# Some new $k$-fractional trapezium-like integral inequalities via generalized relative semi-$(r; m, h_1, h_2)$-preinvex mappings and applications

Artion Kashuri[1], Muhammad Adil Khan[2] and Rozana Liko[3]

[1,3]Department of Mathematics, Faculty of Technical Science, University Ismail Qemali, Vlora, Albania
[2]Department of Mathematics, University of Peshawer, Pakistan
E-mail: artionkashuri@gmail.com[1], adilswati@gmail.com[2], rozanaliko86@gmail.com[3]

**Abstract**

In this article, we first presented a new general identity concerning differentiable mappings defined on $m$-invex set via $k$-fractional integrals. By using the concept of generalized relative semi-$(r; m, h_1, h_2)$-preinvexity and the obtained identity as an auxiliary result, some new estimates with respect to trapezium-like integral inequalities via $k$-fractional integrals are established. It is pointed out that some new special cases can be deduced from main results of the article. Applications to special means for trapezium-like integral inequalities via $k$-fractional integrals are provided as well.

*2010 Mathematics Subject Classification.* **26A51**. 26A33, 26D07, 26D10, 26D15
*Keywords.* Hermite-Hadamard inequality, Hölder's inequality, Minkowski inequality, power mean inequality, $k$-fractional integrals, $m$-invex.

## 1 Introduction

The following notations are used throughout this paper. We use $I$ to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and $I^\circ$ to denote the interior of $I$. The set of integrable functions on the interval $[a, b]$ is denoted by $L[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

**Theorem 1.1.** Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on $I$ and $a, b \in I$ with $a < b$. Then the following inequalities hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This inequality (1.1) is also known as trapezium inequality.

The trapezium type inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. For other recent results which generalize, improve and extend the inequality (1.1) through various classes of convex functions interested readers are referred to [[1]-[26],[28],[29],[33]-[38],[41]-[46],[49],[51],[52]].

Let us recall some special functions and evoke some basic definitions as follows.

**Tbilisi Mathematical Journal** 12(3) (2019), pp. 1–19.
Tbilisi Centre for Mathematical Sciences.
*Received by the editors:* 11 February 2018.
*Accepted for publication:* 15 June 2019.

**Definition 1.2.** The Euler beta function is defined for $x, y > 0$ as

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.2}$$

**Definition 1.3.** The incomplete beta function is defined for $x, y > 0$ and $0 < a \leq 1$ as

$$\beta(a;x,y) = \int_0^a t^{x-1}(1-t)^{y-1}dt. \tag{1.3}$$

For $a = 1$, we have $\beta(a;x,y) = \beta(x,y)$.

**Definition 1.4.** Let $f \in L[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad b > x. \tag{1.4}$$

Here $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

In 2013, Sarikaya et al., see [42], established the following theorem by utilizing Riemann-Liouville fractional integrals.

**Theorem 1.5.** Let $f : [a,b] \longrightarrow \mathbb{R}$ be a positive function along with $0 \leq a < b$ and let $f \in L_1[a,b]$. Suppose that $f$ is a convex function on $[a,b]$. Then for $\alpha > 0$, the following inequalities for fractional integrals hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right] \leq \frac{f(a)+f(b)}{2}. \tag{1.5}$$

For $\alpha = 1$, the double inequality (1.5) is reduced in (1.1).

**Definition 1.6.** For $k \in \mathbb{R}^+$ and $x \in \mathbb{C}$, the $k$-gamma function is defined by

$$\Gamma_k(x) = \lim_{n \longrightarrow \infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}. \tag{1.6}$$

Its integral representation is given by

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-\frac{t^k}{k}}dt. \tag{1.7}$$

One can note that

$$\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha). \tag{1.8}$$

For $k = 1$, (1.7) gives integral representation of gamma function.

**Definition 1.7.** [31] Let $f \in L[a,b]$. Then $k$-fractional integrals of order $\alpha, k > 0$ with $a \geq 0$ are defined as

$$I_{a+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a$$

and

$$I_{b-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad b > x. \tag{1.9}$$

For $k = 1$, $k$-fractional integrals give Riemann-Liouville integrals.

**Definition 1.8.** [50]

A set $M_\varphi \subseteq \mathbb{R}^n$ is named as a relative convex ($\varphi$-convex) set, if and only if, there exists a function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that,

$$t\varphi(x) + (1-t)\varphi(y) \in M_\varphi, \ \forall \ x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi, t \in [0,1]. \tag{1.10}$$

**Definition 1.9.** [50] A function $f$ is named as a relative convex ($\varphi$-convex) function on a relative convex ($\varphi$-convex) set $M_\varphi$, if and only if, there exists a function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that,

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + (1-t)f(\varphi(y)), \tag{1.11}$$

$\forall \ x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi, t \in [0,1]$.

**Definition 1.10.** [30] Let $h : [0,1] \longrightarrow \mathbb{R}$ be a non-negative function and $h \neq 0$. The function $f$ on the invex set $K$ is said to be $h$-preinvex with respect to $\eta$, if

$$f\big(x + t\eta(y,x)\big) \leq h(1-t)f(x) + h(t)f(y) \tag{1.12}$$

for each $x, y \in K$ and $t \in [0,1]$, where $f(\cdot) > 0$.

Clearly, when putting $h(t) = t$ in Definition 1.10, $f$ becomes a preinvex function, see [40]. If the mapping $\eta(y,x) = y - x$ in Definition 1.10, then the non-negative function $f$ reduces to $h$-convex mappings, see [48].

**Definition 1.11.** [47] Let $f : K \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a non-negative function. $f : K \longrightarrow \mathbb{R}$ is said to be a $tgs$-convex function on $K$, if

$$f\big((1-t)x + ty\big) \leq t(1-t)[f(x) + f(y)] \tag{1.13}$$

holds for all $x, y \in K$ and $t \in (0,1)$.

**Definition 1.12.** [27] A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to $MT$-convex, if it is non-negative and $\forall\, x, y \in I$ and $t \in (0,1)$ satisfies the following inequality

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.14}$$

**Definition 1.13.** [38] A function: $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be $m$-$MT$-convex, if $f$ is positive and for $\forall\, x, y \in I$, and $t \in (0,1)$, among $m \in [0,1]$, satisfies the following inequality

$$f\big(tx + m(1-t)y\big) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.15}$$

**Definition 1.14.** [39] Let $K \subseteq \mathbb{R}$ be an open $m$-invex set respecting $\eta : K \times K \longrightarrow \mathbb{R}$ and $h_1, h_2 : [0,1] \longrightarrow [0, +\infty)$ are continuous functions. A function $f : K \longrightarrow \mathbb{R}$ is said to be generalized $(m, h_1, h_2)$-preinvex with respect to $\eta$, if

$$f\big(mx + t\eta(y, mx)\big) \leq m h_1(t) f(x) + h_2(t) f(y) \tag{1.16}$$

is valid for all $x, y \in K$ and $t \in [0,1]$, for some fixed $m \in (0,1]$.

Motivated by the above literatures, the main objective of this article is to establish some new estimates on trapezium-like integral type inequalities via $k$-fractional integrals associated with generalized relative semi-$(r; m, h_1, h_2)$-preinvex mappings. It is pointed out that some new special cases will be deduced from main results of the article. Applications to special means for trapezium-like integral inequalities via $k$-fractional integrals will be deduced as well.

## 2 Main results

The following definitions will be used in this section.

**Definition 2.1.** [9] A set $K \subseteq \mathbb{R}^n$ is named as $m$-invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y, mx) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.

**Remark 2.2.** In Definition 2.1, under certain conditions, the mapping $\eta(y, mx)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the $m$-invex set degenerates an invex set on $K$.

We next introduce a new class called generalized relative semi-$(r; m, h_1, h_2)$-preinvex mappings.

**Definition 2.3.** Let $K \subseteq \mathbb{R}$ be an open $m$-invex set with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}$. Suppose $h_1, h_2 : [0,1] \longrightarrow [0, +\infty)$ and $\varphi : I \longrightarrow K$ are continuous. A mapping $f : K \longrightarrow (0, +\infty)$ is said to be generalized relative semi-$(r; m, h_1, h_2)$-preinvex, if

$$f\big(m\varphi(x) + t\eta(\varphi(y), m\varphi(x))\big) \leq M_r(h_1(t), h_2(t); mf(x), f(y)) \tag{2.1}$$

holds for all $x, y \in I$ and $t \in [0,1]$ and for some fixed $m \in (0,1]$, where

$$M_r(h_1(t), h_2(t); mf(x), f(y)) := \begin{cases} \Big[m h_1(t) f^r(x) + h_2(t) f^r(y)\Big]^{\frac{1}{r}}, & \text{if } r \neq 0; \\ \big[mf(x)\big]^{h_1(t)} \big[f(y)\big]^{h_2(t)}, & \text{if } r = 0, \end{cases}$$

is the weighted power mean of order $r$ for positive numbers $f(x)$ and $f(y)$.

**Remark 2.4.** In Definition 2.3, if we choose $m = r = 1$, these definition reduces to the definition considered by Noor in [32] and Preda et. al. in [13].

**Remark 2.5.** In Definition 2.3, if we choose $r = 1$ and $\varphi(x) = x$, then we get Definition 1.14.

**Remark 2.6.** For $r = 1$, let us discuss some special cases in Definition 2.3 as follows.

(I) Taking $h_1(t) = (1-t)^s$, $h_2(t) = t^s$ for $s \in (0,1]$, then we get generalized relative semi-$(m,s)$-Breckner-preinvex mappings.
(II) Taking $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ for $s \in (0,1]$, then we get generalized relative semi-$(m,s)$-Godunova-Levin-Dragomir-preinvex mappings.
(III) Taking $h_1(t) = h(1-t)$, $h_2(t) = h(t)$, then we get generalized relative semi-$(m,h)$-preinvex mappings.
(IV) Taking $h_1(t) = h_2(t) = t(1-t)$, then we get generalized relative semi-$(m,tgs)$-preinvex mappings.
(V) Taking $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized relative semi-$m$-$MT$-preinvex mappings.

It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature.

For establishing our main results regarding some new trapezium-like type integral inequalities associated with generalized relative semi-$(r;m,h_1,h_2)$-preinvexity via $k$-fractional integrals, we need the following lemma.

**Lemma 2.7.** Suppose $K = \left[m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right) + \eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)\right] \subseteq \mathbb{R}$ be an open $m$-invex subset with respect to $\eta : K \times K \longrightarrow \mathbb{R}$ with $0 \leq a_1 < ma_2, \lambda \in [0,1] \setminus \left\{\frac{1}{2}\right\}$, for some fixed $m \in (0,1]$, where $\eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right) > 0$. Also, let $\varphi : I \longrightarrow K$ be a continuous function. Assume that $f : K \longrightarrow \mathbb{R}$ be a differentiable mapping on $K^\circ$ such that $f' \in L(K)$. Then for $\alpha, k > 0$, the following equality holds for $k$-fractional integrals:

$$
\begin{aligned}
&-\frac{f\left(m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)}{\eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)} \\
&-\frac{f\left(m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right) + \eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)\right)}{\eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)} \\
&+\frac{\Gamma_k(\alpha+k)}{\eta^{\frac{\alpha}{k}+1}\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)} \\
&\times\Big[I^{\alpha,k}_{\left(m\varphi\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)^+} f\left(m\varphi\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right) + \eta\left(\varphi(\lambda a_1+m(1-\lambda)a_2), m\varphi\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)\right) \\
&+I^{\alpha,k}_{\left(m\varphi\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)+\eta\left(\varphi(\lambda a_1+m(1-\lambda)a_2), m\varphi\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)\right)^-} f\left(m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)\Big] \\
&= \int_0^1 \left((1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\right) \qquad (2.2) \\
&\times f'\left(m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right) + t\eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)\right) dt.
\end{aligned}
$$

We denote

$$I^{\alpha,k}_{f,\eta,\varphi}(\lambda,m,a_1,a_2) := \int_0^1 \left((1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\right) \times f'\left(m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right) + t\eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)\right) dt. \tag{2.3}$$

*Proof.* After split into two integrals the integral (2.3), integrating by parts and changing the variable $u = m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right) + t\eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right), \forall\, t \in [0,1]$, we get

$$\begin{aligned}
I^{\alpha,k}_{f,\eta,\varphi}(\lambda,m,a_1,a_2) &= (1-t)^{\frac{\alpha}{k}} \\
&\times \frac{f\left(m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right) + t\eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)\right)}{\eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)}\Bigg|_0^1 \\
&+ \frac{\alpha}{k\eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)} \int_0^1 (1-t)^{\frac{\alpha}{k}-1} \\
&\times f\left(m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right) + t\eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)\right) dt \\
&- \Bigg[ t^{\frac{\alpha}{k}} \times \frac{f\left(m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right) + t\eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)\right)}{\eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)}\Bigg|_0^1 \\
&- \frac{\alpha}{k\eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)} \int_0^1 t^{\frac{\alpha}{k}-1} \\
&\times f\left(m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right) + t\eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)\right) dt \Bigg] \\
&= -\frac{f\left(m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)}{\eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)} \\
&- \frac{f\left(m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right) + \eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)\right)}{\eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)} \\
&+ \frac{\Gamma_k(\alpha+k)}{\eta^{\frac{\alpha}{k}+1}\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)} \\
&\times \Bigg[ I^{\alpha,k}_{\left(m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)^+} f\left(m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right) + \eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)\right) \\
&+ I^{\alpha,k}_{\left(m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right) + \eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)\right)^-} f\left(m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right) \Bigg].
\end{aligned}$$

This completes the proof of our lemma. Q.E.D.

**Remark 2.8.** In Lemma 2.7, for $m = k = 1$ and $\eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)$ $= \varphi(\lambda a_1 + m(1-\lambda)a_2) - m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)$, where $\varphi$ is identity function, we obtain Lemma 2.1 in [41].

**Corollary 2.9.** In Lemma 2.7, for $\lambda = 0$ and $\eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)$ $= \varphi(\lambda a_1 + m(1-\lambda)a_2) - m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)$, where $\varphi$ is identity function and $m = 1$, we get

$$\frac{f(a_1)+f(a_2)}{a_1-a_2} + \frac{\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\alpha}{k}+1}}\left[I^{\alpha,k}_{a_1^+}f(a_2) + I^{\alpha,k}_{a_2^-}f(a_1)\right]$$
$$= \int_0^1 \left((1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\right) f'(ta_2 + (1-t)a_1)dt. \tag{2.4}$$

**Corollary 2.10.** In Lemma 2.7, for $\lambda = 1$ and $\eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)$ $= m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right) - \varphi(\lambda a_1 + m(1-\lambda)a_2)$, where $\varphi$ is identity function and $m = 1$, we get

$$\frac{f(a_2)+f(2a_2-a_1)}{a_1-a_2} + \frac{\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\alpha}{k}+1}}\left[I^{\alpha,k}_{a_2^+}f(2a_2-a_1) + I^{\alpha,k}_{(2a_2-a_1)^-}f(a_2)\right]$$
$$= \int_0^1 \left((1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\right) f'((1+t)a_2 - ta_1)dt. \tag{2.5}$$

Using Lemma 2.7, we now state the following theorems for the corresponding version for power of first derivative.

**Theorem 2.11.** Let $\alpha, k > 0$ and $0 < r \leq 1$. Suppose that $K = \left[m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right),\right.$ $\left. m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right) + \eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)\right] \subseteq \mathbb{R}$ be an open $m$-invex subset with respect to $\eta : K \times K \longrightarrow \mathbb{R}$ with $0 \leq a_1 < ma_2$, $\lambda \in [0,1]\backslash\left\{\frac{1}{2}\right\}$, for some fixed $m \in (0,1]$, where $\eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right) > 0$. Also, let $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$ and $\varphi : I \longrightarrow K$ are continuous. Assume that $f : K \longrightarrow (0,+\infty)$ be a differentiable mapping on $K^\circ$ such that $f' \in L(K)$. If $f'^q$ is generalized relative semi-$(r; m, h_1, h_2)$-preinvex mapping, $q > 1$, $p^{-1} + q^{-1} = 1$, then the following inequality for $k$-fractional integrals holds:

$$\left|I^{\alpha,k}_{f,\eta,\varphi}(\lambda, m, a_1, a_2)\right| \leq \left[\frac{2k}{\alpha p + k}\left(1 - \frac{1}{2^{\frac{\alpha p}{k}}}\right)\right]^{\frac{1}{p}} \tag{2.6}$$
$$\times\left[m\left(f'\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)^{rq} I^r(h_1(t); r) + (f'(\lambda a_1 + m(1-\lambda)a_2))^{rq} I^r(h_2(t); r)\right]^{\frac{1}{rq}},$$

where

$$I(h_i(t); r) := \int_0^1 h_i^{\frac{1}{r}}(t)dt, \quad \forall\, i = 1, 2.$$

*Proof.* From Lemma 2.7, generalized relative semi-$(r; m, h_1, h_2)$-preinvexity of $f'^q$, Hölder inequality, Minkowski inequality, the fact that $(c_2 - c_1)^\theta \leq c_2^\theta - c_1^\theta$ for any $0 \leq c_1 \leq c_2$, where $\theta \geq 1$ and properties of the modulus, we have

$$\left|I^{\alpha,k}_{f,\eta,\varphi}(\lambda, m, a_1, a_2)\right| \leq \int_0^1 \left|(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\right|$$
$$\times\left|f'\left(m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right) + t\eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right)\right)\right|dt$$

$$\leq \left[\int_0^1 \left|(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\right|^p dt\right]^{\frac{1}{p}}$$

$$\times\left[\int_0^1 \left|f'\left(m\varphi\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)+t\eta\left(\varphi(\lambda a_1+m(1-\lambda)a_2), m\varphi\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)\right)\right|^q dt\right]^{\frac{1}{q}}$$

$$\leq \left[\int_0^{\frac{1}{2}} \left((1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\right)^p dt + \int_{\frac{1}{2}}^1 \left(t^{\frac{\alpha}{k}} - (1-t)^{\frac{\alpha}{k}}\right)^p dt\right]^{\frac{1}{p}}$$

$$\times\left[\int_0^1 \left[mh_1(t)\left(f'\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)^{rq} + h_2(t)(f'(\lambda a_1+m(1-\lambda)a_2))^{rq}\right]^{\frac{1}{r}} dt\right]^{\frac{1}{q}}$$

$$\leq \left[\int_0^{\frac{1}{2}} \left((1-t)^{\frac{\alpha p}{k}} - t^{\frac{\alpha p}{k}}\right) dt + \int_{\frac{1}{2}}^1 \left(t^{\frac{\alpha p}{k}} - (1-t)^{\frac{\alpha p}{k}}\right) dt\right]^{\frac{1}{p}}$$

$$\times\left[\left(\int_0^1 m^{\frac{1}{r}} h_1^{\frac{1}{r}}(t)\left(f'\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)^q dt\right)^r + \left(\int_0^1 h_2^{\frac{1}{r}}(t)(f'(\lambda a_1+m(1-\lambda)a_2))^q dt\right)^r\right]^{\frac{1}{rq}}$$

$$= \left[\frac{2k}{\alpha p+k}\left(1-\frac{1}{2^{\frac{\alpha p}{k}}}\right)\right]^{\frac{1}{p}}$$

$$\times\left[m\left(f'\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)^{rq} I^r(h_1(t);r) + (f'(\lambda a_1+m(1-\lambda)a_2))^{rq} I^r(h_2(t);r)\right]^{\frac{1}{rq}}.$$

So, the proof of this theorem is complete. Q.E.D.

We point out some special cases of Theorem 2.11.

**Corollary 2.12.** In Theorem 2.11, for $p=q=2$, we get the following inequality for generalized relative semi-$(r;m,h_1,h_2)$-preinvex mappings:

$$\left|I^{\alpha,k}_{f,\eta,\varphi}(\lambda,m,a_1,a_2)\right| \leq \sqrt{\frac{2k}{2\alpha+k}\left(1-\frac{1}{4^{\frac{\alpha}{k}}}\right)} \tag{2.7}$$

$$\times\left[m\left(f'\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)^{2r} I^r(h_1(t);r) + (f'(\lambda a_1+m(1-\lambda)a_2))^{2r} I^r(h_2(t);r)\right]^{\frac{1}{2r}}.$$

**Corollary 2.13.** In Theorem 2.11, for $\lambda=0$ and $\eta\left(\varphi(\lambda a_1+m(1-\lambda)a_2), m\varphi\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)$ $=\varphi(\lambda a_1+m(1-\lambda)a_2)-m\varphi\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)$, where $\varphi$ is identity function and $m=1$, we get the following inequality for generalized relative semi-$(r;1,h_1,h_2)$-preinvex mappings:

$$\left|\frac{f(a_1)+f(a_2)}{a_1-a_2}+\frac{\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\alpha}{k}+1}}\left[I^{\alpha,k}_{a_1^+}f(a_2)+I^{\alpha,k}_{a_2^-}f(a_1)\right]\right|\leq\left[\frac{2k}{\alpha p+k}\left(1-\frac{1}{2^{\frac{\alpha p}{k}}}\right)\right]^{\frac{1}{p}} \tag{2.8}$$

$$\times\left[(f'(a_1))^{rq}I^r(h_1(t);r)+(f'(a_2))^{rq}I^r(h_2(t);r)\right]^{\frac{1}{rq}}.$$

**Corollary 2.14.** In Corollary 2.13, for $h_1(t)=h(1-t)$, $h_2(t)=h(t)$ and $f'\leq K$, we get the following inequality for generalized relative semi-$(r;1,h)$-preinvex mappings:

$$\left|\frac{f(a_1)+f(a_2)}{a_1-a_2}+\frac{\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\alpha}{k}+1}}\left[I^{\alpha,k}_{a_1^+}f(a_2)+I^{\alpha,k}_{a_2^-}f(a_1)\right]\right|$$

$$\leq 2^{\frac{1}{rq}}K\left[\frac{2k}{\alpha p+k}\left(1-\frac{1}{2^{\frac{\alpha p}{k}}}\right)\right]^{\frac{1}{p}}I^{\frac{1}{q}}(h(t);r). \tag{2.9}$$

**Corollary 2.15.** In Corollary 2.14, for $h_1(t)=(1-t)^s$, $h_2(t)=t^s$, we get the following inequality for generalized relative semi-$(r;1,s)$-Breckner-preinvex mappings:

$$\left|\frac{f(a_1)+f(a_2)}{a_1-a_2}+\frac{\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\alpha}{k}+1}}\left[I^{\alpha,k}_{a_1^+}f(a_2)+I^{\alpha,k}_{a_2^-}f(a_1)\right]\right|$$

$$\leq 2^{\frac{1}{rq}}K\left[\frac{2k}{\alpha p+k}\left(1-\frac{1}{2^{\frac{\alpha p}{k}}}\right)\right]^{\frac{1}{p}}\left(\frac{r}{r+s}\right)^{\frac{1}{q}}. \tag{2.10}$$

**Corollary 2.16.** In Corollary 2.14, for $h_1(t)=(1-t)^{-s}$, $h_2(t)=t^{-s}$ and $0<s<r$, we get the following inequality for generalized relative semi-$(r;1,s)$-Godunova-Levin-Dragomir-preinvex mappings:

$$\left|\frac{f(a_1)+f(a_2)}{a_1-a_2}+\frac{\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\alpha}{k}+1}}\left[I^{\alpha,k}_{a_1^+}f(a_2)+I^{\alpha,k}_{a_2^-}f(a_1)\right]\right|$$

$$\leq 2^{\frac{1}{rq}}K\left[\frac{2k}{\alpha p+k}\left(1-\frac{1}{2^{\frac{\alpha p}{k}}}\right)\right]^{\frac{1}{p}}\left(\frac{r}{r-s}\right)^{\frac{1}{q}}. \tag{2.11}$$

**Corollary 2.17.** In Theorem 2.11, for $\lambda=0, m=1, h_1(t)=h_2(t)=t(1-t), f'\leq K$ and $\eta\left(\varphi(\lambda a_1+m(1-\lambda)a_2), m\varphi\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)=\varphi(\lambda a_1+m(1-\lambda)a_2)-m\varphi\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)$, where $\varphi$ is identity function, we get the following inequality for generalized relative semi-$(1,tgs)$-preinvex mappings:

$$\left|\frac{f(a_1)+f(a_2)}{a_1-a_2}+\frac{\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\alpha}{k}+1}}\left[I^{\alpha,k}_{a_1^+}f(a_2)+I^{\alpha,k}_{a_2^-}f(a_1)\right]\right|$$

$$\leq 2^{\frac{1}{rq}} K\left[\frac{2k}{\alpha p+k}\left(1-\frac{1}{2^{\frac{\alpha p}{k}}}\right)\right]^{\frac{1}{p}} \beta^{\frac{1}{q}}\left(1+\frac{1}{r}, 1+\frac{1}{r}\right). \tag{2.12}$$

**Corollary 2.18.** In Corollary 2.14, for $h_1(t)=\dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t)=\dfrac{\sqrt{t}}{2\sqrt{1-t}}$ and $r\in\left(\dfrac{1}{2},1\right]$, we get the following inequality for generalized relative semi-$(r;1)$-$MT$-preinvex mappings:

$$\left|\frac{f(a_1)+f(a_2)}{a_1-a_2}+\frac{\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\alpha}{k}+1}}\left[I^{\alpha,k}_{a_1^+}f(a_2)+I^{\alpha,k}_{a_2^-}f(a_1)\right]\right|$$

$$\leq 2^{\frac{1}{rq}} K\left[\frac{2k}{\alpha p+k}\left(1-\frac{1}{2^{\frac{\alpha p}{k}}}\right)\right]^{\frac{1}{p}} \beta^{\frac{1}{q}}\left(1-\frac{1}{2r}, 1+\frac{1}{2r}\right). \tag{2.13}$$

**Theorem 2.19.** Let $\alpha, k>0$ and $0<r\leq 1$. Suppose that $K=\Big[m\varphi\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right),$ $m\varphi\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)+\eta\left(\varphi(\lambda a_1+m(1-\lambda)a_2), m\varphi\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)\Big]\subseteq\mathbb{R}$ be an open $m$-invex subset with respect to $\eta: K\times K\longrightarrow\mathbb{R}$ with $0\leq a_1<ma_2$, $\lambda\in[0,1]\backslash\left\{\frac{1}{2}\right\}$, for some fixed $m\in(0,1]$, where $\eta\left(\varphi(\lambda a_1+m(1-\lambda)a_2), m\varphi\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)>0$. Also, let $h_1, h_2:[0,1]\longrightarrow[0,+\infty)$ and $\varphi: I\longrightarrow K$ are continuous. Assume that $f: K\longrightarrow(0,+\infty)$ be a differentiable mapping on $K^\circ$ such that $f'\in L(K)$. If $f'^q$ is generalized relative semi-$(r;m,h_1,h_2)$-preinvex mapping, $q\geq 1$, then the following inequality for $k$-fractional integrals holds:

$$\left|I^{\alpha,k}_{f,\eta,\varphi}(\lambda,m,a_1,a_2)\right|\leq\left[\frac{2k}{\alpha+k}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\right]^{1-\frac{1}{q}} \tag{2.14}$$

$$\times\Bigg\{\left[m\left(f'\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)^{rq} I^r(h_1(t);\alpha,k,r)\right.$$

$$\left.+(f'(\lambda a_1+m(1-\lambda)a_2))^{rq} I^r(h_2(t);\alpha,k,r)\right]^{\frac{1}{r}}$$

$$+\Big[m\left(f'\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)^{rq}\overline{I}^r(h_1(t);\alpha,k,r)$$

$$\left.+(f'(\lambda a_1+m(1-\lambda)a_2))^{rq}\overline{I}^r(h_2(t);\alpha,k,r)\Big]^{\frac{1}{r}}\right\}^{\frac{1}{q}},$$

where

$$I(h_i(t);\alpha,k,r):=\int_0^{\frac{1}{2}}\left((1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right)h_i^{\frac{1}{r}}(t)dt,\quad\forall\, i=1,2$$

and

$$\overline{I}(h_i(t);\alpha,k,r):=\int_{\frac{1}{2}}^1\left(t^{\frac{\alpha}{k}}-(1-t)^{\frac{\alpha}{k}}\right)h_i^{\frac{1}{r}}(t)dt,\quad\forall\, i=1,2.$$

*Proof.* From Lemma 2.7, generalized relative semi-$(r;m,h_1,h_2)$-preinvexity of $f'^q$, the well-known power mean inequality, Minkowski inequality and properties of the modulus, we have

$$\left|I^{\alpha,k}_{f,\eta,\varphi}(\lambda,m,a_1,a_2)\right| \leq \int_0^1 \left|(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\right|$$

$$\times\left|f'\left(m\varphi\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)+t\eta\left(\varphi(\lambda a_1+m(1-\lambda)a_2),m\varphi\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)\right)\right|dt$$

$$\leq \left[\int_0^1 \left|(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\right|dt\right]^{1-\frac{1}{q}} \times \left[\int_0^1 \left|(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\right|\right.$$

$$\left.\times\left|f'\left(m\varphi\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)+t\eta\left(\varphi(\lambda a_1+m(1-\lambda)a_2),m\varphi\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)\right)\right|^q dt\right]^{\frac{1}{q}}$$

$$\leq \left[\int_0^{\frac{1}{2}} \left((1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\right)dt + \int_{\frac{1}{2}}^1 \left(t^{\frac{\alpha}{k}} - (1-t)^{\frac{\alpha}{k}}\right)dt\right]^{1-\frac{1}{q}}$$

$$\times\left[\int_0^1 \left|(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\right|\left[mh_1(t)\left(f'\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)^{rq}+h_2(t)(f'(\lambda a_1+m(1-\lambda)a_2))^{rq}\right]^{\frac{1}{r}}dt\right]^{\frac{1}{q}}$$

$$\leq \left[\int_0^{\frac{1}{2}} \left((1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\right)dt + \int_{\frac{1}{2}}^1 \left(t^{\frac{\alpha}{k}} - (1-t)^{\frac{\alpha}{k}}\right)dt\right]^{1-\frac{1}{q}}$$

$$\times\left\{\left[\left(\int_0^{\frac{1}{2}} m^{\frac{1}{r}}\left((1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\right)h_1^{\frac{1}{r}}(t)\left(f'\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)^q dt\right)^r\right.\right.$$

$$\left.+\left(\int_0^{\frac{1}{2}} \left((1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\right)h_2^{\frac{1}{r}}(t)(f'(\lambda a+m(1-\lambda)b))^q dt\right)^r\right]^{\frac{1}{r}}$$

$$+\left[\left(\int_{\frac{1}{2}}^1 m^{\frac{1}{r}}\left(t^{\frac{\alpha}{k}} - (1-t)^{\frac{\alpha}{k}}\right)h_1^{\frac{1}{r}}(t)\left(f'\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)^q dt\right)^r\right.$$

$$\left.\left.+\left(\int_{\frac{1}{2}}^1 \left(t^{\frac{\alpha}{k}} - (1-t)^{\frac{\alpha}{k}}\right)h_2^{\frac{1}{r}}(t)(f'(\lambda a_1+m(1-\lambda)a_2))^q dt\right)^r\right]^{\frac{1}{r}}\right\}^{\frac{1}{q}}$$

$$= \left[\frac{2k}{\alpha+k}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\right]^{1-\frac{1}{q}}$$

$$\times\left\{\left[m\left(f'\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)^{rq} I^r(h_1(t);\alpha,k,r)\right.\right.$$

$$\left.\left.+(f'(\lambda a_1+m(1-\lambda)a_2))^{rq} I^r(h_2(t);\alpha,k,r)\right]^{\frac{1}{r}}\right.$$

$$+\Big[m\left(f'\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)^{rq}\overline{I}^{r}(h_1(t);\alpha,k,r)$$

$$+(f'(\lambda a_1+m(1-\lambda)a_2))^{rq}\overline{I}^{r}(h_2(t);\alpha,k,r)\Big]^{\frac{1}{r}}\Bigg\}^{\frac{1}{q}}.$$

So, the proof of this theorem is complete. Q.E.D.

We point out some special cases of Theorem 2.19.

**Corollary 2.20.** In Theorem 2.19, for $q=1$, we get the following inequality for generalized relative semi-$(r;m,h_1,h_2)$-preinvex mappings:

$$\left|I^{\alpha,k}_{f,\eta,\varphi}(\lambda,m,a_1,a_2)\right|\leq\Big[m\left(f'\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)^{r}I^{r}(h_1(t);\alpha,k,r) \tag{2.15}$$

$$+(f'(\lambda a_1+m(1-\lambda)a_2))^{r}I^{r}(h_2(t);\alpha,k,r)\Big]^{\frac{1}{r}}$$

$$+\Big[m\left(f'\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)^{r}\overline{I}^{r}(h_1(t);\alpha,k,r)$$

$$+(f'(\lambda a_1+m(1-\lambda)a_2))^{r}\overline{I}^{r}(h_2(t);\alpha,k,r)\Big]^{\frac{1}{r}}.$$

**Corollary 2.21.** In Theorem 2.19, for $\lambda=0$ and $\eta\left(\varphi(\lambda a_1+m(1-\lambda)a_2),m\varphi\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)\right)$ $=\varphi(\lambda a_1+m(1-\lambda)a_2)-m\varphi\left(\lambda a_2+(1-\lambda)\frac{a_1}{m}\right)$, where $\varphi$ is identity function and $m=1$, we get the following inequality for generalized relative semi-$(r;1,h_1,h_2)$-preinvex mappings:

$$\left|\frac{f(a_1)+f(a_2)}{a_1-a_2}+\frac{\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\alpha}{k}+1}}\left[I^{\alpha,k}_{a_1^{+}}f(a_2)+I^{\alpha,k}_{a_2^{-}}f(a_1)\right]\right|\leq\left[\frac{2k}{\alpha+k}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\right]^{1-\frac{1}{q}} \tag{2.16}$$

$$\times\Bigg\{\Big[(f'(a_1))^{rq}I^{r}(h_1(t);\alpha,k,r)+(f'(a_2))^{rq}I^{r}(h_2(t);\alpha,k,r)\Big]^{\frac{1}{r}}$$

$$+\Big[(f'(a_1))^{rq}\overline{I}^{r}(h_1(t);\alpha,k,r)+(f'(a_2))^{rq}\overline{I}^{r}(h_2(t);\alpha,k,r)\Big]^{\frac{1}{r}}\Bigg\}^{\frac{1}{q}}.$$

**Corollary 2.22.** In Corollary 2.21, for $h_1(t)=h(1-t)$, $h_2(t)=h(t)$ and $f'\leq K$, we get the following inequality for generalized relative semi-$(r;1,h)$-preinvex mappings:

$$\left|\frac{f(a_1)+f(a_2)}{a_1-a_2}+\frac{\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\alpha}{k}+1}}\left[I^{\alpha,k}_{a_1^{+}}f(a_2)+I^{\alpha,k}_{a_2^{-}}f(a_1)\right]\right|$$

$$\leq 2^{\frac{1}{q}}K\left[\frac{2k}{\alpha+k}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\right]^{1-\frac{1}{q}}\times\Big[I^{r}(h(t);\alpha,k,r)+I^{r}(h(1-t);\alpha,k,r)\Big]^{\frac{1}{rq}}. \tag{2.17}$$

**Corollary 2.23.** In Corollary 2.22, for $h_1(t) = (1-t)^s$, $h_2(t) = t^s$, we get the following inequality for generalized relative semi-$(r;1,s)$-Breckner-preinvex mappings:

$$\begin{aligned}\left|\frac{f(a_1)+f(a_2)}{a_1-a_2}+\frac{\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\alpha}{k}+1}}\left[I^{\alpha,k}_{a_1^+}f(a_2)+I^{\alpha,k}_{a_2^-}f(a_1)\right]\right| \\ \leq 2^{\frac{1}{q}}K\left[\frac{2k}{\alpha+k}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\right]^{1-\frac{1}{q}} \\ \times\left\{\left[\beta\left(\frac{1}{2};\frac{s}{r}+1,\frac{\alpha}{k}+1\right)-\frac{1}{\left(\frac{\alpha}{k}+\frac{s}{r}+1\right)2^{\frac{\alpha}{k}+\frac{s}{r}+1}}\right]^r\right. \\ \left.+\left[\left(\frac{1}{\frac{\alpha}{k}+\frac{s}{r}+1}\right)\left(1-\frac{1}{2^{\frac{\alpha}{k}+\frac{s}{r}+1}}\right)-\beta\left(\frac{1}{2};\frac{\alpha}{k}+1,\frac{s}{r}+1\right)\right]^r\right\}^{\frac{1}{rq}}.\end{aligned} \tag{2.18}$$

**Corollary 2.24.** In Corollary 2.22, for $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ and $0 < s < r$, we get the following inequality for generalized relative semi-$(r;1,s)$-Godunova-Levin-Dragomir-preinvex mappings:

$$\begin{aligned}\left|\frac{f(a_1)+f(a_2)}{a_1-a_2}+\frac{\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\alpha}{k}+1}}\left[I^{\alpha,k}_{a_1^+}f(a_2)+I^{\alpha,k}_{a_2^-}f(a_1)\right]\right| \\ \leq 2^{\frac{1}{q}}K\left[\frac{2k}{\alpha+k}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\right]^{1-\frac{1}{q}} \\ \times\left\{\left[\beta\left(\frac{1}{2};1-\frac{s}{r},\frac{\alpha}{k}+1\right)-\frac{1}{\left(\frac{\alpha}{k}-\frac{s}{r}+1\right)2^{\frac{\alpha}{k}-\frac{s}{r}+1}}\right]^r\right. \\ \left.+\left[\left(\frac{1}{\frac{\alpha}{k}-\frac{s}{r}+1}\right)\left(1-\frac{1}{2^{\frac{\alpha}{k}-\frac{s}{r}+1}}\right)-\beta\left(\frac{1}{2};\frac{\alpha}{k}+1,1-\frac{s}{r}\right)\right]^r\right\}^{\frac{1}{rq}}.\end{aligned} \tag{2.19}$$

**Corollary 2.25.** In Theorem 2.19, for $\lambda = 0, m = 1, h_1(t) = h_2(t) = t(1-t), f' \leq K$ and $\eta\left(\varphi(\lambda a_1 + m(1-\lambda)a_2), m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)\right) = \varphi(\lambda a_1 + m(1-\lambda)a_2) - m\varphi\left(\lambda a_2 + (1-\lambda)\frac{a_1}{m}\right)$, where $\varphi$ is identity function, we get the following inequality for generalized relative semi-$(1,tgs)$-preinvex mappings:

$$\begin{aligned}\left|\frac{f(a_1)+f(a_2)}{a_1-a_2}+\frac{\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\alpha}{k}+1}}\left[I^{\alpha,k}_{a_1^+}f(a_2)+I^{\alpha,k}_{a_2^-}f(a_1)\right]\right| \\ \leq 2^{\left(\frac{1}{r}+1\right)\frac{1}{q}}K\left[\frac{2k}{\alpha+k}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\right]^{1-\frac{1}{q}} \\ \times\left[\beta\left(\frac{1}{2};1+\frac{1}{r},\frac{\alpha}{k}+\frac{1}{r}+1\right)-\beta\left(\frac{1}{2};\frac{\alpha}{k}+\frac{1}{r}+1,1+\frac{1}{r}\right)\right]^{\frac{1}{q}}.\end{aligned} \tag{2.20}$$

**Corollary 2.26.** In Corollary 2.22, for $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$ and $r \in \left(\frac{1}{2}, 1\right]$, we get the following inequality for generalized relative semi-$(r;1)$-$MT$-preinvex mappings:

$$\left| \frac{f(a_1)+f(a_2)}{a_1-a_2} + \frac{\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\alpha}{k}+1}} \left[ I^{\alpha,k}_{a_1^+} f(a_2) + I^{\alpha,k}_{a_2^-} f(a_1) \right] \right|$$

$$\leq 2^{\left(\frac{1}{r}+1\right)\frac{1}{q}} K \left[ \frac{2k}{\alpha+k} \left( 1 - \frac{1}{2^{\frac{\alpha}{k}}} \right) \right]^{1-\frac{1}{q}} \tag{2.21}$$

$$\times \left\{ \left[ \beta\left(\frac{1}{2}; 1-\frac{1}{2r}, \frac{\alpha}{k}+\frac{1}{2r}+1\right) - \beta\left(\frac{1}{2}; \frac{\alpha}{k}-\frac{1}{2r}+1, 1+\frac{1}{2r}\right) \right]^r \right.$$

$$\left. + \left[ \beta\left(\frac{1}{2}; 1+\frac{1}{2r}, \frac{\alpha}{k}-\frac{1}{2r}+1\right) - \beta\left(\frac{1}{2}; \frac{\alpha}{k}+\frac{1}{2r}+1, 1-\frac{1}{2r}\right) \right]^r \right\}^{\frac{1}{rq}}.$$

**Remark 2.27.** For $k = 1$, by our Theorems 2.11 and 2.19 and their corresponding Corollaries, respectively 2.12-2.18 and 2.20-2.26, we can get some new special trapezium-like integral inequalities associated with generalized relative semi-$(r; m, h_1, h_2)$-preinvex mappings via fractional integrals of order $\alpha > 0$. Also, for $\alpha = k = 1$, by our Theorems 2.11 and 2.19 and their corresponding Corollaries, respectively 2.12-2.18 and 2.20-2.26, we can obtain some new special trapezium-like integral inequalities associated with generalized relative semi-$(r; m, h_1, h_2)$-preinvex mappings via classical integrals. Finally, for $\lambda = 1$, using equality (2.5) of Corollary 2.10 and Theorems 2.11 and 2.19 and their corresponding Corollaries, respectively 2.12-2.18 and 2.20-2.26, we can get also some new special trapezium-like integral inequalities associated with generalized relative semi-$(r; m, h_1, h_2)$-preinvex mappings via $k$-fractional integrals.

## 3 Applications to special means

**Definition 3.1.** [2] A function $M : \mathbb{R}^2_+ \longrightarrow \mathbb{R}_+$, is called a Mean function if it has the following properties:

1. Homogeneity: $M(ax, ay) = aM(x, y)$, for all $a > 0$,
2. Symmetry: $M(x, y) = M(y, x)$,
3. Reflexivity: $M(x, x) = x$,
4. Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x, y) \leq M(x', y')$,
5. Internality: $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$.

We consider some means for arbitrary positive real numbers $\alpha, \beta$ $(\alpha \neq \beta)$.

1. The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha+\beta}{2}.$$

2. The geometric mean:
$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}.$$

3. The harmonic mean:
$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}.$$

4. The power mean:
$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2}\right)^{\frac{1}{r}}, \quad r \geq 1.$$

5. The identric mean:
$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$

6. The logarithmic mean:
$$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\ln(\beta) - \ln(\alpha)}.$$

7. The generalized log-mean:
$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\right]^{\frac{1}{p}}; \quad p \in \mathbb{R} \setminus \{-1, 0\}.$$

8. The weighted $p$-power mean:
$$M_p\begin{pmatrix} \alpha_1, & \alpha_2, & \cdots & , \alpha_n \\ u_1, & u_2, & \cdots & , u_n \end{pmatrix} = \left(\sum_{i=1}^{n} \alpha_i u_i^p\right)^{\frac{1}{p}},$$

where $0 \leq \alpha_i \leq 1$, $u_i > 0$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} \alpha_i = 1$.

It is well known that $L_p$ is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$. Now, let $a_1$ and $a_2$ be positive real numbers such that $a_1 < a_2$. Let $h_1, h_2 : [0,1] \longrightarrow [0, +\infty)$, $\varphi : I \longrightarrow K$ are continuous, $\eta : K \times K \longrightarrow \mathbb{R}$ with $\lambda \in [0,1] \setminus \left\{\frac{1}{2}\right\}$ and $M := M(\varphi(\lambda a_1 + (1-\lambda)a_2), \varphi(\lambda a_2 + (1-\lambda)a_1)) : [\varphi(\lambda a_1 + (1-\lambda)a_2), \varphi(\lambda a_1 + (1-\lambda)a_2) + \eta(\varphi(\lambda a_1 + (1-\lambda)a_2), \varphi(\lambda a_2 + (1-\lambda)a_1))] \times [\varphi(\lambda a_1 + (1-\lambda)a_2), \varphi(\lambda a_1 + (1-\lambda)a_2) + \eta(\varphi(\lambda a_1 + (1-\lambda)a_2), \varphi(\lambda a_2 + (1-\lambda)a_1))] \longrightarrow \mathbb{R}_+$, which is one of the above mentioned means. Therefore one can obtain various inequalities using the results of Sect. 2 for these means as follows: Replace $\eta(\varphi(\lambda a_1 + (1-\lambda)a_2), \varphi(\lambda a_2 + (1-\lambda)a_1)) = M(\varphi(\lambda a_1 + (1-\lambda)a_2), \varphi(\lambda a_2 + (1-\lambda)a_1))$ for value $m = 1$ in (2.6) and (2.14), one can obtain the following interesting inequalities involving means:

$$\left|I^{\alpha,k}_{f,M(\cdot,\cdot),\varphi}(\lambda, 1, a_1, a_2)\right| = \left| - \frac{f(\varphi(\lambda a_2 + (1-\lambda)a_1))}{M(\varphi(\lambda a_1 + (1-\lambda)a_2), \varphi(\lambda a_2 + (1-\lambda)a_1))}\right.$$

$$-\frac{f\left(\varphi\left(\lambda a_2+(1-\lambda)a_1\right)+M\left(\varphi(\lambda a_1+(1-\lambda)a_2),\varphi\left(\lambda a_2+(1-\lambda)a_1\right)\right)\right)}{M\left(\varphi(\lambda a_1+(1-\lambda)a_2),\varphi\left(\lambda a_2+(1-\lambda)a_1\right)\right)}$$

$$+\frac{\Gamma_k(\alpha+k)}{M^{\frac{\alpha}{k}+1}\left(\varphi(\lambda a_1+(1-\lambda)a_2),\varphi\left(\lambda a_2+(1-\lambda)a_1\right)\right)}$$

$$\times\Big[I^{\alpha,k}_{(\varphi(\lambda a_2+(1-\lambda)a_1))^+}f\left(\varphi\left(\lambda a_2+(1-\lambda)a_1\right)+M\left(\varphi(\lambda a_1+(1-\lambda)a_2),\varphi\left(\lambda a_2+(1-\lambda)a_1\right)\right)\right)$$

$$+I^{\alpha,k}_{(\varphi(\lambda a_2+(1-\lambda)a_1)+M(\varphi(\lambda a_1+(1-\lambda)a_2),\varphi(\lambda a_2+(1-\lambda)a_1)))^-}f\left(\varphi\left(\lambda a_2+(1-\lambda)a_1\right)\right)\Big]\Big|$$

$$\leq\left[\frac{2k}{\alpha p+k}\left(1-\frac{1}{2^{\frac{\alpha p}{k}}}\right)\right]^{\frac{1}{p}} \tag{3.1}$$

$$\times\left[\left(f'\left(\lambda a_2+(1-\lambda)a_1\right)\right)^{rq}I^r(h_1(t);r)+\left(f'(\lambda a_1+(1-\lambda)a_2)\right)^{rq}I^r(h_2(t);r)\right]^{\frac{1}{rq}},$$

$$\left|I^{\alpha,k}_{f,M(\cdot,\cdot),\varphi}(\lambda,1,a_1,a_2)\right|\leq\left[\frac{2k}{\alpha+k}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\right]^{1-\frac{1}{q}} \tag{3.2}$$

$$\times\Bigg\{\Big[\left(f'\left(\lambda a_2+(1-\lambda)a_1\right)\right)^{rq}I^r(h_1(t);\alpha,k,r)+\left(f'(\lambda a_1+(1-\lambda)a_2)\right)^{rq}I^r(h_2(t);\alpha,k,r)\Big]^{\frac{1}{r}}$$

$$+\Big[\left(f'\left(\lambda a_2+(1-\lambda)a_1\right)\right)^{rq}\overline{I}^r(h_1(t);\alpha,k,r)+\left(f'(\lambda a_1+(1-\lambda)a_2)\right)^{rq}\overline{I}^r(h_2(t);\alpha,k,r)\Big]^{\frac{1}{r}}\Bigg\}^{\frac{1}{q}}.$$

Letting $M(\varphi(\lambda a_1+(1-\lambda)a_2),\varphi\left(\lambda a_2+(1-\lambda)a_1\right)):=A,G,H,P_r,I,L,L_p,M_p$ in (3.1) and (3.2), we get inequalities involving means for a particular choices of differentiable generalized relative semi-$(r;1,h_1,h_2)$-preinvex mapping $f$ at certain powers. The details are left to the interested reader.

## References


[1] M.U. Awan, M.A. Noor, V.N. Mishra and K.I. Noor, *Some characterization of general preinvex functions*, Inter. J. Anal. Appl., **15**(1) (2017), 46–56.

[2] P.S. Bullen, *Handbook of Means and Their Inequalities*, Kluwer Academic Publishers, Dordrecht, (2003).

[3] F. Chen, *A note on Hermite-Hadamard inequalities for products of convex functions via Riemann-Liouville fractional integrals*, Ital. J. Pure Appl. Math., **33** (2014), 299–306.

[4] Y.M. Chu, M.A. Khan, T. Ali and S.S. Dragomir, *Inequalities for $\alpha$-fractional differentiable functions*, J. Inequal. Appl., (2017) 2017:93, pp. 12.

[5] Y.M. Chu, M.A. Khan, T.U. Khan and T. Ali, *Generalizations of Hermite-Hadamard type inequalities for MT-convex functions*, J. Nonlinear Sci. Appl., **9**(5) (2016), 4305–4316.

[6] Y.M. Chu, G.D. Wang and X.H. Zhang, *Schur convexity and Hadamard's inequality*, Math. Inequal. Appl., **13**(4) (2010), 725–731.

[7] Z. Dahmani, *On Minkowski and Hermite-Hadamard integral inequalities via fractional integration,* Ann. Funct. Anal., **1**(1) (2010), 51–58.

[8] S.S. Dragomir, J. Pečarić and L.E. Persson, *Some inequalities of Hadamard type*, Soochow J. Math., **21** (1995), 335–341.

[9] T.S. Du, J.G. Liao and Y.J. Li, *Properties and integral inequalities of Hadamard-Simpson type for the generalized $(s,m)$-preinvex functions,* J. Nonlinear Sci. Appl., **9** (2016), 3112–3126.

[10] D.I. Duca and L. Lupsa, *Saddle points for vector valued functions: existence, necessary and sufficient theorems,* J. Glob. Optim., **53** (2012), 431–440.

[11] G. Farid, A. Javed and A.U. Rehman, *On Hadamard inequalities for n-times differentiable functions which are relative convex via Caputo k-fractional derivatives,* Nonlinear Anal. Forum, To appear.

[12] G. Farid and A.U. Rehman, *Generalizations of some integral inequalities for fractional integrals*, Ann. Math. Sil., **31** (2017), pp. 14.

[13] C. Fulga and V. Preda, *Nonlinear programming with $\varphi$-preinvex and local $\varphi$-preinvex functions*, Eur. J. Oper. Res., **192** (2009), 737–743.

[14] A. Kashuri and R. Liko, *On Hermite-Hadamard type inequalities for generalized $(s,m,\varphi)$-preinvex functions via k-fractional integrals*, Adv. Inequal. Appl., **6** (2017), 1–12.

[15] A. Kashuri and R. Liko, *Generalizations of Hermite-Hadamard and Ostrowski type inequalities for $MT_m$-preinvex functions*, Proyecciones, **36**(1) (2017), 45–80.

[16] A. Kashuri and R. Liko, *Hermite-Hadamard type fractional integral inequalities for generalized $(r;s,m,\varphi)$-preinvex functions*, Eur. J. Pure Appl. Math., **10**(3) (2017), 495–505.

[17] A. Kashuri and R. Liko, *Hermite-Hadamard type fractional integral inequalities for twice differentiable generalized $(s,m,\varphi)$-preinvex functions*, Konuralp J. Math., **5**(2) (2017), 228–238.

[18] A. Kashuri and R. Liko, *Hermite-Hadamard type inequalities for generalized $(s,m,\varphi)$-preinvex functions via k-fractional integrals*, Tbil. Math. J., **10**(4) (2017), 73–82.

[19] A. Kashuri and R. Liko, *Hermite-Hadamard type fractional integral inequalities for $MT_{(m,\varphi)}$-preinvex functions*, Stud. Univ. Babeş-Bolyai, Math., **62**(4) (2017), 439–450.

[20] A. Kashuri and R. Liko, *Hermite-Hadamard type fractional integral inequalities for twice differentiable generalized beta-preinvex functions*, J. Fract. Calc. Appl., **9**(1) (2018), 241–252.

[21] M.A. Khan, T. Ali, S.S. Dragomir and M.Z. Sarikaya, *Hermite-Hadamard type inequalities for conformable fractional integrals*, Revista de la Real Academia de Ciencias Exactas, Fsicas y Naturales. Serie A. Matemticas, (2017), doi:10.1007/s13398-017-0408-5.

[22] M.A. Khan, Y.M. Chu, A. Kashuri and R. Liko, *Hermite-Hadamard type fractional integral inequalities for $MT_{(r;g,m,\varphi)}$-preinvex functions*, J. Comput. Anal. Appl., **26**(8) (2019), 1487–1503.

[23] M.A. Khan, Y.M. Chu, T.U. Khan and J. Khan, *Some new inequalities of Hermite-Hadamard type for s-convex functions with applications*, Open Math., **15** (2017), 1414–1430.

[24] M.A. Khan, Y. Khurshid, T. Ali and N. Rehman, *Inequalities for three times differentiable functions*, J. Math., Punjab Univ., **48**(2) (2016), 35–48.

[25] M.A. Khan, Y. Khurshid and T. Ali, *Hermite-Hadamard inequality for fractional integrals via $\eta$-convex functions*, Acta Math. Univ. Comenianae, **79**(1) (2017), 153–164.

[26] M.A. Khan, Y.M. Chu, A. Kashuri, R. Liko and G. Ali, *New Hermite-Hadamard inequalities for conformable fractional integrals*, J. Funct. Spaces, **2018** (2018) Article ID 6928130, pp. 9.

[27] W. Liu, W. Wen and J. Park, *Ostrowski type fractional integral inequalities for MT-convex functions*, Miskolc Math. Notes, **16**(1) (2015), 249–256.

[28] W. Liu, W. Wen and J. Park, *Hermite-Hadamard type inequalities for MT-convex functions via classical integrals and fractional integrals*, J. Nonlinear Sci. Appl., **9** (2016), 766–777.

[29] C. Luo, T.S. Du, M.A. Khan, A. Kashuri and Y. Shen, *Some k-fractional integrals inequalities through generalized $\varphi$-m-MT-preinvexity*, J. Comput. Anal. Appl., **27**(4) (2019), 690–705.

[30] M. Matłoka, *Inequalities for h-preinvex functions*, Appl. Math. Comput., **234** (2014), 52–57.

[31] S. Mubeen and G.M. Habibullah, *k-Fractional integrals and applications*, Int. J. Contemp. Math. Sci., **7** (2012), 89–94.

[32] M.A. Noor, *Hermite-Hadamard integral inequalities for log-preinvex functions*, J. Math. Anal. Approx. Theory, **2** (2007), 126–131.

[33] M.A. Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl., **251** (2000), 217–229.

[34] M.A. Noor, *On Hermite-Hadamard integral inequalities for involving two* log*-preinvex functions*, J. Inequal. Pure Appl. Math., **3** (2007), 75–81.

[35] M.A. Noor, *Variational like inequalities*, Optimization, **30** (1994), 323–330.

[36] M.A. Noor, K.I. Noor, M.U. Awan and J. Li, *On Hermite-Hadamard type inequalities for h-preinvex functions*, Filomat, **28**(7) (2014), 1463–1474.

[37] M.A. Noor, K.I. Noor and S. Iftikhar, *Integral inequalities for differentiable relative preinvex functions (survey)*, TWMS J. Pure Appl. Math., **7**(1) (2016), 3–19.

[38] O. Omotoyinbo and A. Mogbodemu, *Some new Hermite-Hadamard integral inequalities for convex functions*, Int. J. Sci. Innovation Tech., **1**(1) (2014), 1–12.

[39] C. Peng, C. Zhou and T.S. Du, *Riemann-Liouville fractional Simpson's inequalities through generalized $(m, h_1, h_2)$-preinvexity*, Ital. J. Pure Appl. Math., **38** (2017), 345–367.

[40] R. Pini, *Invexity and generalized convexity*, Optimization, **22** (1991), 513–525.

[41] M.Z. Sarikaya and H. Budak, *Generalized Hermite-Hadamard type integral inequalities for fractional integrals*, Filomat, **30**(5) (2016), 1315–1326.

[42] M.Z. Sarikaya, E. Set, H. Yaldiz and H. Başak, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Model, **57** (2013), 2403–2407.

[43] E. Set and A. Gözpinar, *A study on Hermite-Hadamard type inequalities for s-convex functions via conformable fractional integrals*, (Submitted).

[44] E. Set, A. Gözpinar and J. Choi, *Hermite-Hadamard type inequalities for twice differentiable m-convex functions via conformable fractional integrals*, Far East J. Math. Sci., **101**(4) (2017), 873–891.

[45] E. Set, M.Z. Sarikaya and A. Gözpinar, *Some Hermite-Hadamard type inequalities for convex functions via conformable fractional integrals and related inequalities*, Creat. Math. Inform., **26**(2) (2017), 221–229.

[46] H.N. Shi, *Two Schur-convex functions related to Hadamard-type integral inequalities*, Publ. Math. Debrecen, **78**(2) (2011), 393–403.

[47] M. Tunç, E. Göv and Ü. Şanal, *On tgs-convex function and their inequalities*, Facta Univ. Ser. Math. Inform., **30**(5) (2015), 679–691.

[48] S. Varošanec, *On h-convexity*, J. Math. Anal. Appl., **326**(1) (2007), 303–311.

[49] H. Wang, T.S. Du and Y. Zhang, *k-fractional integral trapezium-like inequalities through $(h, m)$-convex and $(\alpha, m)$-convex mappings*, J. Inequal. Appl., (2017) 2017:311 doi.org/10.1186/s13660-017-1586-6.

[50] E.A. Youness, *E-convex sets, E-convex functions, and E-convex programming*, J. Optim. Theory Appl., **102** (1999), 439–450.

[51] X.M. Zhang, Y.M. Chu and X.H. Zhang, *The Hermite-Hadamard type inequality of GA-convex functions and its applications*, J. Inequal. Appl., (2010), Article ID 507560, pp. 11.

[52] Y.C. Zhang, T.S. Du and J. Pan, *On new inequalities of Fejér-Hermite-Hadamard type for differentiable $(\alpha, m)$-preinvex mappings*, ScienceAsia, **43** (2017), 258–266.