $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, we have $-a_i \in \mathcal{R}^+$ and condition (H_1) is satisfied. Also, condition (H_2) is obviously verified. Moreover, by a simple computation, we have

$$\alpha = \max_{1\le i\le 2}\left\{c_i^+ + \frac{1}{a_i^-}\left(a_i^+c_i^+ + \sum_{j=1}^{2}\left(b_{ij}^+L_j^f + c_{ij}^+L_j^g\right)\right)\right\} = \max\left\{\frac{131}{240}, \frac{12}{21}\right\} < 1,$$

so, the condition (H_3) is satisfied. Therefore, (1.1) has a unique almost automorphic solution that is globally exponentially stable (see Figures 1-4).

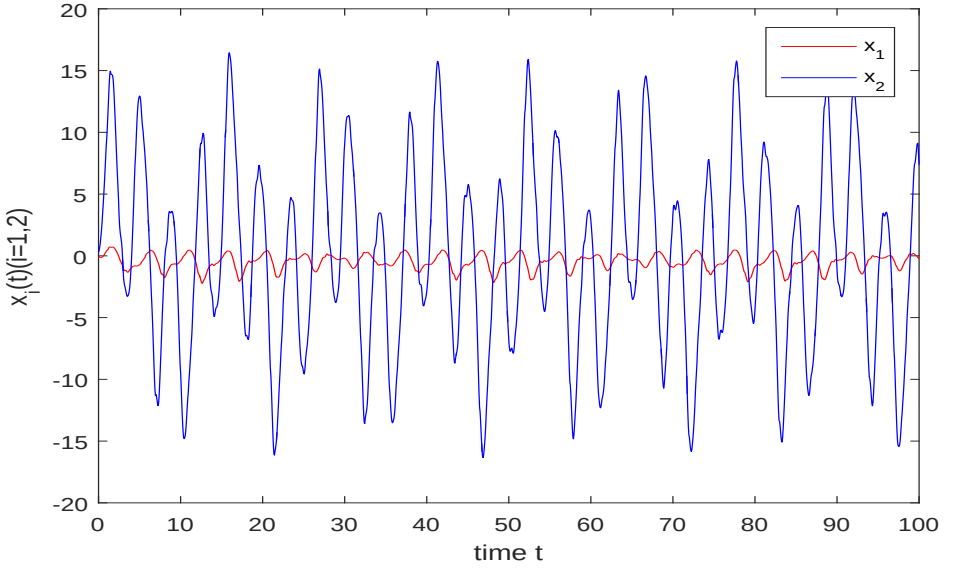


FIGURE 1. When $\mathbb{T} = \mathbb{R}$, almost automorphic solution of system (1.1) with initial condition $((x_1(0), x_2(0))^T = (0.2, 0.3)^T$.

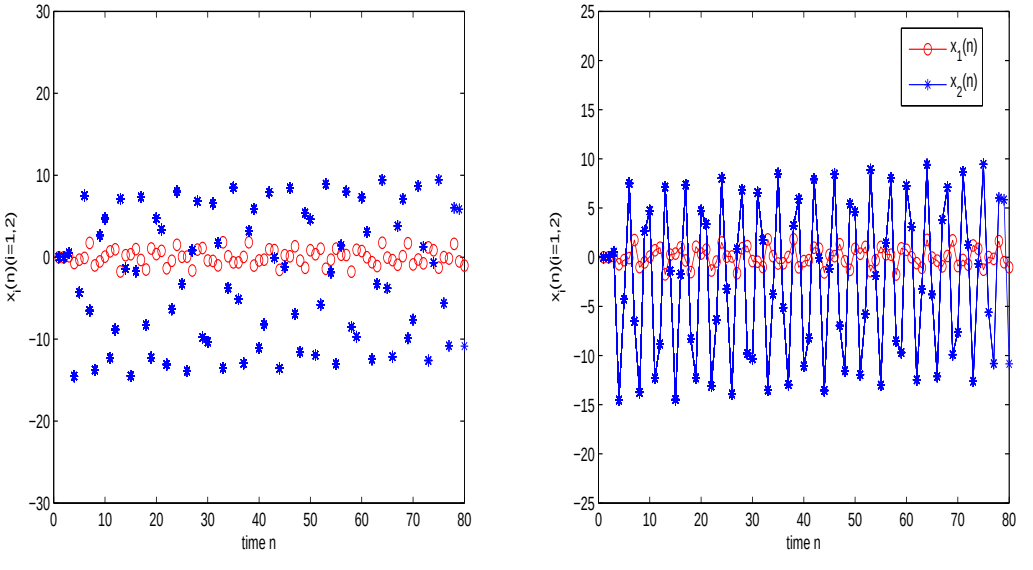


FIGURE 2. When $\mathbb{T} = \mathbb{Z}$, almost automorphic solution of system (1.1) with initial condition $(x_1(1), x_2(1))^T = (0.1, 0.5)^T$.

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