Product of Sheffer sequences: properties and examples*

Mumtaz Riyasat[1], Subuhi Khan[2] and Shakir Shah[3]

Department of Mathematics, Aligarh Muslim University, Aligarh, India
[1]Corresponding author
E-mail: mumtazrst@gmail.com[1], subuhi2006@gmail.com[2], shakirshah59@gmail.com[3]

Abstract

This article is written with an objective to explore the product of two Sheffer sequences. This article is an attempt to explore such type of product which extends the possibility to consider hybrid type Sheffer polynomials. It is important to remark that although this product can be viewed as the umbral composition of two Sheffer sequences but the results related to this product cannot be deduced from the results of the Sheffer sequences. The set of this product of Sheffer sequences is also a non-abelian group and thus convoluting different members of Sheffer class allows us to consider a number of hybrid type special polynomials as its members. Certain results for this class including the quasi-monomiality and determinant form are established. The article concludes with the possibility of considering the product of n-Sheffer sequences.

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1 Introduction and preliminaries

Sheffer polynomial sequences arise in numerous problems of applied mathematics, theoretical physics, approximation theory and several other mathematical branches. Sheffer sequences are studied systematically by theory of modern umbral calculus, see for example [8,9]. Some definitions and results to be used in this work are reviewed from [12].

Let $\mathbb{K}$ be a field of characteristic zero and $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{K}$. An element of $\mathcal{F}$ has the form

$$f(t)=\sum_{k=0}^{\infty}a_k t^k, \tag{1.1}$$

where $a_k\in\mathbb{K}$ for all $k\in\mathbb{N}:=\{0,1,2,\ldots\}$. The order $o(f(t))$ of a power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. The series $f(t)$ has a multiplicative inverse, denoted by $f(t)^{-1}$ or $\frac{1}{f(t)}$, if and only if $o(f(t))=0$. In this case, $f(t)$ is called an *invertible series*. The series $f(t)$ has a compositional inverse, denoted by $\overline{f}(t)$ and satisfying $f(\overline{f}(t))=\overline{f}(f(t))=t$, if and only if $o(f(t))=1$. The series $f(t)$ with $o(f(t))=1$ is called a *delta series*.

Definition 1.1. Let $g(t)$ be an invertible series and $f(t)$ be a delta series of the forms:

$$f(t)=\sum_{1=0}^{\infty}f_n\frac{t^n}{n!},\ f_0=0;\ f_1\neq 0,\qquad g(t)=\sum_{n=0}^{\infty}g_n\frac{t^n}{n!},\ g_0\neq 0, \tag{1.2}$$

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then the sequence $(s_n(x))_{n\in\mathbb{N}}$ is *Sheffer* for the pair $(g(t), f(t))$, if and only if

$$\langle g(t)f(t)^k|s_n(x)\rangle = c_n\delta_{n,k}, \tag{1.3}$$

for all $n, k \geq 0$, where $\delta_{n,k}$ is the Kronecker delta.

Definition 1.2. For the pair $(g(t), f(t))$, the exponential generating function for the Sheffer sequences $s_n(x)$ is given by

$$\frac{1}{g(\overline{f}(t))}e^{x(\overline{f}(t))} = \sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!}, \tag{1.4}$$

for all x in $\mathbb{C}$. Particularly, the Sheffer sequence for $(1, f(t))$ is called the *associated Sheffer sequence* and the Sheffer sequence for $(g(t), t)$ is called the *Appell sequence* [9] for $g(t)$.

Next, we recall the concept of Riordan arrays introduced by Shapiro *et. al.* [10].

Definition 1.3. Let $g(t)$ be an invertible series and $f(t)$ be a delta series. A *generalized Riordan array* with respect to the sequence $(c_n)_{n\in\mathbb{N}}$ is a pair $(g(t), f(t))$, which defines an infinite, lower triangular array $(a_{n,k})_{0\leq k\leq n<\infty}$ according to the following rule:

$$a_{n,k} = \left[\frac{t^n}{c_n}\right] g(t)\frac{(f(t))^k}{c_k}, \tag{1.5}$$

where the functions $\frac{g(t)(f(t))^k}{c_k}$ are called the *column generating functions* of the Riordan array. Particularly, the *classical Riordan arrays* correspond to the case $c_n = 1$ and the exponential Riordan arrays correspond to the case of $c_n = n!$.

For any fixed sequence $(c_n)_{n\in\mathbb{N}}$, the set of all Riordan arrays $(g(t), f(t))$ is a group under matrix multiplication and is called a *Riordan group* with respect to $(c_n)_{n\in\mathbb{N}}$. The identity of this group is $(1, t)$ and the inverse of the array $(g(t), f(t))$ is $\left(\frac{1}{g(\overline{f}(t))}, \overline{f}(t)\right)$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$. Moreover, for any fixed sequence $(c_n)_{n\in\mathbb{N}}$, the Riordan group and the Sheffer group are isomorphic [11].

It is shown in [12] that, the Sheffer sequences can be expressed by determinants. Let $(s_n(x))_{n\in\mathbb{N}}$ be Sheffer sequence for $(g(t), f(t))$ satisfying the following condition:

$$x^n = \sum_{k=0}^{n} a_{n,k}s_k(x), \tag{1.6}$$

then $a_{n,k}$ is the (n, k) entry of the Riordan array $(g(t), f(t))$ [11]. We recall the following determinant definition of Sheffer sequences [12]:

Let $(s_n(x))_{n\in\mathbb{N}}$ be Sheffer for $(g(t), f(t))$, then we have

$$s_0(x) = \tfrac{1}{a_{0,0}},$$

$$s_n(x) = \frac{(-1)^n}{a_{0,0}\ a_{1,1}\ \cdots\ a_{n,n}} \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} & x^n \\ a_{0,0} & a_{1,0} & a_{2,0} & \cdots & a_{n-1,0} & a_{n,0} \\ 0 & a_{1,1} & a_{2,1} & \cdots & a_{n-1,1} & a_{n,1} \\ 0 & 0 & a_{2,2} & \cdots & a_{n-1,2} & a_{n,2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n,n-1} \end{vmatrix}, \tag{1.7}$$

$$= \frac{(-1)^n}{a_{0,0}\ a_{1,1}\ \cdots\ a_{n,n}} \det \begin{pmatrix} X_{n+1} \\ S_{n\times(n+1)} \end{pmatrix},$$

where $X_{n+1} = (1, x, x^2, \ldots, x^{n-1}, x^n)$, $S_{n\times(n+1)} = (a_{j-1,i-1})_{1\le i\le n,\ 1\le j\le n+1}$ and $a_{n,k}$ is the (n,k) entry of the Riordan array $(g(t), f(t))$.

The Sheffer sequences $(s_n(x))_{n\in\mathbb{N}}$ defined by equation (1.7) also satisfy the following definition:

$$s_n(x) = \sum_{k=0}^{n} b_{n,k}\ x^k, \tag{1.8}$$

where $b_{n,k}$ is the (n,k) entry of the Riordan array $\left(\frac{1}{g(\overline{f}(t))}, \overline{f}(t)\right)$ and $(s_n(x))_{n\in\mathbb{N}}$ is Sheffer for $(g(t), f(t))$ [11].

Definition 1.4. Let $(s_n(x))_{n\in\mathbb{N}}$ be Sheffer for $(g(t), f(t))$ and $(r_n(x))_{n\in\mathbb{N}}$ be Sheffer for $(h(t), l(t))$ with

$$r_n(x) = \sum_{k=0}^{n} r_{n,k} x^k, \qquad s_k(x) = \sum_{l=0}^{k} r_{k,l} x^l$$

then the umbral composition of $r_n(x)$ with $s_n(x)$ is the sequence $(r_n(s(x)))_{n\in\mathbb{N}}$ given by

$$(r_n(s(x)) = \sum_{k=0}^{n} r_{n,k}\ s_k(x) = \sum_{0\le k\le l\le n} r_{n,k}\ s_{k,l}\ x^l. \tag{1.9}$$

The composition of two Sheffer sequences $(r_n(s(x)))_{n\in\mathbb{N}}$ is Sheffer for the pair $(g(t)h(f(t)), l(f(t)))$ and the set of all Sheffer sequences is a group under umbral composition [8] and is called the *Sheffer group* with respect to $(c_n)_{n\in\mathbb{N}}$. Under this operation, the set of all Sheffer sequences is a non-abelian group.

The umbral composition of two Sheffer sequences can be treated as product of two Sheffer sequences. In view of this fact, we introduce the generating function and series representation for the product of two Sheffer sequences.

Let $s_n(x)$ and $r_n(x)$ be two Sheffer sequences with Riordan arrays $(g(t), f(t))$ and $(h(t), l(t))$, respectively defined by the generating functions of the following forms:

$$\frac{1}{g(\overline{f}(t))}\exp(x(\overline{f}(t))) = \sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!}, \tag{1.10}$$

$$\frac{1}{h(\overline{l}(t))}\exp(x(\overline{l}(t))) = \sum_{n=0}^{\infty} r_n(x)\frac{t^n}{n!}. \tag{1.11}$$

Expanding the exponential function and then replacing the powers $x^0,\ x^1, \ldots, x^n$ by the polynomials $r_0(x),\ r_1(x), \ldots, r_n(x)$, respectively in both sides of equation (1.10) and then using equation (1.11) in the l.h.s. and denoting the resultant product of Sheffer sequence in the r.h.s. of the resultant equation by

$$s_n(r_1(x)) := s_n(r_1(x)), \tag{1.12}$$

we find the following:

Definition 1.5. Let $s_n(x)$ and $r_n(x)$ be two Sheffer sequences with Riordan arrays $(g(t), f(t))$ and $(h(t), l(t))$. The exponential generating function for the product of Sheffer sequences $s_n(r_1(x))$ is given by

$$\frac{1}{g(\overline{f}(t))}\ \frac{1}{h(\overline{l}(\overline{f}(t)))}\exp(x\overline{l}(\overline{f}(t))) = \sum_{n=0}^{\infty} s_n(r_1(x))\frac{t^n}{n!}. \tag{1.13}$$

The product of Sheffer sequences $s_n(r_1(x))$ is Sheffer for the pair $\big(h(t)g(l(t)), f(l(t))\big)$.

Definition 1.6. The series representation for the product of Sheffer sequences $(s_n(r_1(x)))_{n\in\mathbb{N}}$ is given by

$$s_n(r_1(x)) = \sum_{k=0}^{n} b_{n,k}\ r_k(x), \tag{1.14}$$

where $b_{n,k}$ is the (n,k) entry of the Riordan array $\Big(\frac{1}{g(\overline{f}(t))}, \overline{f}(t)\Big)$ and $(s_n(r_1(x)))_{n\in\mathbb{N}}$ is Sheffer for the pair $\big(h(t)g(l(t)), f(l(t))\big)$.

It is important to remark that although this product can be viewed as the umbral composition of two Sheffer sequences but the results of this product cannot be deduced from the corresponding results of the Sheffer sequences. This fact stresses the importance to introduce such type of product. The article is organized as follows. In Section 2, the determinant approach, quasi-monomial properties and conjugate representation for the product of Sheffer sequence are derived via Riordan arrays. In Section 3, examples of some members belonging to the product of Sheffer sequence are given. In Section 4, a brief view of product of n-Sheffer sequences is presented.

2 Properties

The Sheffer sequences are shown to be quasi-monomial [4]. The product of two Sheffer sequences have never been shown to be quasi-monomial. To frame the product of Sheffer sequence $(s_n(r_1(x)))_{n\in\mathbb{N}}$ within the context of monomiality principle, the following result is proved:

Theorem 2.1. The product of Sheffer sequences $(s_n(r_1(x)))_{n\in\mathbb{N}}$ is quasi-monomial with respect to the following multiplicative and derivative operators:

$$\hat{M}_{s_n(r_1(x))} = \left(x - \frac{g'(l(D_x))}{g(l(D_x))} f'(l(D_x)) - \frac{h'(D_x)}{h(D_x)}\right) \frac{1}{f'(l(D_x))\ l'(D_x)} \tag{2.1}$$

and

$$\hat{P}_{s_n(r_1(x))} = f(l(D_x)), \tag{2.2}$$

respectively, where $D_x := \frac{d}{dx}$.

Proof. Differentiating equation (1.13) partially with respect to t and then simplifying the resultant equation, it follows that

$$\left(x - \frac{h'(\bar{l}(\overline{f}(t)))}{h(\bar{l}(\overline{f}(t)))} - \frac{g'(\overline{f}(t))}{g(\overline{f}(t))} l'(\bar{l}(\overline{f}(t)))\right) \frac{1}{l'(\bar{l}(f(t)))\ \overline{f}'(\overline{f}(t))} \tag{2.3}$$
$$\times \left\{\frac{1}{g(\overline{f}(t))}\ \frac{1}{h(\bar{l}(\overline{f}(t)))} \exp(x\bar{l}(\overline{f}(t)))\right\} = \sum_{n=0}^{\infty} s_n(r_1(x)) \frac{t^n}{n!}.$$

Since $g(t)$ and $h(t)$ are invertible series of t, therefore $\frac{g'(t)}{g(t)}$ and $\frac{h'(t)}{h(t)}$ possess power series expansions of t. Therefore, the following identity for the product of Sheffer sequences $(s_n(r_1(x)))_{n\in\mathbb{N}}$ holds true:

$$f(l(D_x)) \left\{\frac{1}{g(\overline{f}(t))}\ \frac{1}{h(\bar{l}(\overline{f}(t)))} \exp(x\bar{l}(\overline{f}(t)))\right\} = t\ \left(\frac{1}{g(\overline{f}(t))}\ \frac{1}{h(\bar{l}(\overline{f}(t)))} \exp(x\bar{l}(\overline{f}(t)))\right). \tag{2.4}$$

Making use of equations (2.4) and (1.13) in the l.h.s. of equation (2.3), rearranging the summations and then equating the coefficients of the same powers of t in both sides of the resultant equation, it follows that

$$\left(\left(x - \frac{g'(l(D_x))}{g(l(D_x))} f'(l(D_x)) - \frac{h'(D_x)}{h(D_x)}\right) \frac{1}{f'(l(D_x))\ l'(D_x)}\right) \{s_n(r_1(x))\} = s_{n+1}(r_1(x)), \tag{2.5}$$

which in view of monomiality principle equation $\hat{M}\{p_n(x)\} = p_{n+1}(x)$ for the product of Sheffer sequences yields assertion (2.1).

In order to prove assertion (2.2), generating function (1.13) is used in both sides of the identity (2.4), so that we have

$$f(l(D_x)) \left\{\sum_{n=0}^{\infty} s_n(r_1(x)) \frac{t^n}{n!}\right\} = \sum_{n=1}^{\infty} s_{n-1}(r_1(x)) \frac{t^n}{(n-1)!}, \tag{2.6}$$

which on rearranging the summation in the r.h.s. and then equating the coefficients of the same powers of t in both sides of the resultant equation and in view of monomiality principle equation $\hat{P}\{p_n(x)\} = n\ p_{n-1}(x)$ for the product of Sheffer sequences yields assertion (2.2).

Q.E.D.

Remark 2.1. Using expressions (2.1) and (2.2) of the operators in monomiality principle equation $\hat{M}\hat{P}\{p_n(x)\} = n\ p_n(x)$, we deduce the following consequence of Theorem 2.1:

Corollary 2.2. The product of Sheffer sequences $(s_n(r_1(x)))_{n\in\mathbb{N}}$ satisfy the following differential equation:

$$\left(\left(\frac{x}{f'(l(D_x))} - \frac{g'(l(D_x))}{g(l(D_x))}\right)\frac{f(l(D_x))}{l'(D_x)} - \frac{h'(D_x)}{h(D_x)}\frac{f(l(D_x))}{f'(l(D_x))\ l'(D_x)} - n\right) s_n(r_1(x)) = 0. \tag{2.7}$$

The determinant form for the product of Sheffer sequences $(s_n(r_1(x)))_{n\in\mathbb{N}}$ is established by proving the following theorem:

Theorem 2.3. The product of Sheffer sequences $(s_n(r_1(x)))_{n\in\mathbb{N}}$ of degree n is defined by

$$\begin{aligned} s_0(r_1(x)) &= \tfrac{1}{a_{0,0}}, \\ s_n(r_1(x)) &= \tfrac{(-1)^n}{a_{0,0}\ a_{1,1}\ \cdots\ a_{n,n}} \begin{vmatrix} 1 & r_1(x) & r_2(x) & \cdots & r_{n-1}(x) & r_n(x) \\ a_{0,0} & a_{1,0} & a_{2,0} & \cdots & a_{n-1,0} & a_{n,0} \\ 0 & a_{1,1} & a_{2,1} & \cdots & a_{n-1,1} & a_{n,1} \\ 0 & 0 & a_{2,2} & \cdots & a_{n-1,2} & a_{n,2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n,n-1} \end{vmatrix}, \\ &= \tfrac{(-1)^n}{a_{0,0}\ a_{1,1}\ \cdots\ a_{n,n}} \det\begin{pmatrix} R_{n+1}(x) \\ M_{n\times(n+1)} \end{pmatrix}, \end{aligned} \tag{2.8}$$

where $R_{n+1}(x) = (1, r_1(x), \ldots, r_{n-1}(x), r_n(x))$, $M_{n\times(n+1)} = (a_{j-1,i-1})_{1\le i\le n,\ 1\le j\le n+1}$ and $a_{n,k}$ is the (n,k) entry of the Riordan array $\big(g(t), f(t)\big)$.

Proof. Replacing x^n by $r_n(x)$ in the l.h.s. and x by $r_1(x)$ in r.h.s. of equation (1.6), it follows that

$$r_n(x) = \sum_{k=0}^{n} a_{n,k}\ s_k(r_1(x)), \tag{2.9}$$

where $a_{n,k}$ is the (n,k) entry of the Riordan array $\big(g(t), f(t)\big)$.

The above identity leads to the following system of infinite equations in unknowns $s_n(r_1(x))$ for

$n = 0, 1, \ldots :$

$$\begin{cases} a_{0,0}\ s_0(r_1(x)) = 1, \\ \\ a_{1,0}\ s_0(r_1(x)) + a_{1,1}\ s_1(r_1(x)) = r_1(x), \\ \\ a_{2,0}\ s_0(r_1(x)) + a_{2,1}\ s_1(r_1(x)) + a_{2,2}\ s_2(r_1(x)) = r_2(x), \\ \vdots \\ a_{n,0}\ s_0(r_1(x)) + a_{n,1}\ s_1(r_1(x)) + a_{n,2}\ s_2(r_1(x)) + \ldots + a_{n,n}\ s_n(r_1(x)) = r_n(x). \\ \vdots \end{cases} \tag{2.10}$$

From first equation of system (2.10), first part of assertion (2.8) is proved. Also, the special form of system (2.10) (lower triangular) allows us to work out the unknowns $(s_n(r_1(x)))_{n\in\mathbb{N}}$. Operating with the first $n+1$ equations simply by applying the Cramer's rule, we have

$$s_n(r_1(x)) = \frac{1}{a_{0,0}\ a_{1,1}\ \ldots\ a_{n,n}} \begin{vmatrix} a_{0,0} & 0 & 0 & \cdots & 0 & 1 \\ \\ a_{1,0} & a_{1,1} & 0 & \cdots & 0 & r_1(x) \\ \\ a_{2,0} & a_{2,1} & a_{2,2} & \cdots & 0 & r_2(x) \\ \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ a_{n-1,0} & a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & r_{n-1}(x) \\ \\ a_{n,0} & a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & r_n(x) \end{vmatrix}. \tag{2.11}$$

Now, bringing the $(n+1)$-th column to the first place by n transpositions of adjacent columns and noting that the determinant of a square matrix is the same as that of its transpose, second part of assertion (2.8) is proved.

Q.E.D.

To derive the conjugate representation for the product of Sheffer sequences $(s_n(r_1(x)))_{n\in\mathbb{N}}$, the following result is proved:

Theorem 2.4. The following conjugate representation for the product of Sheffer sequences $(s_n(r_1(x)))_{n\in\mathbb{N}}$ holds true:

$$s_n(r_1(x)) = \sum_{k=0}^{n} \frac{\left\langle (g(\overline{f}(t))h(\overline{l}(\overline{f}(t))))^{-1}(\overline{l}(\overline{f}(t)))^k \mid r_n(x) \right\rangle}{c_k} r_k(x). \tag{2.12}$$

Proof. We know that the product of Sheffer sequences $(s_n(r_1(x)))_{n\in\mathbb{N}}$ is defined by equation (1.14), where $b_{n,k}$ is the (n,k) entry of the Riordan array $\left(\frac{1}{g(\overline{f}(t))h(\overline{l}(\overline{f}(t)))}, \overline{l}(\overline{f}(t))\right)$. Then, according to the above fact and in view of equation (1.5), we have

$$a_{n,k} = \left[\frac{t^n}{c_n}\right] \frac{1}{g(\overline{f}(t))h(\overline{l}(\overline{f}(t)))} \frac{(\overline{l}(\overline{f}(t)))^k}{c_k}, \tag{2.13}$$

which on simplification gives

$$a_{n,k} = \frac{1}{c_k}\left[\frac{t^n}{c_n}\right](g(\overline{f}(t))h(\overline{l}(\overline{f}(t))))^{-1}((\overline{l}(\overline{f}(t))))^k, \tag{2.14}$$

that is

$$a_{n,k} = \frac{1}{c_k}\left\langle (g(\overline{f}(t))h(\overline{l}(\overline{f}(t))))^{-1}(\overline{l}(\overline{f}(t)))^k \mid s_n(x)\right\rangle. \tag{2.15}$$

Use of equation (2.15) in equation (1.14) yields assertion (2.12).

Q.E.D.

In the next section, examples of some members belonging to the product of Sheffer sequences are considered.

3 Examples

By making suitable selections for the pair of functions $(g(t), f(t))$ and $(h(t), l(t))$ in the results derived for product of Sheffer sequences, we get the results for the product of particulars members of the Sheffer family. The following examples will illustrate the process:

Example 3.1. Taking $g(t) = h(t) = \left(\frac{2}{1+\sqrt{1-t^2}}\right)^{\lambda}$, $f(t) = l(t) = \frac{-t}{1+\sqrt{1-t^2}}$ and $\bar{f}(t) = \bar{l}(t) = \frac{-2t}{1+t^2}$ (for which the sheffer sequences $(s_n(x))_{n\in\mathbb{N}}$ reduce to the Gegenbauer polynomials $C_n^{(\lambda)}(x)$ [1]) in the l.h.s. of generating function (1.13), we find the resultant product of Gegenbauer polynomials $C_n^{(\lambda)}(C_1^{(\lambda)}(x))$. The results for the product of Gegenbauer polynomials are given in Table 1 below.

Table 1. Results for $\mathbf{C_n^{(\lambda)}(C_1^{(\lambda)}(x))}$

S.No.	Results	Expressions
I.	Generating function	$\left(\frac{1+t^2}{1+6t^2+t^4}\right)^{\lambda}\exp\left(\frac{4xt(1+t^2)}{1+6t^2+t^4}\right) = \sum_{n=0}^{\infty} C_n^{(\lambda)}(C_1^{(\lambda)}(x))\frac{t^n}{n!}$
II.	Series definition	$C_n^{(\lambda)}(C_1^{(\lambda)}(x)) = \sum_{k=0}^{[\frac{n}{2}]}\sum_{l=0}^{[\frac{n-2k}{2}]} \frac{(-1)^{k+l}\Gamma(n-k+\lambda)\Gamma(n-2k-l+\lambda)2^{2n-4k-2l}x^{n-2k-2l}}{(\Gamma(\lambda))^2\ k!\ l!\ (n-2k)!\ (n-2k-2l)!}$
III.	Multiplicative and derivative operators	$\hat{M}_{C_n^{(\lambda)}(C_1^{(\lambda)}(x))} = \Big(x - \frac{2^{\lambda-1}\ \lambda\ D_x(1+\sqrt{1-D_x^2})^{\lambda+2}}{(\sqrt{2(1-D_x^2+\sqrt{1-D_x^2})})^2\ (1+\sqrt{1-D_x^2}+\sqrt{2(1-D_x^2+\sqrt{1-D_x^2})})^{\lambda+1}}$ $-\frac{\lambda D_x}{\sqrt{1-D_x^2}(1+\sqrt{1-D_x^2})}\Big)\frac{1+\sqrt{1-D_x^2}}{\sqrt{2(1-D_x^2+\sqrt{1-D_x^2})}\ (1+\sqrt{1-D_x^2}+\sqrt{2(1-D_x^2+\sqrt{1-D_x^2})})}\frac{1}{\sqrt{(1-D_x^2)}}$ $\hat{P}_{C_n^{(\lambda)}(C_1^{(\lambda)}(x))} = \frac{-D_x}{1+\sqrt{1-D_x^2}+\sqrt{2(1-D_x^2+\sqrt{1-D_x^2})}}$
IV.	Differential equation	$\Big(\Big(x + \frac{2^{\lambda-1}\ \lambda\ D_x(1+\sqrt{1-D_x^2})^{\lambda+2}}{(\sqrt{2(1-D_x^2+\sqrt{1-D_x^2})})^2\ (1+\sqrt{1-D_x^2}+\sqrt{2(1-D_x^2+\sqrt{1-D_x^2})})^{\lambda+1}} + \frac{\lambda D_x}{\sqrt{1-D_x^2}(1+\sqrt{1-D_x^2})}\Big)$ $\frac{D_x(1+\sqrt{1-D_x^2})}{\sqrt{2(1-D_x^2+\sqrt{1-D_x^2})}\ (1+\sqrt{1-D_x^2}+\sqrt{2(1-D_x^2+\sqrt{1-D_x^2})})^2} \times \frac{1}{\sqrt{1-D_x^2}} - n\Big)C_n^{(\lambda)}(C_1^{(\lambda)}(x)) = 0$
V.	Conjugate representation	$C_n^{(\lambda)}(C_1^{(\lambda)}(x)) = \sum_{k=0}^{n} \frac{\langle((1+t^2)^{-3\lambda}(1+6t^2+t^4)^{-\lambda})^{-1}(4t(1+t^2)(1+t^4+6t^2)^{-1})^k|C_n^{(\lambda)}(x)\rangle}{c_k} C_k^{(\lambda)}(x)$

Taking $a_{n,k} = \begin{cases} 0 & \text{if } n-k \text{ is odd,} \\ \frac{c_n(-1)^k(\lambda+k)}{c_k 2^n(\lambda+n)}\binom{\lambda+n}{n-k/2} & \text{if } n-k \text{ is even,} \end{cases}$ where $c_n = 1/\binom{\lambda}{n}$ (for which the determinant form (1.7) of the $(s_n(x))_{n\in\mathbb{N}}$ reduces to the determinant form of the $C_n^{(\lambda)}(x)$) and $r_n(x) = C_n^{(\lambda)}(x)$ in equation (2.8), the following determinant form for the product of Gegenbauer polynomials $C_n^{(\lambda)}(C_1^{(\lambda)}(x))$ is obtained:

$$C_0^{(\lambda)}(C_1^{(\lambda)}(x)) = 1,$$

$$C_n^{(\lambda)}(C_1^{(\lambda)}(x)) = (-1)^{\frac{n(n+3)}{2}} 2^{\frac{n(n+1)}{2}} \begin{vmatrix} 1 & C_1^{(\lambda)}(x) & C_2^{(\lambda)}(x) & \cdots & C_{n-1}^{(\lambda)}(x) & C_n^{(\lambda)}(x) \\ 1 & 0 & \frac{\lambda}{2\lambda(\lambda+1)} & \cdots & 0 & \frac{n!\lambda(\lambda+n-1)!}{2^n\left(\frac{n}{2}\right)!\lambda(\lambda+1)\ldots(\lambda+n-1)\left(\lambda+\frac{n}{2}\right)!} \\ 0 & -\frac{1}{2} & 0 & \cdots & & 0 \\ 0 & 0 & \frac{1}{4} & \cdots & 0 & \frac{n(n-1)\ldots3(\lambda+2)(\lambda+n-1)!}{2^n\left(\frac{n-2}{2}\right)!(\lambda+2)\ldots(\lambda+n-1)\left(\lambda+\frac{n+2}{2}\right)!} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \left(\frac{-1}{2}\right)^{n-1} & 0 \end{vmatrix}. \tag{3.1}$$

Remark 3.1. For $\lambda = 1$, the Gegenbauer polynomials $C_n^{(\lambda)}(x)$ reduce to the Chebyshev polynomials of the second kind $U_n(x)$ [1]. Therefore, taking $\lambda = 1$ in the results of the product Gegenbauer polynomials $C_n^{(\lambda)}(C_1^{(\lambda)}(x))$, the corresponding results for the product of Chebyshev polynomials of the second kind $U_n(U_1(x))$ can be obtained.

Remark 3.2. For $\lambda = 1/2$, the Gegenbauer polynomials $C_n^{(\lambda)}(x)$ reduce to the Legendre polynomials $P_n(x)$ [1]. Therefore, taking $\lambda = 1/2$ in the results of the product Gegenbauer polynomials $C_n^{(\lambda)}(C_1^{(\lambda)}(x))$, the corresponding results for the product of Legendre polynomials $P_n(P_1(x))$ can be obtained.

Example 3.2. Taking $g(t) = h(t) = 1$, $f(t) = l(t) = \log(1+t)$ and $\bar{f}(t) = \bar{l}(t) = e^t - 1$ (for which the Sheffer sequences $(s_n(x))_{n\in\mathbb{N}}$ reduce to the exponential polynomials $\varphi_n(x)$ [9]) in the l.h.s. of generating function (1.13), we find the resultant product of the exponential polynomials $\varphi_n(\varphi_1(x))$. The results for the product of exponential polynomials are given in Table 2 below.

Table 2. Results for $\varphi_{\mathbf{n}}(\varphi_{\mathbf{1}}(\mathbf{x}))$

S.No.	Results	Expressions
I.	Generating function	$\exp(x(e^{e^t-1}-1)) = \sum_{n=0}^{\infty} \varphi_n(\varphi_1(x)) \frac{t^n}{n!}$
II.	Series definition	$\varphi_n(\varphi_1(x)) = \sum_{k=0}^{n} \sum_{l=0}^{k} S(n,k)S(k,l)x^l;\ S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n$
III.	Multiplicative and derivative operators	$\hat{M}_{\varphi_n(\varphi_1(x))} = x(1+\log(1+D_x))(1+D_x)$ $\hat{P}_{\varphi_n(\varphi_1(x))} = \log(1+\log(1+D_x))$
IV.	Differential equation	$(x(1+\log(1+D_x))(1+D_x)\log(1+\log(1+D_x)) - n)\varphi_n(\varphi_1(x)) = 0$
V.	Conjugate representation	$\varphi_n(\varphi_1(x)) = \sum_{k=0}^{n} \frac{\langle (e^{e^t-1}-1))^k \mid \varphi_n(x) \rangle}{c_k} \varphi_k(x)$

Taking $a_{n,k} = s(n,k)$ with $a_{n,0} = \delta_{n,0}$ (for which the determinant form (1.7) of the $(s_n(x))_{n\in\mathbb{N}}$ reduces to the determinant form of the exponential polynomials $\varphi_n(x)$) and $r_n(x) = \varphi_n(x)$ in equation (2.8), the following determinant form for the product of exponential polynomials $\varphi_n(\varphi_1(x))$ is obtained:

$$\varphi_0(\varphi_1(x)) = 1,$$
$$\varphi_n(\varphi_1(x)) = (-1)^{n+1} \begin{vmatrix} \varphi_1(x) & \varphi_2(x) & \cdots & \varphi_{n-1}(x) & \varphi_n(x) \\ s(1,1) & s(2,1) & \cdots & s(n-1,1) & s(n,1) \\ 0 & s(2,2) & \cdots & s(n-1,2) & s(n,2) \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & s(n-1,n-1) & s(n,n-1) \end{vmatrix}, \tag{3.2}$$

where $s(n,k) = (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}$ are the Stirling numbers of the first kind.

Example 3.3. Taking $g(t) = h(t) = e^{\nu t^2/2}$, $f(t) = l(t) = t$ and $\bar{f}(t) = \bar{l}(t) = t$ (for which the Sheffer sequences $(s_n(x))_{n\in\mathbb{N}}$ reduce to the Hermite polynomials $H_n^{(\nu)}(x)$ of variance ν [9]) in the l.h.s. of generating function (1.13), we find the resultant product of Hermite polynomials $H_n^{(\nu)}(H_1^{(\nu)}(x))$ of variance ν. The results for the product of Hermite polynomials are given in Table 3 below.

Table 3. Results for $\mathbf{H_n^{(\nu)}(H_1^{(\nu)}(x))}$

S.No.	Results	Expressions
I.	Generating function	$e^{xt-\nu t^2} = \sum_{n=0}^{\infty} H_n^{(\nu)}(H_1^{(\nu)}(x))\frac{t^n}{n!}$
II.	Series definition	$H_n^{(\nu)}(H_1^{(\nu)}(x)) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \sum_{l=0}^{\left[\frac{n-2k}{2}\right]} \frac{(-\nu)^{k+l}\, x^{n-2k-2l}}{k!\, l!\, (n-2k-2l)!}$
III.	Multiplicative and derivative operators	$\hat{M}_{H_n^{(\nu)}(H_1^{(\nu)}(x))} = x - 2\nu D_x$ $\hat{P}_{H_n^{(\nu)}(H_1^{(\nu)}(x))} = D_x$
IV.	Differential equation	$(2\nu D_x^2 - xD_x + n)H_n^{(\nu)}(H_1^{(\nu)}(x)) = 0$
V.	Conjugate representation	$H_n^{(\nu)}(H_1^{(\nu)}(x)) = \sum_{k=0}^{n} \frac{\langle e^{-\nu t^2} t^k | H_n^{(\nu)}(x)\rangle}{c_k} H_k^{(\nu)}(x)$

Taking $a_{n,k} = \binom{n}{k} g_{n-k}$ where $g_k = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \frac{k!\left(\frac{\nu}{2}\right)^{k/2}}{\left(\frac{k}{2}\right)!} & \text{if } k \text{ is even,} \end{cases}$ (for which the determinant form (1.7) of the $(s_n(x))_{n\in\mathbb{N}}$ reduces to the determinant form of the Hermite polynomials $H_n^{(\nu)}(x)$ of variance ν) and $r_n(x) = H_n^{(\nu)}(x)$ in equation (2.8), the following determinant form for the product of Hermite polynomials $H_n^{(\nu)}(H_1^{(\nu)}(x))$ of variance ν is obtained:

$$H_0^{(\nu)}(H_1^{(\nu)}(x)) = 1,$$

$$H_n^{(\nu)}(H_1^{(\nu)}(x)) = (-1)^n \begin{vmatrix} 1 & H_1^{(\nu)}(x) & H_2^{(\nu)}(x) & \cdots & H_{n-1}^{(\nu)}(x) & H_n^{(\nu)}(x) \\ 1 & 0 & 1 & \cdots & \frac{(n-1)!\left(\frac{\nu}{2}\right)^{n-1/2}}{\left(\frac{n-1}{2}\right)!} & \frac{n!\left(\frac{\nu}{2}\right)^{n/2}}{\left(\frac{n}{2}\right)!} \\ 0 & 1 & 0 & \cdots & \binom{n-1}{1}\frac{(n-2)!\left(\frac{\nu}{2}\right)^{n-2/2}}{\left(\frac{n-2}{2}\right)!} & \binom{n}{1}\frac{(n-1)!\left(\frac{\nu}{2}\right)^{n-1/2}}{\left(\frac{n-1}{2}\right)!} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}\frac{(n-3)!\left(\frac{\nu}{2}\right)^{n-3/2}}{\left(\frac{n-3}{2}\right)!} & \binom{n}{2}\frac{(n-2)!\left(\frac{\nu}{2}\right)^{n-2/2}}{\left(\frac{n-2}{2}\right)!} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{vmatrix}. \tag{3.3}$$

We note that for $\nu = 1$ the product of Hermite polynomials of variance ν reduces to the product of Hermite polynomials $H_n(H_1(x))$.

Example 3.4. Taking $g(t) = h(t) = \frac{1}{(1-t)^{\alpha+1}}$ and $f(t) = l(t) = \bar{f}(t) = \bar{l}(t) = \frac{t}{t-1}$ (for which the Sheffer sequences $(s_n(x))_{n\in\mathbb{N}}$ reduce to the Laguerre polynomials $L_n^{(\alpha)}(x)$ [9]) in the l.h.s. of generating function (1.13), we find the resultant product of the Laguerre polynomials of order α,

$L_n^{(\alpha)}(L_1^{(\alpha)}(x))$. The results for the product of Laguerre polynomials are given in Table 4 below.

Table 4. Results for $\mathbf{L_n^{(\alpha)}(L_1^{(\alpha)}(x))}$

S.No.	Results	Expressions
I.	Generating function	$\exp(xt) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(L_1^{(\alpha)}(x))\frac{t^n}{n!}$
II.	Series definition	$L_n^{(\alpha)}(L_1^{(\alpha)}(x)) = \sum_{k=0}^{n}\sum_{l=0}^{k}\frac{(-1)^{k+l}}{k!\,l!}\binom{n+\alpha}{n-k}\binom{k+\alpha}{k-l}x^l$
III.	Multiplicative and derivative operators	$\hat{M}_{L_n^{(\alpha)}(L_1^{(\alpha)}(x))} = \frac{1}{(1-D_x)}\left((1+\alpha)(D_x^4 - 4D_x^3 + 6D_x^2 - 4D_x) - xD_x + x\right)$ $\hat{P}_{L_n^{(\alpha)}(L_1^{(\alpha)}(x))} = D_x$
IV.	Differential equation	$\left((1+\alpha)(D_x^5 - 4D_x^4 + 6D_x^3 - 4D_x^2) - xD_x^2 + xD_x - n(1-D_x)\right)L_n^{(\alpha)}(L_1^{(\alpha)}(x)) = 0$
V.	Conjugate representation	$L_n^{(\alpha)}(L_1^{(\alpha)}(x)) = \sum_{k=0}^{n}\frac{\langle((t-1)^{\alpha+1}(1-t)^{-\alpha-1}t^k\,\vert L_n^{(\alpha)}(x)\rangle}{c_k}L_k^{(\alpha)}(x)$

Taking $a_{n,k} = (-1)^k\frac{n!}{k!}\binom{n+\alpha}{n-k}$ (for which the determinant form (1.7) of the Sheffer sequences reduces to the determinant form of the Laguerre polynomials $L_n^{(\alpha)}(x)$ of order α) and $r_n(x) = L_n^{(\alpha)}(x)$ $(n = 1, 2, \ldots)$ in equation (2.8), the following determinant form of the product of Laguerre polynomials of order α $L_n^{(\alpha)}(L_1^{(\alpha)}(x))$ is obtained:

$$L_0^{(\alpha)}(L_1^{(\alpha)}(x)) = 1,$$

$$L_n^{(\alpha)}(L_1^{(\alpha)}(x)) = (-1)^{\frac{n(n+3)}{2}}\begin{vmatrix} 1 & L_1^{(\alpha)}(x) & L_2^{(\alpha)}(x) & \cdots & L_{n-1}^{(\alpha)}(x) & L_n^{(\alpha)}(x) \\ 1 & (\alpha+1) & (\alpha+2)_2 & \cdots & (\alpha+n-1)_{n-1} & (\alpha+n)_n \\ 0 & -1 & -2(\alpha+2) & \cdots & -(n-1)(\alpha+n-1)_{n-2} & -n(\alpha+n)_{n-1} \\ 0 & 0 & 1 & \cdots & \frac{(n-1)(n-2)}{2}(\alpha+n-1)_{n-3} & \frac{n(n-1)}{2}(\alpha+n)_{n-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & (-1)^{n-1} & (-1)^{n-1}n(n+\alpha) \end{vmatrix}. \tag{3.4}$$

We note that for $\alpha = 0$, the product of Laguerre polynomials of order α reduce to product of Laguerre polynomials $L_n(L_1(x))$.

By making different selections for the pair of functions $(g(t), f(t))$ and $(h(t), l(t))$ in the results of product of Sheffer sequences, we get the results for the product of different members of Sheffer family. To illustrate this process, the following examples are considered:

Example 3.5. Taking $(g(t), f(t)) = (e^{t^2/2}, t)$ of the Hermite polynomials and $(h(t), l(t)) = \left(\frac{2}{1+\sqrt{1-t^2}}, \frac{-t}{1+\sqrt{1-t^2}}\right)$ of the Chebyshev polynomials of the second kind with $\bar{f}(t) = t$ and $\bar{l}(t) = \frac{-2t}{1+t^2}$ in the l.h.s. of generating function (1.13), we find the product of Chebyshev-Hermite polynomials of the second kind $U_n(H_1(x))$. The results for the product of Chebyshev-Hermite polynomials of the second kind are given in Table 5 below.

Table 5. Results for $\mathbf{H_n(U_1(x))}$

S.No.	Results	Expressions
I.	Generating function	$\left(\frac{\sqrt{1+t^2}+\sqrt{1+3t^2}}{2\sqrt{1+t^2}}\right)\exp\left(\frac{-2xt}{1+t^2} - \frac{t^2}{2}\right) = \sum_{n=0}^{\infty} H_n(U_1(x))\frac{t^n}{n!}$
II.	Series definition	$H_n(U_1(x)) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \sum_{l=0}^{\left[\frac{n-2k}{2}\right]} \frac{(-1)^{k+l} n!\ (n-2k-l)! 2^{n-3k-2l} x^{n-2k-2l}}{k!\ l!\ (n-2k)!\ (n-2k-2l)!}$
III.	Multiplicative and derivative operators	$\hat{M}_{H_n(U_1(x))} = \left(x + \frac{D_x}{1+\sqrt{1-D_x^2}} - \frac{D_x}{\sqrt{1-D_x^2}(1+\sqrt{1-D_x^2})^2}\right)\left(-1 - \frac{D_x^2}{\sqrt{1-D_x^2}(1+\sqrt{1-D_x^2})^2}\right)$ $\hat{P}_{H_n(U_1(x))} = -\frac{D_x}{1+\sqrt{1-D_x^2}}$
IV.	Differential equation	$\left(\left(\frac{-xD_x}{1+\sqrt{1-D_x^2}} - \frac{D_x^2}{(1+\sqrt{1-D_x^2})^2} + \frac{D_x^2}{\sqrt{1-D_x^2}(1+\sqrt{1-D_x^2})^3}\right)\left(\frac{D_x}{1+\sqrt{1-D_x^2}} - \frac{D_x^3}{\sqrt{1-D_x^2}(1+\sqrt{1-D_x^2})^3}\right) - n\right) H_n(U_1(x)) = 0$
V.	Conjugate representation	$H_n(U_1(x)) = \sum_{k=0}^{n} \frac{\left\langle (e^{-t^2/2}\left(\frac{\sqrt{1+t^2}+\sqrt{1+3t^2}}{2\sqrt{1+t^2}}\right)\left(\frac{-2t}{1+t^2}\right)^k | U_n(x)\right\rangle}{c_k} U_k(x)$

Taking $a_{n,k} = \binom{n}{k} g_{n-k}$ where $g_k = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \frac{k!\left(\frac{1}{2}\right)^{k/2}}{\left(\frac{k}{2}\right)!} & \text{if } k \text{ is even,} \end{cases}$ of the Hermite polynomials $H_n(x)$ and $r_n(x) = U_n(x)$ in equation (2.8), the following determinant form for the product of Chebyshev-Hermite polynomials of the second kind $H_n(U_1(x))$ is obtained:

$$H_0(U_1(x)) = 1,$$

$$H_n(U_1(x)) = (-1)^n \begin{vmatrix} 1 & U_1(x) & U_2(x) & \cdots & U_{n-1}(x) & U_n(x) \\ 1 & 0 & 1 & \cdots & \frac{(n-1)!\left(\frac{1}{2}\right)^{n-1/2}}{\left(\frac{n-1}{2}\right)!} & \frac{n!\left(\frac{1}{2}\right)^{n/2}}{\left(\frac{n}{2}\right)!} \\ 0 & 1 & 0 & \cdots & \binom{n-1}{1}\frac{(n-2)!\left(\frac{1}{2}\right)^{n-2/2}}{\left(\frac{n-2}{2}\right)!} & \binom{n}{1}\frac{(n-1)!\left(\frac{1}{2}\right)^{n-1/2}}{\left(\frac{n-1}{2}\right)!} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}\frac{(n-3)!\left(\frac{1}{2}\right)^{n-3/2}}{\left(\frac{n-3}{2}\right)!} & \binom{n}{2}\frac{(n-2)!\left(\frac{1}{2}\right)^{n-2/2}}{\left(\frac{n-2}{2}\right)!} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{vmatrix}. \quad (3.5)$$

Example 3.6. Taking $(g(t), f(t)) = (1, \log(1+t))$ of the exponential polynomials and $(h(t), l(t)) = \left(\frac{1}{1-t}, \frac{t}{t-1}\right)$ of the Laguerre polynomials with $\bar{f}(t) = e^t - 1$ and $\bar{l}(t) = \frac{t}{t-1}$ in the l.h.s. of generating function (1.13), we find the product of Laguerre-exponential polynomials $\varphi_n(L_1(x))$. The results for the product of Laguerre-exponential polynomials are given in Table 6 below.

Table 6. Results for $\varphi_n(L_1(x))$

S.No.	Results	Expressions
I.	Generating function	$\left(\frac{1}{2-e^t}\right)\exp\left(x\frac{e^t-1}{e^t-2}\right) = \sum_{n=0}^{\infty} \varphi_n(L_1(x))\frac{t^n}{n!}$
II.	Series definition	$\varphi_n(L_1(x)) = \sum_{k=0}^{n}\sum_{l=0}^{k} \frac{(-1)^l\, S(n,k)k!x^l}{(l!)^2\,(k-l)!}$; $S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$
III.	Multiplicative and derivative operators	$\hat{M}_{\varphi_n(L_1(x))} = x - 2xD_x + D_x - 1 - \frac{1-2D_x}{1-D_x}$ $\hat{P}_{\varphi_n(L_1(x))} = \log\left(\frac{2D_x-1}{D_x-1}\right)$
IV.	Differential equation	$\left(\left(x - 2xD_x + D_x - 1 - \frac{1-2D_x}{1-D_x}\right)\log\left(\frac{2D_x-1}{D_x-1}\right) - n\right)\varphi_n(L_1(x)) = 0$
V.	Conjugate representation	$\varphi_n(L_1(x)) = \sum_{k=0}^{n} \frac{\left\langle (2-e^t)^{-1}\left(\frac{e^t-1}{e^t-2}\right)^k \middle| L_n(x)\right\rangle}{c_k} L_k(x)$

Taking $a_{n,k} = s(n,k)$ with $a_{n,0} = \delta_{n,0}$ and $r_n(x) = L_n(x)$ in equation (2.8), the following determinant form for the product of Laguerre-exponential polynomials $\varphi_n(L_1(x))$ is obtained:

$$\varphi_0(L_1(x)) = 1,$$
$$\varphi_n(L_1(x)) = (-1)^{n+1}\begin{vmatrix} L_1(x) & L_2(x) & \cdots & L_{n-1}(x) & L_n(x) \\ s(1,1) & s(2,1) & \cdots & s(n-1,1) & s(n,1) \\ 0 & s(2,2) & \cdots & s(n-1,2) & s(n,2) \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & s(n-1,n-1) & s(n,n-1) \end{vmatrix}. \tag{3.6}$$

Example 3.7. Taking $(g(t), f(t)) = \left(\frac{1}{1-t}, \frac{t}{t-1}\right)$ of the Laguerre polynomials and $(h(t), l(t)) = (e^{t^2/2}, t)$ of the Hermite polynomials with $\bar{f}(t) = \frac{t}{t-1}$ and $\bar{l}(t) = t$ in the l.h.s. of generating function (1.13), we find the product of Hermite-Laguerre polynomials $L_n(H_1(x))$. The results for the product of Hermite-Laguerre polynomials $L_n(H_1(x))$ are given in Table 7 below.

Table 7. Results for $\mathbf{L_n(H_1(x))}$

S.No.	Results	Expressions
I.	Generating function	$\left(\frac{1}{1-t}\right)\exp\left(\frac{xt}{t-1}-\frac{1}{2}\left(\frac{t}{t-1}\right)^2\right)=\sum_{n=0}^{\infty}L_n(H_1(x))\frac{t^n}{n!}$
II.	Series definition	$L_n(H_1(x))=n!\sum_{k=0}^{n}\sum_{l=0}^{[k/2]}\frac{(-1)^{k+l}}{k!}\frac{(\frac{1}{2})^l}{l!}\frac{x^{k-2l}}{(n-k)!\,(k-2l)!}$
III.	Multiplicative and derivative operators	$\hat{M}_{L_n(H_1(x))}=-x(1-D_x)^2-\frac{1}{1-D_x}+\frac{D_x}{(D_x-1)^2}$ $\hat{P}_{L_n(H_1(x))}=\frac{D_x}{D_x-1}$
IV.	Differential equation	$\left(-xD_x(D_x-1)+D_x+\frac{D_x^2}{(D_x-1)^3}-n\right)L_n(H_1(x))=0$
V.	Conjugate representation	$L_n(H_1(x))=\sum_{k=0}^{n}\frac{\left\langle(1-t)^{-1}\exp\left(-\frac{1}{2}\left(\frac{t}{t-1}\right)^2\right)\left(\frac{t}{t-1}\right)^k\middle\vert H_n(x)\right\rangle}{c_k}H_k(x)$

Taking $a_{n,k}=(-1)^k\frac{n!}{k!}\binom{n}{n-k}$ and $r_n(x)=H_n(x)$ $(n=1,2,\ldots)$ in equation (2.8), the following determinant form for product of Hermite-Laguerre polynomials $L_n(H_1(x))$ is obtained:

$$L_0(H_1(x))=1,$$

$$L_n(H_1(x))=(-1)^{\frac{n(n+3)}{2}}\begin{vmatrix} 1 & H_1(x) & H_2(x) & \cdots & H_{n-1}(x) & H_n(x) \\ 1 & (1) & (2)_2 & \cdots & (n-1)_{n-1} & (n)_n \\ 0 & -1 & -4 & \cdots & -(n-1)(n-1)_{n-2} & -n(n)_{n-1} \\ 0 & 0 & 1 & \cdots & \frac{(n-1)(n-2)}{2}(n-1)_{n-3} & \frac{n(n-1)}{2}(n)_{n-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & (-1)^{n-1} & (-1)^{n-1}n(n) \end{vmatrix}. \tag{3.7}$$

Note. Since the product of two Sheffer sequences is non-abelian, thus we note that the results obtained for the product of Chebyshev-Hermite polynomials $H_n(U_1(x))$, product of Laguerre-exponential polynomials $\varphi_n(L_1(x))$ and for the product of Hermite-Laguerre polynomials $L_n(H_1(x))$ do not commute with the product of Hermite-Chebyshev polynomials $U_n(H_1(x))$, product of exponential-Laguerre polynomials $L_n(\varphi_1(x))$ and for the product of Laguerre-Hermite polynomials $H_n(L_1(x))$.

4 Product of n-Sheffer sequences

The Laguerre polynomials play a key role in applied mathematics and physics as they are involved in the solutions of the wave equation of hydrogen atom. The Laguerre polynomials arise in quantum mechanics, in the radial part of the solution of the Schrödinger equation for a one-electron atom. They also describe the static Wigner functions of oscillator systems in quantum mechanics in phase

space. They further enter in the quantum mechanics of the morse potential and of the $3D$ isotropic harmonic oscillator.

The Laguerre polynomials $L_n^{(\alpha)}(x)$ [9] of order α form the Sheffer sequence for the pair $\left(\frac{1}{(1-t)^{\alpha+1}}, \frac{t}{t-1}\right)$. Due to the fact that $\bar{f}(t) = f(t)$ for the Laguerre polynomials, these polynomials deserve special attention. This fact also help in deriving interesting properties of the Laguerre polynomials by using umbral methods [9]. Now, we consider n-multiplicative Laguerre polynomials and explore their interesting properties.

From generating function Table 4 (I), we note that $L_n^{(\alpha)}(L_1^{(\alpha)}(x)) = x^n$, *i.e.* the product of Laguerre polynomials are self inverse under umbral composition. Now, taking $g(t) = \frac{1}{(1-t)^{\alpha+1}}$, $h(t) = \frac{1}{(1-t)^{\beta+1}}$ and $f(t) = l(t) = \frac{t}{t-1}$ in the l.h.s. of generating function (1.13), we find that the resultant product of Laguerre polynomials $L_n^{(\alpha)}(L_1^{(\beta)}(x))$ is Sheffer for the pair $((1-t)^{\alpha-\beta}, t)$ and is defined by the following generating function:

$$\frac{1}{(1-t)^{\alpha-\beta}} \exp(xt) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(L_1^{(\beta)}(x)) \frac{t^n}{n!}. \tag{4.1}$$

Thus, $L_n^{(\alpha)}(L_1^{(\beta)}(x))$ is Sheffer for $((1-t)^{\alpha-\beta}, t)$ and consequently

$$S_n^{(\lambda)}(x) = (1-t)^{\beta-\alpha} x^n = (-1)^n L_n^{\beta-\alpha-n}(x). \tag{4.2}$$

Now, suppose that

$$L_n^{(\alpha)}(L_1^{(\beta)}(x)) = (1-t)^{-\lambda} x^n = (-1)^n L_n^{(-\lambda-n)}(x), \tag{4.3}$$

so that we have

$$L_n^{(\alpha)}(L_1^{(\beta)}(x)) = S_n^{(\alpha-\beta)}(x). \tag{4.4}$$

Similarly, the product of k Laguerre polynomials of orders α_1, α_2,..., α_k respectively is given by

$$L_n^{(\alpha_1)}(L_n^{(\alpha_2)}(L_n^{(\alpha_3)}(\ldots(L_n^{(\alpha_k)}(x))\ldots))) = \begin{cases} L_n^{(\alpha_1-\alpha_2+\alpha_3-\cdots+\alpha_k)}(x), \ k \text{ odd} \\ S_n^{(\alpha_1-\alpha_2+\alpha_3-\cdots-\alpha_k)}(x), \ k \text{ even}, \end{cases} \tag{4.5}$$

where $S_n^{(\lambda)}(x)$ is Sheffer for $((1-t)^{\lambda}, t)$ and the method to form such type of product is called Rota's composition law [9]. In view of this, the generating function for the product of k Laguerre polynomials of orders α_1, α_2,..., α_k respectively is given by

$$\begin{cases} \frac{1}{(1-t)^{\alpha_1-\alpha_2-\ldots+\alpha_k}} \exp(xt) = \sum\limits_{n=0}^{\infty} L_n^{(\alpha_1-\alpha_2+\alpha_3-\cdots+\alpha_k)}(x), \ k \text{ odd} \\ \frac{1}{(1-t)^{\alpha_1-\alpha_2-\ldots-\alpha_k}} \exp(xt) = \sum\limits_{n=0}^{\infty} S_n^{(\alpha_1-\alpha_2+\alpha_3-\cdots-\alpha_k)}(x), \ k \text{ even} \end{cases} \tag{4.6}$$

We conclude that the product of two Sheffer sequences considered in this paper extends the possibility to introduce hybrid type Sheffer polynomials. The set of this product of Sheffer sequences

is a non-abelian group and thus convoluting different members of Sheffer class allows us to consider a number of hybrid type special polynomials as its members. The results which are derived in Sections 2 and 3 related to this product cannot be deduced from the results of the Sheffer sequences. Hence, the generating function (1.13) in its product form gains special importance due to the fact that by convoluting other members belonging to the Sheffer family, we get a large number of new product of hybrid type special polynomials as its members. We present the list of other members belonging to the Sheffer family in Table 8 below:

Table 8. Members belonging to Sheffer family

S.No.	$\mathcal{A}(\mathbf{t})$; $\mathbf{H}(\mathbf{t})$	$\mathbf{g}(\mathbf{t})$; $\mathbf{f}(\mathbf{t})$	Generating functions	Polynomials
I.	$e^{-t^2}; 2t$	$e^{\frac{t^2}{4}}; \frac{t}{2}$	$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$	Hermite polynomials $H_n(x)$ [7]
II.	$e^{-t^m}; \nu t$	$e^{(\frac{t}{\nu})^m}; \frac{t}{\nu}$	$\exp(\nu xt - t^m) = \sum_{n=0}^{\infty} H_{n,m,\nu}(x)\frac{t^n}{n!}$	Generalized Hermite polynomials $H_{n,m,\nu}(x)$ [6]
III.	$\frac{t}{1-t}$; $\ln\left(\frac{1+t}{1-t}\right)$	$\frac{2}{e^t-1}$; $\frac{e^t-1}{e^t+1}$	$\frac{t}{1-t}\left(\frac{1+t}{1-t}\right)^x = \sum_{n=0}^{\infty} \mathcal{P}_n(x)\frac{t^n}{n!}$	$\mathcal{P}$idduck polynomials $P_n(x)$ [3]
IV.	$e^{\beta t}; 1-e^t$	$(1-t)^{-\beta}$; $\ln(1-t)$	$\exp\left(\beta t + x(1-e^t)\right) = \sum_{n=0}^{\infty} a_n^{(\beta)}(x)\frac{t^n}{n!}$	Actuarial polynomials $a_n^{(\beta)}(x)$ [3]
V.	e^{-t}; $\ln\left(1+\frac{t}{a}\right)$	$\exp\left(a(e^t-1)\right)$; $a(e^t-1)$	$e^{-t}\left(1+\frac{t}{a}\right)^x = \sum_{n=0}^{\infty} c_n(x\,;a)\frac{t^n}{n!}$	Poisson-Charlier polynomials $c_n(x\,;a)$ [5]
VI.	$\left(1+(1+t)^{\lambda}\right)^{-\mu}$; $\ln(1+t)$	$\left(1+e^{\lambda t}\right)^{\mu}$; e^t-1	$\left(1+(1+t)^{\lambda}\right)^{-\mu}(1+t)^x$ $= \sum_{n=0}^{\infty} s_n(x;\lambda,\mu)\frac{t^n}{n!}$	Peters polynomials $s_n(x;\lambda,\mu)$ [3]
VII.	$\frac{t}{\ln(1+t)}; \ln(1+t)$	$\frac{t}{e^t-1}; e^t-1$	$\frac{t}{\ln(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}$	Bernoulli polynomials of the second kind $b_n(x)$ [5]
VIII.	$\frac{2}{2+t}; \ln(1+t)$	$\frac{1}{2}(1+e^t); e^t-1$	$\frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty} r_n(x)\frac{t^n}{n!}$	Related polynomials $r_n(x)$ [5]
IX.	$\frac{1}{\sqrt{1+t^2}}; \arctan(t)$	$sect; \tan t$	$\frac{1}{\sqrt{1+t^2}}\exp(x\arctan(t)) = \sum_{n=0}^{\infty} R_n(x)\frac{t^n}{n!}$	Hahn polynomials $R_n(x)$ [2]
X.	$(1-4t)^{-\frac{1}{2}}$ $\times\left(\frac{2}{1+\sqrt{1-4t}}\right)^{a-1}$; $\frac{-4t}{(1+\sqrt{1-4t}\,)^2}$	$\frac{1+t}{(1-t)^a}$; $\frac{1}{4}-\frac{1}{4}\left(\frac{1+t}{1-t}\right)^2$	$(1-4t)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-4t}}\right)^{a-1}$ $\times\exp\left(\frac{-4xt}{(1+\sqrt{1-4t}\,)^2}\right) = \sum_{n=0}^{\infty} R_n(a,x)t^n$	Shively's pseudo-Laguerre polynomials $R_n(a,x)$ [7]

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