Finding hybrid relatives of the Bessel polynomials

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Abstract

In this work, the 2D-Appell and the Bessel polynomials are combined to introduce the family of the 2D-Appell-Bessel polynomials. The generating function, quasi-monomial properties, series definition and determinant form of these polynomials are established. Examples of some members belonging to this family are considered. The graphs of some hybrid special polynomials are also drawn for suitable values of the indices.

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1 Introduction and preliminaries

The Appell polynomials were introduced by P.E. Appell [1]. These polynomials appear in different applications in pure and applied mathematics [2, 11, 15].

The following definitions of Appell polynomials are equivalent:

$\{A_n(x)\}$ $(n \in \mathbb{N}_0)$ is an Appell set ($A_n(x)$ being of degree exactly n), if either

(i) $\frac{d}{dx}A_n(x) = n\ A_{n-1}(x), \quad n \in \mathbb{N}_0$, or

(ii) there exists an exponential generating function of the form

$$A(t)\exp(xt) = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}, \tag{1.1}$$

where $A(t)$ has (at least the formal) expansion:

$$A(t) = \sum_{n=0}^{\infty} A_n \frac{t^n}{n!}, \quad A_0 \neq 0. \tag{1.2}$$

The classical examples of Appell polynomials besides the trivial example $\{x^n\}_{n=0}^{\infty}$ are the Bernoulli, Euler and Hermite polynomials.

The notions of the monomiality principle and operational techniques are used to unify the special polynomials and to find the mixed type special polynomials, see for example [5]. Recently,

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the 2D-Appell polynomials $A_n(x,y)$ are introduced which are defined by means of the generating function [6]

$$A(t)\varphi(y,t)e^{xt} = \sum_{n=0}^{\infty} A_n(x,y)\frac{t^n}{n!}, \tag{1.3}$$

where

$$\varphi(y,t) = \sum_{n=0}^{\infty} \varphi_n(y)\frac{t^n}{n!}, \quad \varphi_0(y) \neq 0, \quad \varphi(0,t) = 1. \tag{1.4}$$

The 2D-Appell polynomials $A_n(x,y)$ are shown to be quasi-monomial [4, 14] under the action of the following multiplicative and derivative operators [6]:

$$\hat{M}_A = x + \frac{\varphi'(y,D_x)}{\varphi(y,D_x)} + \frac{A'(D_x)}{A(D_x)} \tag{1.5}$$

and

$$\hat{P}_A = D_x, \tag{1.6}$$

respectively.

Consequently, $\hat{M}_A$ and $\hat{P}_A$ satisfy the following recurrences:

$$\hat{M}_A\{A_n(x,y)\} = A_{n+1}(x,y) \tag{1.7}$$

and

$$\hat{P}_A\{A_n(x,y)\} = n\ A_{n-1}(x,y), \tag{1.8}$$

for all $n \in \mathbb{N}$.

The operators $\hat{M}_A$ and $\hat{P}_A$ also satisfy the following commutation relation:

$$[\hat{P}_A, \hat{M}_A] = 1. \tag{1.9}$$

In view of the monomiality principle equation

$$\hat{M}_A\hat{P}_A\{A_n(x,y)\} = n\ A_n(x,y), \tag{1.10}$$

the differential equation satisfied by $A_n(x,y)$ is

$$\left(xD_x + \frac{\varphi'(y,D_x)}{\varphi(y,D_x)}D_x + \frac{A'(D_x)}{A(D_x)}D_x - n\right)A_n(x,y) = 0. \tag{1.11}$$

Also, $A_n(x,y)$ can be explicitly constructed as:

$$A_n(x,y) = \hat{M}_A^n\{1\}, \quad A_0(x,y) = 1. \tag{1.12}$$

Identity (1.12) implies that the exponential generating function of $A_n(x,y)$ can be given in the form:

$$\exp(t\hat{M}_A)\{1\} = \sum_{n=0}^{\infty} A_n(x,y)\frac{t^n}{n!},\ |t| < \infty. \tag{1.13}$$

Krall and Frink [10] studied the polynomials

$$y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!} \left(\frac{x}{2}\right)^k, \tag{1.14}$$

which are called Bessel polynomials in view of their connection with Bessel functions. The $y_n(x)$ satisfy the differential equation

$$x^2y'' + (2x+2)y' + n(n+1)y = 0, \tag{1.15}$$

and their derivatives form an orthogonal sequence of polynomials. Krall and Frink also discuss applications of these polynomials to spherical waves.

Carlitz [3] defined the Bessel polynomials by generating function:

$$\exp\left(x\left(1-\sqrt{1-2t}\right)\right) = \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!} \tag{1.16}$$

and gave many of their properties. These polynomials have the following explicit representation:

$$p_n(x) = \sum_{k=1}^{n} \frac{(2n-k-1)!}{2^{n-k}(k-1)!(n-k)!}x^k. \tag{1.17}$$

The Bessel polynomials $p_n(x)$ are related to the polynomials $y_n(x)$ as:

$$p_n(x) = x^n\ y_{n-1}\left(\frac{1}{x}\right). \tag{1.18}$$

Consider two formal power series:

$$f(t) = \sum_{n=0}^{\infty} f_n\frac{t^n}{n!},\ f_0 = 0,\ f_1 \neq 0 \tag{1.19a}$$

and

$$g(t) = \sum_{n=0}^{\infty} g_n\frac{t^n}{n!},\ g_0 \neq 0. \tag{1.19b}$$

According to Roman [12, p.18 (Theorem 2.3.4)], the polynomial sequence $s_n(x)$ is uniquely determined by two (formal) power series given by equations (1.19a) and (1.19b). The exponential generating function of $s_n(x)$ is then given by

$$\frac{1}{g\big(f^{-1}(t)\big)}\exp\left(xf^{-1}(t)\right) = \sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!}, \tag{1.20}$$

for all x in $\mathbb{C}$, where $f^{-1}(t)$ is the compositional inverse of $f(t)$.

The Sheffer sequence $s_n(x)$ for $\big(1, f(t)\big)$ is called the associated Sheffer sequence $\mathfrak{s}_n(x)$ for $f(t)$ and the Sheffer sequence for $\big(\mathtt{g}(t), t\big)$ becomes the Appell sequence $A_n(x)$ for $\mathtt{g}(t)$ [12, p. 17].

The associated Sheffer sequence $\mathfrak{s}_n(x)$ is defined by the generating function

$$\exp(xf^{-1}(t)) = \sum_{n=0}^{\infty} \mathfrak{s}_n(x)\frac{t^n}{n!}. \tag{1.21}$$

It is to be noted that Bessel polynomials belong to the associated Sheffer sequence for $f(t) = -\frac{1}{2}t^2 + t$.

Recently, certain mixed special polynomial families related to the Appell sequences are studied in a systematic way, see for example [5, 8, 9, 13]. These polynomials are studied thoroughly due to their applications in various fields of mathematics, physics and engineering. This provides motivation to introduce a hybrid family of special polynomials by combining 2D-Appell and Bessel polynomials.

In this paper, the 2D-Appell-Bessel polynomials are introduced and their properties are derived. In Section 2, the generating function, quasi-monomial properties and series definition of these polynomials are established. In Section 3, examples of some members belonging to this family are considered. The graphs of some hybrid special polynomials are also drawn for suitable values of the indices.

2 2D-Appell-Bessel polynomials

In this section, the 2D-Appell-Bessel polynomials are introduced by means of generating function and their quasi-monomial properties are established

To introduce 2D-Appell-Bessel polynomials (2DABP), the following result is proved.

Theorem 2.1. For the 2D-Appell-Bessel polynomials ${}_Ap_n(x, y)$ of degree n, the following generating function holds true:

$$A(1-\sqrt{1-2t})\varphi(y, 1-\sqrt{1-2t})e^{x(1-\sqrt{1-2t})} = \sum_{n=0}^{\infty} {}_Ap_n(x, y)\frac{t^n}{n!}. \tag{2.1}$$

Proof. In order to derive the generating function for the 2DABP, we take the 2D-Appell polynomials $A_n(x, y)$ as base in generating function (1.16) of the Bessel polynomials. Thus, replacing x by the multiplicative operator $\hat{M}_A$ of the 2D-Appell polynomials $A_n(x, y)$ in the l.h.s. of equation (1.16) and denoting the resultant 2D-Appell-Bessel polynomials in the r.h.s. by ${}_Ap_n(x, y)$, we have

$$\exp\left(\hat{M}_A\left(1-\sqrt{1-2t}\right)\right) = \sum_{n=0}^{\infty} {}_Ap_n(x, y)\frac{t^n}{n!}, \tag{2.2}$$

which by virtue of equation (1.13) with t replaced by $\left(1-\sqrt{1-2t}\right)$ and then using equation (1.3) in the resultant equation proves assertion (2.1).

Q.E.D.

Remark 2.1. From equations (1.16) and (2.2), we get the following operational correspondence between the 2D-Appell-Bessel polynomials ${}_Ap_n(x,y)$ and Bessel polynomials $p_n(x)$:

$$p_n(\hat{M}_A) = {}_Ap_n(x,y). \tag{2.3}$$

Remark 2.2. Since for $y=0$, the 2D-Appell polynomials $A_n(x,y)$ reduce to the Appell polynomials $A_n(x)$. Therefore, taking $y=0$ in the l.h.s. of equation (2.1) and denoting the resultant Appell-Bessel polynomials (ABP) ${}_Ap_n(x)$ in the r.h.s., we deduce the following consequence of Theorem 2.1:

Corollary 2.1. For the Appell-Bessel polynomials ${}_Ap_n(x)$, the following generating function holds true:

$$A(1-\sqrt{1-2t})e^{x(1-\sqrt{1-2t})} = \sum_{n=0}^{\infty} {}_Ap_n(x)\frac{t^n}{n!}. \tag{2.4}$$

Remark 2.3. Using equation (2.3) in definition (1.17), the following explicit representation for the 2D-Appell-Bessel polynomials ${}_Ap_n(x,y)$ is obtained:

$${}_Ap_n(x,y) = \sum_{k=1}^{n} \frac{(2n-k-1)!}{2^{n-k}(k-1)!(n-k)!} A_k(x,y), \tag{2.5}$$

which in view of Remark 2.2, gives the following series definition of the Appell-Bessel polynomials ${}_Ap_n(x)$:

$${}_Ap_n(x) = \sum_{k=1}^{n} \frac{(2n-k-1)!}{2^{n-k}(k-1)!(n-k)!} A_k(x). \tag{2.6}$$

Next, the multiplicative and derivative operators associated with the 2D-Appell-Bessel polynomials are obtained:

Theorem 2.2. For the 2D-Appell-Bessel polynomials ${}_Ap_n(x,y)$, the following multiplicative and derivative operators exist:

$$\hat{M}_{Ap} = \left(x + \frac{\varphi'(y,D_x)}{\varphi(y,D_x)} + \frac{A'(D_x)}{A(D_x)}\right)\frac{1}{1-D_x}, \tag{2.7}$$

$$\hat{P}_{Ap} = D_x - \frac{1}{2}{D_x}^2, \tag{2.8}$$

where $D_x := \frac{\partial}{\partial x}$.

Proof. Consider the identity

$$D_x\ \{A(1-\sqrt{1-2t})\varphi(y,1-\sqrt{1-2t})e^{x(1-\sqrt{1-2t})}\}$$
$$=(1-\sqrt{1-2t})\ (A(1-\sqrt{1-2t})\varphi(y,1-\sqrt{1-2t})e^{x(1-\sqrt{1-2t})}), \tag{2.9}$$

which on simplification gives

$$\left(D_x-\frac{1}{2}{D_x}^2\right)\ \{A(1-\sqrt{1-2t})\varphi(y,1-\sqrt{1-2t})e^{x(1-\sqrt{1-2t})}\}$$
$$=t\ (A(1-\sqrt{1-2t})\varphi(y,1-\sqrt{1-2t})e^{x(1-\sqrt{1-2t})}). \tag{2.10}$$

Differentiating equation (2.1) partially with respect to t, we find

$$\left(x+\frac{\varphi'(y,1-\sqrt{1-2t})}{\varphi(y,1-\sqrt{1-2t}))}+\frac{A'(1-\sqrt{1-2t}))}{A(1-\sqrt{1-2t})}\right)\frac{1}{\sqrt{1-2t}}$$
$$\{A(1-\sqrt{1-2t})\varphi(y,1-\sqrt{1-2t})e^{x(1-\sqrt{1-2t})}\}=\sum_{n=0}^{\infty}{}_Ap_{n+1}(x,y)\frac{t^n}{n!}. \tag{2.11}$$

In view of identity (2.9), equation (2.11) takes the form

$$\left(x+\frac{\varphi'(y,D_x)}{\varphi(y,D_x)}+\frac{A'(D_x)}{A(D_x)}\right)\frac{1}{1-D_x}\{A(1-\sqrt{1-2t})\varphi(y,1-\sqrt{1-2t})e^{x(1-\sqrt{1-2t})}\}$$
$$=\sum_{n=0}^{\infty}{}_Ap_{n+1}(x,y)\frac{t^n}{n!}. \tag{2.12}$$

Now making use of generating function (2.1) in the l.h.s. of the above equation and then equating the coefficients of like powers of t in both sides of the resultant equation, we find

$$\left(x+\frac{\varphi'(y,D_x)}{\varphi(y,D_x)}+\frac{A'(D_x)}{A(D_x)}\right)\frac{1}{1-D_x}\{{}_Ap_n(x,y)\}={}_Ap_{n+1}(x,y), \tag{2.13}$$

which proves assertion (2.7).

Further, making use of generating function (2.1) in both sides of identity (2.10) and then equating the coefficients of like powers of t in both sides of resultant equation, we have

$$\left(D_x-\frac{1}{2}{D_x}^2\right)\{{}_Ap_n(x,y)\}=n\ {}_Ap_{n-1}(x,y). \tag{2.14}$$

In view of equation (2.14), assertion (2.8) is proved.

Q.E.D.

As an immediate consequence of Remark 2.2, the following Corollary is deduced:

Corollary 2.2. The Appell-Bessel polynomials ${}_Ap_n(x)$ are quasi-monomial with respect to the following multiplicative and derivative operators:

$$\hat{\mathcal{M}}_{Ap} = \left(x + \frac{A'(D_x)}{A(D_x)}\right)\frac{1}{1-D_x}, \tag{2.15}$$

$$\hat{\mathcal{P}}_{Ap} = D_x - \frac{1}{2}{D_x}^2. \tag{2.16}$$

Theorem 2.3. For the 2D-Appell-Bessel polynomials ${}_Ap_n(x,y)$, the following differential equation is satisfied:

$$\left(\left(x + \frac{\varphi'(y,D_x)}{\varphi(y,D_x)} + \frac{A'(D_x)}{A(D_x)}\right)\left(\frac{D_x - \frac{1}{2}{D_x}^2}{1-D_x}\right) - n\right){}_Ap_n(x,y) = 0. \tag{2.17}$$

Proof. Using equations (2.7) and (2.8) in the corresponding equation (1.10) for the 2D-Appell-Bessel polynomials ${}_Ap_n(x,y)$, we get assertion (2.17).

Q.E.D.

In view of Remark 2.2, the following corollary is deduced:

Corollary 2.3. For the Appell-Bessel polynomials ${}_Ap_n(x)$, the following differential equation is satisfied:

$$\left(\left(x + \frac{A'(D_x)}{A(D_x)}\right)\left(\frac{D_x - \frac{1}{2}{D_x}^2}{1-D_x}\right) - n\right){}_Ap_n(x) = 0. \tag{2.18}$$

Next, we establish the determinant definition of the 2D-Appell-Bessel polynomials.

In 2014, W. Wang gives the following determinantal definition of the associated Sheffer sequences [16, p.234]:

Let $(\mathfrak{s}_n(x))_{n\in\mathbb{N}}$ be associated Sheffer sequence for $f(t)$, then

$$\begin{aligned}
&\mathfrak{s}_0(x) = 1,\\
&\mathfrak{s}_n(x) = \frac{(-1)^n}{a_{0,0}\ a_{1,1}\ \dots\ a_{n,n}}\begin{vmatrix} x & x^2 & \cdots & x^{n-1} & x^n \\ a_{1,1} & a_{2,1} & \cdots & a_{n-1,1} & a_{n,1} \\ 0 & a_{2,2} & \cdots & a_{n-1,2} & a_{n,2} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1} & a_{n,n-1} \end{vmatrix},
\end{aligned} \tag{2.19}$$

where $a_{n,k}$ is the (n,k) entry of the Riordan array $(1, f(t))$ [17].

Since, the Bessel polynomials $p_n(x)$ belong to the associated Sheffer sequence for $f(t) = -\frac{1}{2}t^2 + t$, we have the following determinant definition of the Bessel polynomials:

Let $(p_n(x))_{n\in\mathbb{N}}$ be the Bessel polynomials, then

$$p_0(x)=1,$$
$$p_n(x)=\frac{(-1)^n}{a_{0,0}\ a_{1,1}\ \dots\ a_{n,n}}\begin{vmatrix} x & x^2 & \cdots & x^{n-1} & x^n \\ a_{1,1} & a_{2,1} & \cdots & a_{n-1,1} & a_{n,1} \\ 0 & a_{2,2} & \cdots & a_{n-1,2} & a_{n,2} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1} & a_{n,n-1} \end{vmatrix}, \tag{2.20}$$

where $a_{n,k}$ is the (n,k) entry of the Riordan array $(1,-\frac{1}{2}t^2+t)$.

Now, we establish the determinant definition of the newly-introduced polynomials i.e. the 2D-Appell-Bessel polynomials ${}_Ap_n(x,y)$.

Taking $n=0$ in series definition (2.5), we obtain:

$${}_Ap_0(x,y)=1. \tag{2.21}$$

Expansion of the determinant in equation (2.20) with respect to the first row, gives

$$\begin{aligned} p_n(x)=&\frac{(-1)^n x}{a_{0,0}\ a_{1,1}\ \dots\ a_{n,n}}\begin{vmatrix} a_{2,1} & \cdots & a_{n-1,1} & a_{n,1} \\ a_{2,2} & \cdots & a_{n-1,2} & a_{n,2} \\ \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & a_{n-1,n-1} & a_{n,n-1} \end{vmatrix} \\ &-\frac{(-1)^n x^2}{a_{0,0}\ a_{1,1}\ \dots\ a_{n,n}}\begin{vmatrix} a_{1,1} & \cdots & a_{n-1,1} & a_{n,1} \\ 0 & \cdots & a_{n-1,2} & a_{n,2} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & a_{n-1,n-1} & a_{n,n-1} \end{vmatrix} \\ &+\cdots+\frac{(-1)^{2n+1}x^{n-1}}{a_{0,0}\ a_{1,1}\ \dots\ a_{n,n}}\begin{vmatrix} a_{1,1} & a_{2,1} & \cdots & a_{n,1} \\ 0 & a_{2,2} & \cdots & a_{n,2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n,n-1} \end{vmatrix} \\ &+\frac{x^n}{a_{0,0}\ a_{1,1}\ \dots\ a_{n,n}}\begin{vmatrix} a_{1,1} & a_{2,1} & \cdots & a_{n-1,1} \\ 0 & a_{2,2} & \cdots & a_{n-1,2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1} \end{vmatrix}. \end{aligned} \tag{2.22}$$

Since each minor in equation (2.22) is independent of x, therefore replacing x by the multiplicative operator $\hat{M}_A$ of the 2D-Appell polynomials $A_n(x,y)$ in the l.h.s. and r.h.s. of equation (2.22)

and then using equation (2.3) in l.h.s. and (1.12) in r.h.s., respectively of the resultant equation, we find

$$
\begin{aligned}
{}_{A}p_n(x,y) = {} & \frac{(-1)^n A_1(x,y)}{a_{0,0}\ a_{1,1}\ \ldots\ a_{n,n}} \begin{vmatrix} a_{2,1} & \cdots & a_{n-1,1} & a_{n,1} \\ a_{2,2} & \cdots & a_{n-1,2} & a_{n,2} \\ \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & a_{n-1,n-1} & a_{n,n-1} \end{vmatrix} \\
& - \frac{(-1)^n A_2(x,y)}{a_{0,0}\ a_{1,1}\ \ldots\ a_{n,n}} \begin{vmatrix} a_{1,1} & \cdots & a_{n-1,1} & a_{n,1} \\ 0 & \cdots & a_{n-1,2} & a_{n,2} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & a_{n-1,n-1} & a_{n,n-1} \end{vmatrix} \\
& + \cdots + \frac{(-1)^{2n+1} A_{n-1}(x,y)}{a_{0,0}\ a_{1,1}\ \ldots\ a_{n,n}} \begin{vmatrix} a_{1,1} & a_{2,1} & \cdots & a_{n,1} \\ 0 & a_{2,2} & \cdots & a_{n,2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n,n-1} \end{vmatrix} \\
+ & \\
& \frac{A_n(x,y)}{a_{0,0}\ a_{1,1}\ \ldots\ a_{n,n}} \begin{vmatrix} a_{1,1} & a_{2,1} & \cdots & a_{n-1,1} \\ 0 & a_{2,2} & \cdots & a_{n-1,2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1} \end{vmatrix}. \qquad (2.23)
\end{aligned}
$$

Combining the terms in the r.h.s. of equation (2.23), we obtain

$$
{}_{A}p_n(x,y) = \frac{(-1)^n}{a_{0,0}\ a_{1,1}\ \ldots\ a_{n,n}} \begin{vmatrix} A_1(x,y) & A_2(x,y) & \cdots & A_{n-1}(x,y) & A_n(x,y) \\ a_{1,1} & a_{2,1} & \cdots & a_{n-1,1} & a_{n,1} \\ 0 & a_{2,2} & \cdots & a_{n-1,2} & a_{n,2} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1} & a_{n,n-1} \end{vmatrix}, \qquad (2.24)
$$

where $A_n(x,y)$ are the 2D-Appell polynomials and $a_{n,k}$ is the (n,k) entry of the Riordan array $(1, -\frac{1}{2}t^2 + t)$.

Equation (2.21) together with equation (2.24), gives the determinantal definition for the 2D-Appell-Bessel polynomials ${}_{A}p_n(x,y)$.

In the next section, we consider certain members of the 2D-Appell-Bessel family and derived the corresponding results for these hybrid special polynomials.

3 Examples

First, we consider the following examples by taking $\varphi(y,t)$ of particular members of the 2D-Appell family.

Example 3.1 Since, for $\varphi(y,t) = e^{yt^m}$, the 2D-Appell polynomials $A_n(x,y)$ reduce to the Gould-Hopper-Appell polynomials ${}_HA_n^{(m)}(x,y)$ defined by generating function [6]

$$A(t)e^{xt+yt^m} = \sum_{n=0}^{\infty} {}_HA_n^{(m)}(x,y)\frac{t^n}{n!}. \tag{3.1}$$

Therefore, from equation (2.1), we find that the Gould-Hopper-Appell-Bessel polynomials ${}_HA^{(m)}p_n(x,y)$ are defined by the following generating function:

$$A(1-\sqrt{1-2t})e^{x(1-\sqrt{1-2t})+y(1-\sqrt{1-2t})^m} = \sum_{n=0}^{\infty} {}_HA^{(m)}p_n(x,y)\frac{t^n}{n!}. \tag{3.2}$$

In view of equation (2.3), the following operational correspondence between the Gould-Hopper-Appell-Bessel polynomials ${}_HA^{(m)}p_n(x,y)$ and Bessel polynomials $p_n(x)$ is obtained:

$$p_n(\hat{M}_{H^{(m)}A}) = {}_HA^{(m)}p_n(x,y), \tag{3.3}$$

where $\hat{M}_{H^{(m)}A}$ is the multiplicative operator of the Gould-Hopper-Appell polynomials ${}_HA_n^{(m)}(x,y)$.

By virtue of equation (2.5), the following explicit representation of the Gould-Hopper-Appell-Bessel polynomials ${}_HA^{(m)}p_n(x,y)$ is obtained:

$${}_HA^{(m)}p_n(x,y) = \sum_{k=1}^{n} \frac{(2n-k-1)!}{2^{n-k}(k-1)!(n-k)!} {}_HA_k^{(m)}(x,y). \tag{3.4}$$

From equations (2.7) and (2.8), it follows that the Gould-Hopper-Appell-Bessel polynomials ${}_HA^{(m)}p_n(x,y)$ are quasi-monomial with respect to the following multiplicative and derivative operators:

$$\hat{M}_{H^{(m)}Ap} = \left(x + myD_x^{m-1} + \frac{A'(D_x)}{A(D_x)}\right)\frac{1}{1-D_x}, \tag{3.5}$$

$$\hat{P}_{H^{(m)}Ap} = D_x - \frac{1}{2}{D_x}^2, \tag{3.6}$$

respectively.

Also, from equation (2.17), the following differential equation for the Gould-Hopper-Appell-Bessel polynomials ${}_HA^{(m)}p_n(x,y)$ is obtained:

$$\left(\left(x + myD_x^{m-1} + \frac{A'(D_x)}{A(D_x)}\right)\left(\frac{D_x - \frac{1}{2}{D_x}^2}{1-D_x}\right) - n\right) {}_HA^{(m)}p_n(x,y) = 0. \tag{3.7}$$

Remark 3.1. Since for $m = 2$, the Gould-Hopper-Appell polynomials ${}_HA_n^{(m)}(x,y)$ reduce to the Hermite-Appell polynomials ${}_HA_n(x,y)$. Therefore, taking $m = 2$ in equations (3.2)-(3.7), the corresponding results for the Hermite-Appell-Bessel polynomials ${}_HAp_n(x,y)$ can be obtained.

Remark 3.2. We note that the 2-variable Hermite polynomials $H_n(x,y)$ are related to the classical Hermite polynomials $H_n(x)$ or $He_n(x)$ as:

$$H_n(2x,-1) = H_n(x), \tag{3.8}$$

$$H_n(x,-\frac{1}{2}) = He_n(x). \tag{3.9}$$

Therefore taking $m=2$, $y=-1$ and replacing x by $2x$ (or $y=-\frac{1}{2}$) in equations (3.2)-(3.7), the corresponding results for the classical Hermite-Appell-Bessel polynomials ${}_{HA}p_n(x)$ (or ${}_{HeA}p_n(x)$) can be obtained.

Example 3.2 Since, for $\varphi(y,t) = \frac{1}{1-yt^r}$, the 2D-Appell polynomials $A_n(x,y)$ reduce to the truncated exponential-Appell polynomials ${}_{e^{(r)}}A_n(x,y)$ defined by generating function [7]

$$A(t)\frac{1}{(1-yt^r)}e^{xt} = \sum_{n=0}^{\infty} {}_{e^{(r)}}A_n(x,y)\frac{t^n}{n!}. \tag{3.10}$$

Therefore, from equation (2.1) we find that the truncated exponential-Appell-Bessel polynomials ${}_{e^{(r)}A}p_n(x,y)$ are defined by the following generating function:

$$A(1-\sqrt{1-2t})\frac{1}{(1-y(1-\sqrt{1-2t})^r)}e^{x(1-\sqrt{1-2t})} = \sum_{n=0}^{\infty} {}_{e^{(r)}A}p_n(x,y)\frac{t^n}{n!}. \tag{3.11}$$

In view of equation (2.3), the following operational correspondence between the truncated exponential-Appell-Bessel polynomials ${}_{e^{(r)}A}p_n(x,y)$ and Bessel polynomials $p_n(x)$ holds:

$$p_n(\hat{M}_{e^{(r)}A}) = {}_{e^{(r)}A}p_n(x,y), \tag{3.12}$$

where $\hat{M}_{e^{(r)}A}$ is the multiplicative operator of the truncated exponential-Appell polynomials ${}_{e^{(r)}}A_n(x,y)$.

By virtue of equation (2.5), the following explicit representation of the truncated exponential-Appell-Bessel polynomials ${}_{e^{(r)}A}p_n(x,y)$ is obtained:

$${}_{e^{(r)}A}p_n(x,y) = \sum_{k=1}^{n} \frac{(2n-k-1)!}{2^{n-k}(k-1)!(n-k)!} {}_{e^{(r)}}A_k(x,y). \tag{3.13}$$

From equations (2.7) and (2.8), we get the following multiplicative and derivative operators associated with truncated exponential-Appell-Bessel polynomials ${}_{e^{(r)}A}p_n(x,y)$:

$$\hat{M}_{{}_{e^{(r)}A}p} = \left(x + \frac{ryD_x^{r-1}}{1-yD_x^{r-1}} + \frac{A'(D_x)}{A(D_x)}\right)\frac{1}{1-D_x}, \tag{3.14}$$

$$\hat{P}_{{}_{e^{(r)}A}p} = D_x - \frac{1}{2}{D_x}^2, \tag{3.15}$$

respectively.

Also, from equation (2.17), the following differential equation for the truncated exponential-Appell-Bessel polynomials ${}_{e^{(r)}A}p_n(x,y)$ is obtained:

$$\left(\left(x+\frac{ryD_x^{r-1}}{1-yD_x^{r-1}}+\frac{A'(D_x)}{A(D_x)}\right)\left(\frac{D_x-\frac{1}{2}{D_x}^2}{1-D_x}\right)-n\right){}_{e^{(r)}A}p_n(x,y)=0. \tag{3.16}$$

We know that the function $A(t)$ is determining function for the Appell polynomials $A_n(x)$. Therefore, for suitable selections of $A(t)$, several classical polynomials can be obtained. In particular, we have

- Bernoulli polynomials $B_n(x)$ when

$$A(t)=\left(\frac{t}{e^t-1}\right);$$

- Euler polynomials $E_n(x)$ when

$$A(t)=\left(\frac{2}{e^t+1}\right).$$

In the next examples, we obtain the results for the two specific members of the Gould-Hopper-Appell-Bessel family considered in Example 3.1.

Example 3.3 Since for $A(t)=\left(\frac{t}{e^t-1}\right)$ and $A(t)=\left(\frac{2}{e^t+1}\right)$, the 2D-Appell polynomials become the Gould-Hopper-Bernoulli polynomials ${}_HB_n^{(m)}(x,y)$ and Gould-Hopper-Euler polynomials ${}_HE_n^{(m)}(x,y)$, respectively [6]. Therefore, taking $A(1-\sqrt{1-2t})=\left(\frac{1-\sqrt{1-2t}}{e^{(1-\sqrt{1-2t})}-1}\right)$ in equations (3.2)-(3.7), we get the following results for the Gould-Hopper-Bernoulli-Bessel polynomials ${}_HB^{(m)}p_n(x,y)$:

Table 1. Results for the ${}_HB^{(m)}p_n(x,y)$

S.No.	Results	Mathematical Expressions
1.	Generating function	$\left(\frac{1-\sqrt{1-2t}}{e^{(1-\sqrt{1-2t})}-1}\right)e^{x(1-\sqrt{1-2t})+y(1-\sqrt{1-2t})^m}=\sum_{n=0}^{\infty}{}_HB^{(m)}p_n(x,y)\frac{t^n}{n!}$
2.	Operational rule	${}_HB^{(m)}p_n(\hat{M})={}_HB^{(m)}p_n(x,y)$
3.	Series definition	${}_HB^{(m)}p_n(x,y)=\sum_{k=1}^{n}\frac{(2n-k-1)!}{2^{n-k}(k-1)!(n-k)!}{}_HB_k^{(m)}(x,y)$
4.	Multiplicative and derivative operators	$\hat{M}:=\left(x+myD_x^{m-1}+\frac{e^{D_x}(1-D_x)-1}{D_x(e^{D_x}-1)}\right)\frac{1}{1-D_x}$ $\hat{P}:=D_x-\frac{1}{2}{D_x}^2$
5.	Differential equation	$\left(\left(x+myD_x^{m-1}+\frac{e^{D_x}(1-D_x)-1}{D_x(e^{D_x}-1)}\right)\left(\frac{D_x-\frac{1}{2}{D_x}^2}{1-D_x}\right)-n\right){}_HB^{(m)}p_n(x,y)=0$

Similarly, taking $A(1-\sqrt{1-2t})=\left(\frac{2}{e^{1-\sqrt{1-2t}}+1}\right)$ in equations (3.2)-(3.7), we get the following results for the Gould-Hopper-Euler-Bessel polynomials ${}_HE^{(m)}p_n(x,y)$:

Table 2. Results for the ${}_{H}E^{(m)}p_n(x,y)$

S.No.	Results	Mathematical Expressions
1.	Generating function	$\left(\frac{2}{e^{1-\sqrt{1-2t}}+1}\right)e^{x(1-\sqrt{1-2t})+y(1-\sqrt{1-2t})^m} = \sum_{n=0}^{\infty} {}_{H}E^{(m)}p_n(x,y)\frac{t^n}{n!}$
2.	Operational rule	${}_{p_n(\hat{M}_{H}E^{(m)})} = {}_{H}E^{(m)}p_n(x,y)$
3.	Series definition	${}_{H}E^{(m)}p_n(x,y) = \sum_{k=1}^{n}\frac{(2n-k-1)!}{2^{n-k}(k-1)!(n-k)!}{}_{H}E_k^{(m)}(x,y)$
4.	Multiplicative and derivative operators	$\hat{M} := \left(x + myD_x^{m-1} - \frac{eD_x}{(e^{D_x}+1)}\right)\frac{1}{1-D_x}$ $\hat{P} := D_x - \frac{1}{2}{D_x}^2$
5.	Differential equation	$\left(\left(x + myD_x^{m-1} - \frac{eD_x}{(e^{D_x}+1)}\right)\left(\frac{D_x-\frac{1}{2}{D_x}^2}{1-D_x}\right) - n\right){}_{H}E^{(m)}p_n(x,y) = 0.$

Remark 3.3. Taking $m = 2$ in the results obtained in Tables 1 and 2, the corresponding results for the Hermite-Bernoulli-Bessel polynomials ${}_{H}Bp_n(x,y)$ and Hermite-Euler-Bessel polynomials ${}_{H}Ep_n(x,y)$ can be obtained.

Remark 3.4. Taking $m = 2$, $y = -1$ and $x \to 2x$ (or $y = -\frac{1}{2}$) in the results obtained in Tables 1 and 2, the corresponding results for the classical Hermite-Bernoulli-Bessel polynomials ${}_{H}Bp_n(x)$ and classical Hermite-Euler-Bessel polynomials ${}_{H}Ep_n(x)$ (or ${}_{He}Bp_n(x)$ and ${}_{He}Ep_n(x)$) can be obtained.

In the next example, we obtain the results for two specific members of the truncated exponential-Appell-Bessel family considered in Example 3.2.:

Example 3.4 Taking $A(1-\sqrt{1-2t}) = \left(\frac{1-\sqrt{1-2t}}{e^{(1-\sqrt{1-2t})}-1}\right)$ in equations (3.11)-(3.16), we get the following results for the truncated exponential-Bernoulli-Bessel polynomials ${}_{e^{(r)}}Bp_n(x,y)$:

Table 3. Results for the ${}_{e^{(r)}}Bp_n(x,y)$

S.No.	Results	Mathematical Expressions
1.	Generating function	$\left(\frac{1-\sqrt{1-2t}}{e^{(1-\sqrt{1-2t})}-1}\right)\frac{1}{(1-y(1-\sqrt{1-2t})^r)}e^{x(1-\sqrt{1-2t})} = \sum_{n=0}^{\infty}{}_{e^{(r)}}Bp_n(x,y)\frac{t^n}{n!}$
2.	Operational rule	${}_{p_n(\hat{M}_{e^{(r)}B})} = {}_{e^{(r)}}Bp_n(x,y)$
3.	Series definition	${}_{e^{(r)}}Bp_n(x,y) = \sum_{k=1}^{n}\frac{(2n-k-1)!}{2^{n-k}(k-1)!(n-k)!}{}_{e^{(r)}}B_k(x,y)$
4.	Multiplicative and derivative operators	$\hat{M} := \left(x + \frac{ryD_x^{r-1}}{1-yD_x^{r-1}} + \frac{e^{D_x}(1-D_x)-1}{D_x(e^{D_x}-1)}\right)\frac{1}{1-D_x}$ $\hat{P} := D_x - \frac{1}{2}{D_x}^2$
5.	Differential equation	$\left(\left(x + \frac{ryD_x^{r-1}}{1-yD_x^{r-1}} + \frac{e^{D_x}(1-D_x)-1}{D_x(e^{D_x}-1)}\right)\left(\frac{D_x-\frac{1}{2}{D_x}^2}{1-D_x}\right) - n\right){}_{e^{(r)}}Bp_n(x,y) = 0$

Similarly, taking $A(1-\sqrt{1-2t}) = \left(\frac{2}{e^{1-\sqrt{1-2t}}+1}\right)$ in equations (3.11)-(3.16), we get the following results for the truncated exponential-Euler-Bessel polynomials ${}_{e^{(r)}}Ep_n(x,y)$:

Table 4. Results for the ${}_{e^{(r)}}Ep_n(x,y)$

S.No.	Results	Mathematical Expressions
1.	Generating function	$\left(\frac{2}{e^{1-\sqrt{1-2t}}+1}\right)\frac{1}{(1-y(1-\sqrt{1-2t})^r)}e^{x(1-\sqrt{1-2t})} = \sum_{n=0}^{\infty} {}_{e^{(r)}}E^{p_n(x,y)}\frac{t^n}{n!}$
2.	Operational rule	$p_n(\hat{M}_{e^{(r)}E}) = {}_{e^{(r)}}E^{p_n(x,y)}$
3.	Series definition	${}_{e^{(r)}}E^{p_n(x,y)} = \sum_{k=1}^{n}\frac{(2n-k-1)!}{2^{n-k}(k-1)!(n-k)!}{}_{e^{(r)}}E_k(x,y)$
4.	Multiplicative and derivative operators	$\hat{M} := \left(x + \frac{ryD_x^{r-1}}{1-yD_x^{r-1}} - \frac{e^{D_x}}{(e^{D_x}+1)}\right)\frac{1}{1-D_x}$ $\hat{P} := D_x - \frac{1}{2}{D_x}^2$
5.	Differential equation	$\left(\left(x + \frac{ryD_x^{r-1}}{1-yD_x^{r-1}} - \frac{e^{D_x}}{(e^{D_x}+1)}\right)\left(\frac{D_x-\frac{1}{2}{D_x}^2}{1-D_x}\right) - n\right){}_{e^{(r)}}E^{p_n(x,y)} = 0$

We remark that corresponding to each member belonging to the Appell family, there exists a new special polynomial belonging to the Appell-Bessel family. The generating function and other properties of these special polynomials can be obtained from the results derived in Section 3.

Since, for $y = 0$, the 2D-Appell-Bessel polynomials ${}_Ap_n(x,y)$ become the Appell-Bessel polynomials ${}_Ap_n(x)$. Therefore, taking $y = 0$, in the results obtained in Table 1 or Table 3, we get the following results for the Bernoulli-Bessel polynomials ${}_Bp_n(x)$:

Table 5. Results for the ${}_Bp_n(x)$

S.No.	Results	Mathematical Expressions
1.	Generating function	$\left(\frac{1-\sqrt{1-2t}}{e^{(1-\sqrt{1-2t})}-1}\right)e^{x(1-\sqrt{1-2t})} = \sum_{n=0}^{\infty}{}_Bp_n(x)\frac{t^n}{n!}$
2.	Operational rule	$p_n(\hat{M}_B) = {}_Bp_n(x)$
3.	Series definition	${}_Bp_n(x) = \sum_{k=1}^{n}\frac{(2n-k-1)!}{2^{n-k}(k-1)!(n-k)!}B_k(x)$
4.	Multiplicative and derivative operators	$\hat{M} := \left(x + \frac{e^{D_x(1-D_x)}-1}{D_x(e^{D_x}-1)}\right)\frac{1}{1-D_x}$ $\hat{P} := D_x - \frac{1}{2}{D_x}^2$
5.	Differential equation	$\left(\left(x + \frac{e^{D_x(1-D_x)}-1}{D_x(e^{D_x}-1)}\right)\left(\frac{D_x-\frac{1}{2}{D_x}^2}{1-D_x}\right) - n\right){}_Bp_n(x) = 0$

Similarly, taking $y = 0$, in the results obtained in Table 2 or Table 4, we get the following results for the Euler-Bessel polynomials ${}_Ep_n(x)$:

Table 6. Results for the ${}_Ep_n(x)$

S.No.	Results	Mathematical Expressions
1.	Generating function	$\left(\frac{2}{e^{1-\sqrt{1-2t}}+1}\right)e^{x(1-\sqrt{1-2t})} = \sum_{n=0}^{\infty}{}_Ep_n(x)\frac{t^n}{n!}$
2.	Operational rule	$p_n(\hat{M}_E) = {}_Ep_n(x)$
3.	Series definition	${}_Ep_n(x) = \sum_{k=1}^{n}\frac{(2n-k-1)!}{2^{n-k}(k-1)!(n-k)!}E_k(x)$
4.	Multiplicative and derivative operators	$\hat{M} := \left(x - \frac{e^{D_x}}{(e^{D_x}+1)}\right)\frac{1}{1-D_x}$ $\hat{P} := D_x - \frac{1}{2}{D_x}^2$
5.	Differential equation	$\left(\left(x - \frac{e^{D_x}}{(e^{D_x}+1)}\right)\left(\frac{D_x-\frac{1}{2}{D_x}^2}{1-D_x}\right) - n\right){}_Ep_n(x) = 0$

In the next section, the surface plots and graphs of some hybrid special polynomials considered in Section 3 are drawn for suitable values of the indices.

Appendix

Recall the following series given in [6, 7]:

Table A1. Explicit representations

Polynomials	Explicit representations
Gould-Hopper-Bernoulli polynomials	${}_{H}B_{n}^{(m)}(x,y) = n!\sum_{r=0}^{[n/m]} \frac{B_{n-mr}(x)\ y^{r}}{(n-mr)!r!}$
Gould-Hopper-Euler polynomials	${}_{H}E_{n}^{(m)}(x,y) = n!\sum_{r=0}^{[n/m]} \frac{E_{n-mr}(x)\ y^{r}}{(n-mr)!r!}$
Hermite-Bernoulli polynomials	${}_{H}B_{n}(x,y) = n!\sum_{r=0}^{[n/2]} \frac{B_{n-2r}(x)\ y^{r}}{(n-2r)!r!}$
Hermite-Euler polynomials	${}_{H}E_{n}(x,y) = n!\sum_{r=0}^{[n/2]} \frac{E_{n-2r}(x)\ y^{r}}{(n-2r)!r!}$
Truncated exponential-Bernoulli polynomials	${}_{e^{(2)}}B_{n}(x,y) = n!\sum_{k=0}^{[\frac{n}{2}]} \frac{B_{n-2k}(x)y^{k}}{(n-2k)!}$
Truncated exponential-Euler polynomials	${}_{e^{(2)}}E_{n}(x,y) = n!\sum_{k=0}^{[\frac{n}{2}]} \frac{E_{n-2k}(x)y^{k}}{(n-2k)!}$

The values of the Bernoulli polynomials $B_n(x)$ and the Euler polynomials $E_n(x)$ for $n = 0, 1, 2, 3, 4, 5$ are given in the following Table.

Table A2. Expressions of first six $B_n(x)$ and $E_n(x)$

n	0	1	2	3	4	5
$B_n(x)$	1	$x-\frac{1}{2}$	$x^2-x+\frac{1}{6}$	$x^3-\frac{3}{2}x^2+\frac{x}{2}$	$x^4-2x^3+x^2-\frac{1}{30}$	$x^5-\frac{5}{2}x^4+\frac{5}{3}x^3-\frac{x}{6}$
$E_n(x)$	1	$x-\frac{1}{2}$	x^2-x	$x^3-\frac{3}{2}x^2+\frac{1}{6}$	$x^4-2x^3+\frac{2}{3}x$	$x^5-\frac{5}{2}x^4+\frac{5}{3}x^2-\frac{1}{2}$

Now, using the explicit expressions given in Tables A1 and A2 in the series definitions of the Gould-Hopper-Bernoulli-Bessel polynomials ${}_{H}B^{(m)}p_n(x,y)$, Gould-Hopper-Euler-Bessel polynomials ${}_{H}E^{(m)}p_n(x,y)$, Hermite-Bernoulli-Bessel polynomials ${}_{H}Bp_n(x,y)$ and Hermite-Euler-Bessel polynomials ${}_{H}Ep_n(x,y)$ for $m = 3$ and $n = 5$, the following expressions are obtained:

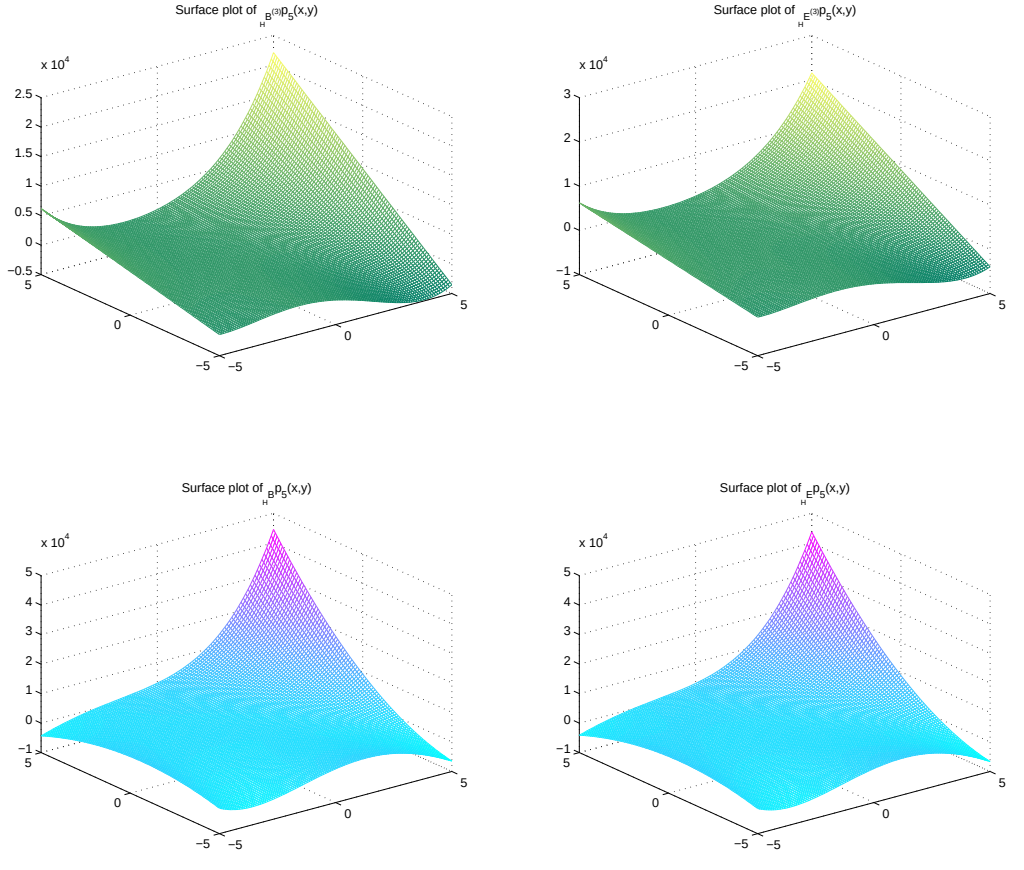
$$ {}_{H}B^{(3)}p_5(x,y) = x^5 + \frac{15}{2}x^4 + \frac{80}{3}x^3 + \frac{95}{2}x^2 + \frac{67}{3}x + 160y + 60x^2y + 180xy - \frac{106}{3}, \tag{3.17} $$

$$ {}_{H}E^{(3)}p_5(x,y) = x^5 + \frac{15}{2}x^4 + 25x^3 + \frac{235}{6}x^2 + \frac{20}{3}x + 150y + 60x^2y + 180xy - \frac{91}{2}, \tag{3.18} $$

$$ {}_{H}Bp_5(x,y) = x^5 + \frac{15}{2}x^4 + \frac{80}{3}x^3 + \frac{95}{2}x^2 + \frac{67}{3}x + 155y + 40xy + 90y^2 + 90x^2y + 60xy^2 + 20x^3y - \frac{106}{3}, \tag{3.19} $$

$$ {}_{H}Ep_5(x,y) = x^5 + \frac{15}{2}x^4 + 25x^3 + \frac{235}{6}x^2 + \frac{20}{3}x + \frac{415}{3}y + 30xy + 90y^2 + 60xy^2 + 90x^2y + 20x^3y - \frac{91}{2}. \tag{3.20} $$

From equations (3.17)-(3.20), making use of MATLAB, the following surface plots are drawn:



Similarly, using the explicit expressions given in Tables A1 and A2 in the series definitions of the truncated exponential-Bernoulli-Bessel polynomials ${}_{e^{(r)}B}p_n(x,y)$ and truncated exponential-Euler-Bessel polynomials ${}_{e^{(r)}E}p_n(x,y)$ for $r=2$ and $n=5$ the following expressions are obtained:

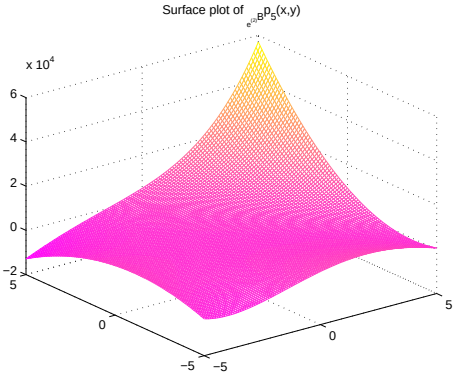
$$ {}_{e^{(2)}B}p_5(x,y) = x^5+\frac{15}{2}x^4+\frac{80}{3}x^3+\frac{95}{2}x^2+\frac{67}{3}x+95y+180y^2+160xy+90x^2y+120xy^2+20x^3y-\frac{106}{3}, \tag{3.21} $$

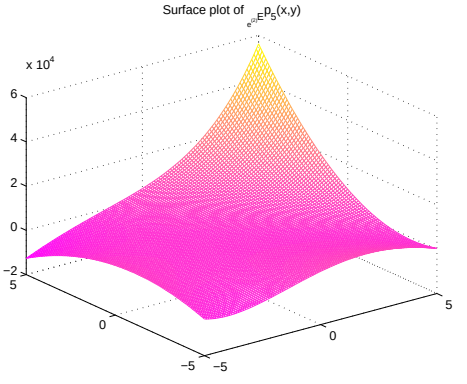
$$ {}_{e^{(2)}E}p_5(x,y) = x^5+\frac{15}{2}x^4+25x^3+40x^2+\frac{20}{3}x+\frac{230}{3}y+180y^2+150xy+90x^2y+120xy^2+20x^3y-\frac{91}{2}. \tag{3.22} $$

Applying MATLAB to equations (3.21) and (3.22), the following surface plots are drawn:

Further, for $n=5$, the following expressions for the Bernoulli-Bessel ${}_Bp_5(x)$ and Euler-Bessel polynomials ${}_Ep_5(x)$ are obtained:

$$ {}_Bp_5(x) = x^5+\frac{15}{2}x^4+\frac{80}{3}x^3+\frac{95}{2}x^2-\frac{68}{3}x-\frac{106}{3} \tag{3.23} $$

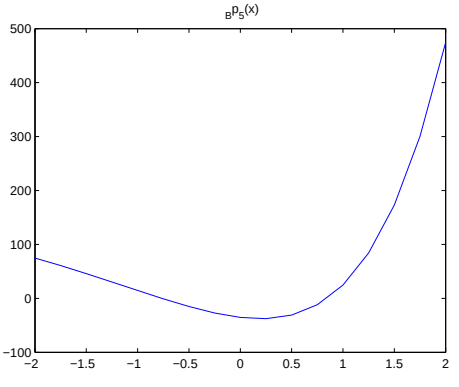


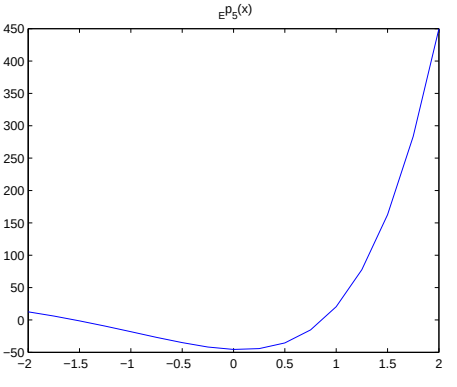


and

$$ {}_{E}p_5(x) = x^5 + \frac{15}{2}x^4 + 25x^3 + \frac{235}{6}x^2 + \frac{20}{3}x - \frac{273}{6}. \tag{3.24} $$

From equations (3.23) and (3.24), the following graphs are drawn:





The special functions are the solutions of differential equations associated with many problems arising in different areas of science and engineering. The properties of new classes of non-trivial special functions proposed in this article can be used to solve the existing as well as new emerging problems in certain branches of science.

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