A Lumped Galerkin finite element method for the generalized Hirota-Satsuma coupled KdV and coupled MKdV equations

Nuri Murat Yagmurlu[1], Berat Karaagac[2] and Alaattin Esen[3]

[1,3]Department of Mathematics, Inonu University, 44280, Malatya, Turkey
[2]Department of Mathematics Education, Adıyaman University, 02040, Adıyaman, Turkey
E-mail: bkaraaagac@adiyaman.edu.tr[2]

Abstract

In the present study, a Lumped Galerkin finite element method using quadratic B-splines has been applied to the generalized Hirota-Satsuma coupled Korteweg de Vries (KdV) and coupled modified Korteweg-de Vries (mKdV) equations. The numerical solutions of discretized equations using Lumped Galerkin finite element method have been obtained using the fourth order Runge-Kutta method which is widely used for the solution of ordinary differential equation system. The numerical solutions obtained for various space and time values have been compared with exact ones using the error norms L_2 and L_∞. Lumped Galerkin finite element method is an effective one which can be applied to a wide range of nonlinear evolution equations.

2010 Mathematics Subject Classification. **65N30**. 65D07, 65L06, 35Q53
Keywords. the generalized Hirota-Satsuma coupled KdV and coupled mKdV, finite element method, Galerkin method, B-spline, Runge-Kutta.

1 Introduction

Nonlinear evolution equations, i.e partial differential equations of which one independent variable is generally time 't' arise in such diverse areas as physics,chemistry, biology, mechanics and material sciences. Because of this reason, obtaining the exact and numerical solutions of nonlinear evolution equations have a key role for those in these fields.

In this study, we have considered the generalized Hirota-satsuma coupled Korteweg de Vries equation introduced by Wu *et. al* [17]

$$\begin{aligned} u_t &= (1/2)\, u_{xxx} - 3uu_x + 3\,(vw)_x\,, \\ v_t &= -v_{xxx} + 3uv_x, \\ w_t &= -w_{xxx} + 3uw_x \end{aligned} \tag{1.1}$$

and coupled modified Korteweg de Vries equation

$$\begin{aligned} u_t &= (1/2)\, u_{xxx} - 3u^2u_x + (3/2)v_{xx} + 3\,(uv)_x - 3\lambda u_x, \\ v_t &= -v_{xxx} - 3vv_x - 3u_xv_x + 3u^2v_x + 3\lambda v_x. \end{aligned} \tag{1.2}$$

In Eq. (1.1), choosing $w = v$ and$w = v^*$ transform Eq. (1.1) into Hirota-Satsuma equation [7] and new complex coupled KdV equation [17], respectively. Additionally, Eq. (1.2) transforms into generalized KdV and mKdV equation by choosing $u = 0$ and $v = 0$, respectively [5]. Lumped Galerkin finite element method have been studied in many papers; among others, Esen and Kutluay [3, 8], and Esen [4] have applied lumped Galerkin method to various problems. Moreover,

Tbilisi Mathematical Journal 12(3) (2019), pp. 159–173.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 07 May 2018.
Accepted for publication: 10 August 2019.

Kutluay and Ucar [9, 10] have solved coupled KdV equation using different base functions. Generalized Hirota-Satsuma coupled KdV and coupled mKdV equations have been widely investigated by several scholars using various methods. For example; Fan [5] has obtained four different soliton solutions for Eqs. (1.1) and (1.2) using extended tanh-function method and symbolic computation. Yu *et al.* [18] have obtained many new solutions including the solitary wave solutions and triangular functional (singly periodic) solutions of the generalized Hirota-Satsuma coupled KdV equation using Jacobi Elliptic Function Method and symbolic computation. Yang and Zhang [19] have obtained exact travelling wave solutions of Eq. (1.1) using coupled Projective Riccati equation. Zayed *et al.* [20] have applied an algebraic method devised to construct new exact soliton solutions for two different types of the generalized Hirota-Satsuma coupled KdV equation. Kaya [11] has used Adomian Decomposition Method (ADM) to obtain solitary wave solutions of the generalized Hirota-Satsuma coupled KdV equation given by Eq. (1.1). Moreover, soliton solutions of Eq. (1.1) have been obtained by Ganji and Rafei [6] with Hes variational iteration method. Abbasbandy [1] has applied Homotopy Analysis Method to Eq. (1.1). Khater *et. al.* [12] have obtained the numerical solutions for several problems including Eqs. (1.1) and (1.2) using Spectral Collocation Method based on Lagrange polynomials. Khattak *et al.* [13] have obtained the numerical solutions of Hirota-Satsuma and Jaulent-Miodek equations using Meshless Collocation Method known as Kansa's Method. Kangalgil and Ayaz [14] have applied Differential Transform Method (DTM) to the generalized Hirota-satsuma coupled KdV and mKdV equations. Guo-Zhong *et al.* [21] have obtained approximate analytic solutions of Eqs. (1.1) and (1.2) by applying Variational Iteration Method (VIM). Arife *et. al.* [2] have obtained the numerical solutions of the generalized Hirota-Satsuma coupled KdV and coupled mKdV equations using Homotopy Analysis Method (HAM). Kumar and Pankaj [15] have used Laplace-Modified Decomposition Method in order to obtain the analytical and numerical solutions of Eq. (1.1).

In this study, Lumped Galerkin finite element method has been successfully applied to the generalized Hirota-Satsuma coupled KdV and coupled mKdV equations given by Eqs. (1.1) and (1.2), respectively. During the application process, element shape functions and weight functions have been chosen as quadratic B-spline bases. Then, the numerical solutions of the newly obtained system of equations are found using the fourth order Runge-Kutta method.

2 Lumped Galerkin finite element formulation for the generalized Hirota-Satsuma coupled KdV equation

In the present section, lumped Galerkin finite element formulation has been constructed for the generalized Hirota-Satsuma coupled KdV equation given by Eq. (1.1) with the help of quadratic B-spline basis functions. Let us consider the generalized Hirota-Satsuma coupled KdV equation [5,13] given in Eq. (1.1) with the boundary conditions

$$\begin{aligned} &u(L_1,t) = u(L_2,t) = f_1(t), \qquad && v(L_1,t) = v(L_2,t) = f_2(t), \\ &u_x(L_1,t) = u_x(L_2,t) = f_1^*(t), \qquad && v_x(L_1,t) = v_x(L_2,t) = f_2^*(t), \\ &\qquad\qquad w(L_1,t) = w(L_2,t) = f_3(t) \\ &\qquad\qquad w_x(L_1,t) = w_x(L_2,t) = f_3^*(t) \end{aligned} \tag{2.1}$$

and the initial conditions

$$u(x,0)=g_1(x), \qquad v(x,0)=g_2(x), \qquad w(x,0)=g_3(x) \tag{2.2}$$

where f_i, f_i^* and g_i are sufficiently differentiable and known functions for $i=1,2,3$ and subscript x denotes the partial derivatives with respect to space. The solution domain of the problem is interval $[L_1,L_2]$ such that $x_0=L_1$, $x_N=L_2$. The interval has been divided into N equal subintervals such that $h=x_{i+1}-x_i$. Each subinterval $[x_i,x_{i+1}]$ is called an element. In this manner, the interval is divided into N finite elements by nodal points represented by $\{x_i\}_{i=0}^{N}$. Let us assume that $W(x)$ defines weight function, then the weak form of the generalized Hirota-Satsuma coupled KdV equation given by Eq. (1.1) on the interval $[L_1,L_2]$ becomes

$$\begin{aligned}
&\int_{L_1}^{L_2}(Wu_t+(1/2)W_xu_{xx}+3Wuu_x-3Wvw_x-3Wwv_x)\,dx=(1/2)Wu_{xx}|_{L_1}^{L_2},\\
&\int_{L_1}^{L_2}(Wv_t-W_xv_{xx}-3Wuv_x)\,dx=-Wv_{xx}|_{L_1}^{L_2},\\
&\int_{L_1}^{L_2}(Ww_t-W_xw_{xx}-3Wuw_x)\,dx=-Ww_{xx}|_{L_1}^{L_2}
\end{aligned} \tag{2.3}$$

and the weak form of a typical element on $[x_i,x_{i+1}]$ given by Eq. (1.1) becomes

$$\begin{aligned}
&\int_{x_i}^{x_{i+1}}(Wu_t+(1/2)W_xu_{xx}+3Wuu_x-3Wvw_x-3Wwv_x)\,dx=(1/2)Wu_{xx}|_{x_i}^{x_{i+1}},\\
&\int_{x_i}^{x_{i+1}}(Wv_t-W_xv_{xx}-3Wuv_x)\,dx=-Wv_{xx}|_{x_i}^{x_{i+1}},\\
&\int_{x_i}^{x_{i+1}}(Ww_t-W_xw_{xx}-3Wuw_x)\,dx=-Ww_{xx}|_{x_i}^{x_{i+1}}.
\end{aligned} \tag{2.4}$$

The approximate solutions $U_N(x,t)$, $V_N(x,t)$ and $W_N(x,t)$ of the problem corresponding to exact solutions $u(x,t)$, $v(x,t)$ and $w(x,t)$, respectively, can also be written as follows on the whole domain in terms of quadratic B-spline base functions given in Ref. [16]

$$\begin{gathered}
U_N(x,t)=\sum_{j=-1}^{N}\delta_j(t)\varphi_j(x), \qquad V_N(x,t)=\sum_{j=-1}^{N}\sigma_j(t)\varphi_j(x),\\
W_N(x,t)=\sum_{j=-1}^{N}\tau_j(t)\varphi_j(x).
\end{gathered} \tag{2.5}$$

If local coordinate transformation $\xi=x-x_i$, $0\le\xi\le h$ is applied to quadratic B-spline base functions, we obtain

$$\varphi_{m-1}=\tfrac{1}{h^2}(1-\xi)^2, \qquad \varphi_m=\tfrac{1}{h^2}\left(h^2+2h\xi-2\xi^2\right), \qquad \varphi_{m+1}=\tfrac{1}{h^2}\xi^2. \tag{2.6}$$

The approximate solutions given by Eq. (2.5) on a typical element $[x_i,x_{i+1}]$ can be stated in terms of the basis given in Eq. (2.6) as follows

$$U_N(\xi,t)=\sum_{j=m-1}^{m+1}\delta_j(t)\varphi_j(\xi),\qquad V_N(\xi,t)=\sum_{j=-1}^{N}\sigma_j(t)\varphi_j(\xi),$$
$$W_N(\xi,t)=\sum_{j=m-1}^{m+1}\tau_j(t)\varphi_j(\xi). \tag{2.7}$$

Since lumped Galerkin finite element method has been applied, we take $W_i=\varphi_j$ for $i,j=m-1,m,m+1$, and put the following equations into the weak form of the problem given by Eq. (2.4)

$$zm=3u,\qquad \widetilde{zm}=3v,\qquad gm=3w. \tag{2.8}$$

Together with the approximate solutions given by Eq. (2.7), we obtain the following equations

$$\begin{aligned}
&\sum_{j=m-1}^{m+1}\left\{\left(\int_0^h\varphi_i^e\varphi_j^e d\xi\right)\dot{\delta}_j^e+\tfrac{1}{2}\left(\int_0^h(\varphi_i^e)'\left(\varphi_j^e\right)'d\xi\right)\delta_j^e+zm\left(\int_0^h(\varphi_i^e)\left(\varphi_j^e\right)'d\xi\right)\delta_j^e\right.\\
&\qquad\left.-gm\left(\int_0^h(\varphi_i^e)\left(\varphi_j^e\right)'d\xi\right)\sigma_j^e-\widetilde{zm}\left(\int_0^h(\varphi_i^e)\left(\varphi_j^e\right)'d\xi\right)\tau_j^e\right\}\\
&\qquad=\sum_{j=m-1}^{m+1}\tfrac{1}{2}\left\{(\varphi_i^e)\left(\varphi_j^e\right)''\delta_j^e\right\}_0^h,\\
&\sum_{j=m-1}^{m+1}\left\{\left(\int_0^h\varphi_i^e\varphi_j^e d\xi\right)\dot{\sigma}_j^e-\left(\int_0^h(\varphi_i^e)'\left(\varphi_j^e\right)''d\xi\right)\sigma_j^e-zm\left(\int_0^h(\varphi_i^e)\left(\varphi_j^e\right)'d\xi\right)\sigma_j^e\right\}\\
&\qquad=-\sum_{j=m-1}^{m+1}\left\{(\varphi_i^e)\left(\varphi_j^e\right)''\sigma_j^e\right\}_0^h,\\
&\sum_{j=m-1}^{m+1}\left\{\left(\int_0^h\varphi_i^e\varphi_j^e d\xi\right)\dot{\tau}_j^e-\left(\int_0^h(\varphi_i^e)'\left(\varphi_j^e\right)''d\xi\right)\tau_j^e-zm\left(\int_0^h(\varphi_i^e)\left(\varphi_j^e\right)'d\xi\right)\tau_j^e\right\}\\
&\qquad=-\sum_{j=m-1}^{m+1}\left\{(\varphi_i^e)\left(\varphi_j^e\right)''\tau_j^e\right\}_0^h.
\end{aligned} \tag{2.9}$$

Here zm, $\widetilde{zm}$ and gm are lumped parameters and are used for the linearization of the terms $3uu_x, 3vw_x$ and $3wv_x$, respectively. When $u=(u_m+u_{m+1})/2$, $v=(v_m+v_{m+1})/2$ and $w=(w_m+w_{m+1})/2$ are taken, the values of the parameters zm, $\widetilde{zm}$ and gm at the nodal points x_m are obtained as follows

$$\begin{aligned}
zm&=3\left(\delta_{m-1}+2\delta_m+\delta_{m+1}\right)/2,\\
\widetilde{zm}&=3\left(\sigma_{m-1}+2\sigma_m+\sigma_{m+1}\right)/2,\\
gm&=3\left(\tau_{m-1}+2\tau_m+\tau_{m+1}\right)/2.
\end{aligned}$$

After some calculations in the system (2.9), the element matrices in the system become

$$\begin{aligned}
A_{ij}^e&=\int_0^h\varphi_i^e\varphi_j^e d\xi,\qquad B_{ij}^e=\int_0^h(\varphi_i^e)'(\varphi_j^e)''d\xi,\\
C_{ij}^e&=\int_0^h\varphi_i^e(\varphi_j^e)'d\xi,\quad D_{ij}^e=\left\{\varphi_i^e(\varphi_j^e)''\right\}_0^h
\end{aligned} \tag{2.10}$$

where

$$A_{ij}^{e}=\frac{h}{30}\begin{bmatrix}6&13&1\\13&54&13\\1&13&6\end{bmatrix},\qquad B_{ij}^{e}=\frac{2}{h^{2}}\begin{bmatrix}-1&2&-1\\0&0&0\\1&-2&1\end{bmatrix},$$
$$C_{ij}^{e}=\frac{1}{6}\begin{bmatrix}-3&2&1\\-8&0&8\\-1&-2&3\end{bmatrix},\qquad D_{ij}^{e}=\frac{2}{h^{2}}\begin{bmatrix}-1&2&-1\\0&0&0\\1&-2&1\end{bmatrix}. \tag{2.11}$$

The system (2.9) can be expressed in matrix format as follows

$$\begin{aligned}&A^{e}\dot{\delta}^{e}+\left(\left(1/2\right)\left(B^{e}-D^{e}\right)+zmC^{e}\right)\delta^{e}-gmC^{e}\sigma^{e}-\widetilde{zm}C^{e}\tau^{e}=0,\\&A^{e}\sigma^{e}+\left(-B^{e}+D^{e}-zmC^{e}\right)\sigma^{e}=0,\\&A^{e}\tau^{e}+\left(-B^{e}+D^{e}-zmC^{e}\right)\tau^{e}=0\end{aligned}$$

where element parameters $\delta^{e}=\left(\delta_{m-1}^{e},\delta_{m}^{e},\delta_{m+1}^{e}\right)^{T}$, $\sigma^{e}=\left(\sigma_{m-1}^{e},\sigma_{m}^{e},\sigma_{m+1}^{e}\right)^{T}$ and $\tau^{e}=\left(\tau_{m-1}^{e},\tau_{m}^{e},\tau_{m+1}^{e}\right)^{T}$ are defined on a typical element $[x_i,x_{i+1}]$ and A^e, B^e,D^e and C^e are element matrices. If we combine all of the elements over the domain, the following equations

$$\begin{aligned}&\dot{\delta}=-A^{-1}\left(\left(1/2\right)\left(B-D\right)+C1\left(zm\right)\right)\delta-C2\left(\widetilde{zm}\right)\sigma-C3\left(gm\right)\tau,\\&\dot{\sigma}=-A^{-1}\left(-B+D-C1\left(zm\right)\right)\sigma,\\&\dot{\tau}=-A^{-1}\left(-B+D-C1\left(zm\right)\right)\tau\end{aligned} \tag{2.12}$$

are obtained. Here $\delta=\left(\delta_{-1},\ \delta_{0},...,\delta_{N-1},\ \delta_{N}\right)^{T}$, $\sigma=\left(\sigma_{-1},\ \sigma_{0},...,\sigma_{N-1},\ \sigma_{N}\right)^{T}$ and $\tau=\left(\tau_{-1},\tau_{0},...,\tau_{N-1},\ \tau_{N}\right)^{T}$ are shape parameters and A, B, $C1\left(zm\right)$, $C2\left(\widetilde{zm}\right)$ and $C3\left(gm\right)$ are matrices defined on the whole domain. The generalized rows of the obtained matrices are given below;

$$\begin{aligned}&A:\tfrac{h}{30}\left(1,26,66,26,1\right),\\&B:\tfrac{2}{h^{2}}\left(1,-2,0,2,-1\right),\\&C:\tfrac{1}{6}\left(-1,-10,0,10,1\right),\\&D:\tfrac{2}{h^{2}}\left(1,-2,0,2,-1\right),\\&C1\left(z_m\right):\tfrac{1}{6}\left(-zm_0,-2zm_0-8zm_1,3zm_0-3zm_2,8zm_1+2zm_2,zm_2\right),\\&C2\left(\widetilde{zm}\right):\tfrac{1}{6}\left(-\widetilde{zm}_0,-2\widetilde{zm}_0-8\widetilde{zm}_1,3\widetilde{zm}_0-3\widetilde{zm}_2,8\widetilde{zm}_1+2\widetilde{zm}_2,\widetilde{zm}_2\right),\\&C3\left(gm\right):\tfrac{1}{6}\left(-gm_0,-2gm_0-8gm_1,3gm_0-3gm_2,8gm_1+2gm_2,gm_2\right).\end{aligned}$$

For $m=0\left(1\right)N$, we obtain

$$\begin{aligned}&zm_0=3\left(\delta_{m-1}+2\delta_m+\delta_{m+1}\right)/2,\qquad &&\widetilde{zm}_0=3\left(\sigma_{m-1}+2\sigma_m+\sigma_{m+1}\right)/2,\\&zm_1=3\left(\delta_m+2\delta_{m+1}+\delta_{m+2}\right)/2,\qquad &&\widetilde{zm}_1=3\left(\sigma_m+2\sigma_{m+1}+\sigma_{m+2}\right)/2,\\&zm_2=3\left(\delta_{m+1}+2\delta_{m+2}+\delta_{m+3}\right)/2,\qquad &&\widetilde{zm}_2=3\left(\sigma_{m+1}+2\sigma_{m+2}+\sigma_{m+3}\right)/2\end{aligned}$$

$$\begin{aligned}&gm_0=3\left(\tau_{m-1}+2\tau_m+\tau_{m+1}\right)/2,\\&gm_1=3\left(\tau_m+2\tau_{m+1}+\tau_{m+2}\right)/2,\\&gm_2=3\left(\tau_{m+1}+2\tau_{m+2}+\tau_{m+3}\right)/2.\end{aligned}$$

In the end of all these computations, a system of equations consisting of $(3N+6)$ equations and $(3N+6)$ unknowns is obtained. Before obtaining the values of parameters $\delta(t), \sigma(t)$ and $\tau(t)$ and using them in approximate solution forms in Eq. (2.5), boundary conditions are applied as a fundamental step of finite elements method. Using boundary conditions

$$\begin{aligned} u(x_m,t) &= U_N(x_m,t) = \delta_{m-1} + \delta_m, \\ v(x_m,t) &= V_N(x_m,t) = \sigma_{m-1} + \sigma_m, \\ w(x_{mj},t) &= W_N(x_m,t) = \tau_{m-1} + \tau_m \end{aligned}$$

for $m = 0$; δ_{-1}, σ_{-1} and τ_{-1} are obtained in terms of δ_0, σ_0 and τ_0, and for $m = N$, δ_N, σ_N and τ_N are obtained in terms of $\delta_{N-1}, \sigma_{N-1}$ and τ_{N-1}. After applying boundary conditions, the system (2.12) transforms into an $(3N \times 3N)$ system. With the help of this system, the parameters $\delta^{n+1}, \sigma^{n+1}$ and τ^{n+1} are obtained successively in terms of the parameters δ^n, σ^n and τ^n using the fourth order Runge-Kutta method. To start the iteration, the values of the parameters δ^0, σ^0 and τ^0 should be obtained. Initial values of the parameters are obtained using the following equations

$$\begin{aligned} u(x_j,0) = U_N(x_j,0) = \sum_{j=-1}^{N} \delta_j^0 \varphi_j, \quad v(x_j,0) = V_N(x_j,0) = \sum_{j=-1}^{N} \sigma_j^0 \varphi_j, \\ w(x_j,0) = W_N(x_j,0) = \sum_{j=-1}^{N} \tau_j^0 \varphi_j. \end{aligned}$$

Since this new system consists of $(3N+3)$ equations and $(3N+6)$ unknowns, for this system to be solvable, three auxiliary equations should be added. This is achieved using the derivatives of initial conditions given as follows

$$\begin{aligned} u'(x_m) = U_N'(x_m) = \tfrac{2}{h}(\delta_m - \delta_{m-1}), \quad v'(x_m) = V_N'(x_m) = \tfrac{2}{h}(\sigma_m - \sigma_{m-1}), \\ w'(x_m) = W_N'(x_m) = \tfrac{2}{h}(\tau_m - \tau_{m-1}) \end{aligned}$$

and result in $(3N+6) \times (3N+6)$ type system. Solving this system of equations using a known method, the values of parameters δ^0, σ^0 and τ^0 obtained.

3 Lumped Galerkin finite element formulation for the generalized Hirota-Satsuma coupled modified KdV equation

In this section, a lumped Galerkin finite element formulation is formed for the generalized Hirota-Satsuma coupled modified KdV equation given by Eq. (1.2). While applying the method, again quadratic B-spline base functions are used and the newly obtained system is solved by the fourth order Runge-Kutta method. The model problem given in (1.2) with the boundary conditions

$$\begin{aligned} u(L_1,t) = u(L_2,t) = \psi_1(t), \qquad v(L_1,t) = v(L_2,t) = \psi_2(t) \\ u_x(L_1,t) = u_x(L_2,t) = \psi_1^*(t), \quad v_x(L_1,t) = v_x(L_2,t) = \psi_2^*(t) \end{aligned} \tag{3.1}$$

and the initial conditions

$$u(x,0) = \varphi_1(x), \quad v(x,0) = \varphi_2(x) \tag{3.2}$$

where ψ_i, ψ_i^* and φ_i are sufficiently differentiable and known functions for $i = 1, 2$. As in the previous section, the solution interval of the problem $[L_1, L_2]$ is divided into N

finite elements and $(N+1)$ nodal points are obtained. Then the weak formulation of the problem given by Eq. (1.2) has been obtained over a typical element $[x_i, x_{i+1}]$ by taking the weight function as $W(x)$ as follows

$$\begin{aligned}
&\int_{x_i}^{x_{i+1}} \left(W u_t + (1/2) W_x u_{xx} + 3W u^2 u_x + (3/2) W_x v_x - 3W u v_x \right. \\
&\qquad \left. -3W v u_x + 3\lambda W u_x\right) dx = (1/2) W u_{xx}\big|_{x_i}^{x_{i+1}} - (3/2) W v_x\big|_{x_i}^{x_{i+1}} \\
&\int_{x_i}^{x_{i+1}} \left(W v_t - W_x v_{xx} + 3W v v_x + 3W u_x v_x \right. \\
&\qquad \left. -3W u^2 v_x - 3\lambda W v_x\right) dx = -W v_{xx}\big|_{x_i}^{x_{i+1}} .
\end{aligned} \tag{3.3}$$

The approximate solutions $U_N(x,t)$ and $V_N(x,t)$ corresponding to exact ones $u(x,t)$ and $v(x,t)$, respectively. Using local transformation $\xi = x - x_i$, $0 \le \xi \le h$ and base functions given in Eq. (2.6), the approximations on a typical element $[x_i, x_{i+1}]$ are found as follows

$$U_N = \sum_{j=m-1}^{m+1} \delta_j^e \varphi_j^e, \qquad V_N = \sum_{j=m-1}^{m+1} \sigma_j^e \varphi_j^e. \tag{3.4}$$

In order to apply Lumped Galerkin finite element method, if we choose $W_i = \varphi_j$, $tm = 3u^2$, $zm = 3u$, $\widetilde{zm} = 3v$ and $\widetilde{gm} = 3u_x$ in Eq. (3.3) and use these approximations in Eq. (3.3), the following system of equation is obtained

$$\begin{aligned}
&\sum_{j=m-1}^{m+1} \left\{ \left(\int_0^h \varphi_i^e \varphi_j^e d\xi \right) \dot{\delta}_j^e + \tfrac{1}{2} \left(\int_0^h (\varphi_i^e)' (\varphi_j^e)'' d\xi \right) \delta_j^e + tm \left(\int_0^h (\varphi_i^e) (\varphi_j^e)' d\xi \right) \delta_j^e \right. \\
&\quad + \tfrac{3}{2} \left(\int_0^h (\varphi_i^e)' (\varphi_j^e)' d\xi \right) \sigma_j^e - zm \left(\int_0^h (\varphi_i^e) (\varphi_j^e)' d\xi \right) \sigma_j^e - \widetilde{zm} \left(\int_0^h (\varphi_i^e) (\varphi_j^e)' d\xi \right) \delta_j^e \\
&\quad \left. + 3\lambda \left(\int_0^h (\varphi_i^e) (\varphi_j^e)' d\xi \right) \delta_j^e \right\} = \sum_{j=m-1}^{m+1} \left\{ \tfrac{1}{2} (\varphi_i^e) (\varphi_j^e)'' \delta_j^e + \tfrac{3}{2} (\varphi_i^e) (\varphi_j^e)' \sigma_j^e \right\}_0^h ,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&\sum_{j=m-1}^{m+1} \left\{ \left(\int_0^h \varphi_i^e \varphi_j^e d\xi \right) \dot{\sigma}_j^e - \left(\int_0^h (\varphi_i^e)' (\varphi_j^e)'' d\xi \right) \sigma_j^e \right. \\
&\quad \left. + (\widetilde{zm} + \widetilde{gm} - tm - 3\lambda) \left(\int_0^h (\varphi_i^e) (\varphi_j^e)' d\xi \right) \sigma_j^e \right\} = - \sum_{j=m-1}^{m+1} \left\{ (\varphi_i^e) (\varphi_j^e)'' \sigma_j^e \right\}_0^h .
\end{aligned}$$

Here $tm, zm, \widetilde{zm}$ and $\widetilde{gm}$ are lumped parameters and are used in the linearizations of the terms $3u^2u_x, 3uv_x$,$3vu_x$ and $3u_xv_x$, respectively. By choosing $u = (u_m + u_{m+1})/2$, $v = (v_m + v_{m+1})/2$ and $u_x = ((u_m)_x + (u_{m+1})_x)/2$ the values of the parameters tm, zm, $\widetilde{zm}$ and $\widetilde{gm}$ at the nodal points x_m become

$$\begin{aligned}
tm &= \tfrac{3}{4} (\delta_{m-1} + 2\delta_m + \delta_{m+1})^2 , \quad zm = \tfrac{3}{2} (\delta_{m-1} + 2\delta_m + \delta_{m+1}) , \\
\widetilde{zm} &= \tfrac{3}{2} (\sigma_{m-1} + 2\sigma_m + \sigma_{m+1}) , \quad \widetilde{gm} = \tfrac{6}{h} (\delta_{m+1} - \delta_{m-1}) .
\end{aligned}$$

If the matrices in Eq. (3.5) are computed, in addition to matrices in (2.10), the following matrices are also obtained

$$E_{ij}^e = \int_0^h (\varphi_i^e)' (\varphi_j^e)' d\xi = \tfrac{2}{3h} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad F_{ij}^e = \left\{ \varphi_i^e (\varphi_j^e)' \right\}_0^h = \tfrac{2}{h} \begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & 1 \end{bmatrix}. \tag{3.6}$$

Using these matrices, the system given by Eq. (3.5) on a typical element $[x_i, x_{i+1}]$ is given by

$$A^e\dot{\delta}^e + \left(\tfrac{1}{2}\left(B^e - D^e\right) + \left(tm - \widetilde{zm} + 3\lambda\right)C^e\right)\delta^e + \left(\tfrac{3}{2}\left(E^e - F^e\right) - zmC^e\right)\sigma^e = 0,$$

$$A^e\dot{\sigma}^e + \left(-B^e + D^e + \left(-\widetilde{zm} + \widetilde{gm} - tm - 3\lambda\right)C^e\right)\sigma^e = 0.$$

Here $\delta^e = \left(\delta^e_{m-1},\ \delta^e_m,\ \delta^e_{m+1}\right)^T$ and $\sigma = \left(\sigma^e_{m-1},\ \sigma^e_m,\ \sigma^e_{m+1}\right)^T$ are element parameters on a typical element and $A^e_{ij}, B^e_{ij}, C^e_{ij}, D^e_{ij}, E^e_{ij}$ and F^e_{ij} are element matrices. After assembling all elements on the solution domain, the following system is obtained

$$\begin{aligned}\dot{\delta} &= -A^{-1}\left[\left(\tfrac{1}{2}\left(B - D\right) + C4\left(tm\right) - C2\left(\widetilde{zm}\right) + 3\lambda C\right)\delta + \left(\tfrac{3}{2}\left(E - F\right) - C1\left(zm\right)\right)\sigma\right],\\ \dot{\sigma} &= -A^{-1}\left(-B + D - C2\left(\widetilde{zm}\right) + C5\left(\widetilde{gm}\right) - C4\left(tm\right) - 3\lambda C\right)\sigma.\end{aligned} \tag{3.7}$$

The generalized rows of matrices E, F, $C4$ and $C5$ are found to be

$$\begin{aligned}&E : \tfrac{2}{3h}\left(-1, -2, 6, -2, -1\right),\\ &F : \tfrac{2}{h}\left(0, 0, 0, 0, 0\right),\\ &C4\left(t_m\right) : \tfrac{1}{6}\left(-tm_0, -2tm_0 - 8tm_1, 3tm_0 - 3tm_2, 8tm_1 + 2tm_2, tm_2\right),\\ &C5\left(\widetilde{gm}\right) : \tfrac{1}{6}\left(-\widetilde{gm}_0, -2\widetilde{gm}_0 - 8\widetilde{gm}_1, 3\widetilde{gm}_0 - 3\widetilde{gm}_2, 8\widetilde{gm}_1 + 2\widetilde{gm}_2, \widetilde{gm}_2\right).\end{aligned}$$

Let us again state that here $\delta = \left(\delta_{-1},\ \delta_0, ..., \delta_{N-1},\ \delta_N\right)^T$ and $\sigma = \left(\sigma_{-1},\ \sigma_0, ..., \sigma_{N-1},\ \sigma_N\right)^T$ are element parameters to be found on the entire domain and A, B, C, D, E and F are element matrices. Thus with the system (3.7) a matrix system in the type of $(2N+4)\times(2N+4)$ is obtained. Applying the following boundary conditions

$$\begin{aligned}&u\left(x_0, t\right) = U_N\left(x_0, t\right) = \delta_{-1} + \delta_0, \qquad v\left(x_0, t\right) = V_N\left(x_0, t\right) = \sigma_{-1} + \sigma_0,\\ &u\left(x_N, t\right) = U_N\left(x_N, t\right) = \delta_{N-1} + \delta_N, \quad v\left(x_N, t\right) = V_N\left(x_N, t\right) = \sigma_{N-1} + \sigma_N\end{aligned}$$

parameters δ_{-1}, σ_{-1} and δ_N, σ_N are eliminated from the system (3.7), thus converting it into $(2N)\times(2N)$ type system. Therefore, the values of δ^{n+1} and σ^{n+1} can be found using the fourth order Runge-Kutta method and the values of parameters δ^n and σ^n iteratively. To start iterative procedure, the required initial values are obtained from the initial conditions of the problem

$$u\left(x_j, 0\right) = U_N\left(x_j, 0\right) = \sum_{j=-1}^{N}\delta_j^0\varphi_j, \quad v\left(x_j, 0\right) = V_N\left(x_j, 0\right) = \sum_{j=-1}^{N}\sigma_j^0\varphi_j \tag{3.8}$$

just as in the previous section. The systems obtained from the Eq. (3.8) consists of $(2N+2)$ equations and $(2N+4)$ unknowns. For the system to be solvable, we use the derivatives of the initial conditions in order to add two equations to the system

$$u'\left(x_m\right) = U'_N\left(x_m\right) = \tfrac{2}{h}\left(\delta_m - \delta_{m-1}\right), \quad v'\left(x_m\right) = V'_N\left(x_m\right) = \tfrac{2}{h}\left(\sigma_m - \sigma_{m-1}\right)$$

The newly obtained system is of the form $(2N+2)\times(2N+2)$. Solving this system results in δ^0 and σ^0, and using these values iteration process can be initiated.

TABLE 1. The error norms L_2 and L_∞ for $u(x,t)$ with the initial conditions IC_1 for $c_0 = c_1 = \beta = 0.5, k = 0.1$.

Δx		0.1		0.5		1	
Δt	t	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
0.001	1	0.1312234	0.0427866	0.1281391	0.0418264	0.1185364	0.0382687
0.01		0.1312236	0.0427839	0.1281627	0.0411626	0.1185615	0.0376259
0.001	2.5	0.3365496	0.1153305	0.3290871	0.1126634	0.3059166	0.1051623
0.01		0.3365498	0.1153353	0.3290878	0.1126641	0.3059172	0.1051626
0.001	5	0.7140327	0.2525354	0.7004935	0.2479466	0.6586975	0.2335403
0.01		0.7140331	0.2525354	0.7004949	0.2479476	0.6586985	0.2335409

4 Numerical results

In this section, the numerical solutions of the generalized Hirota-Satsuma coupled KdV and coupled modified KdV equations for various space and time steps have been obtained. The accuracy and efficiency of the used method are tested using the following error norms known as L_2 and L_∞ and given in the literatures follows

$$L_2 = \sqrt{h\sum_{j=0}^{N}\left|u_j - (U_N)_j\right|^2}, \qquad L_\infty = \|u - U_N\|_\infty = \max_{0\le j\le N}|u_j - (U_N)_j| \tag{4.1}$$

respectively.

4.1 Numerical results for the generalized Hirota-Satsuma coupled KdV equation

As a numerical example, we consider the generalized Hirota-Satsuma coupled KdV equation given in Eq.(1.1) with exact bell-type solutions for u, kink-type solutions for v and w [5, 13, 21]

$$\begin{aligned} u(x,t) &= (1/3)\left(\beta - 2k^2\right) + 2k^2\tanh^2\left(k\left(x+\beta t\right)\right), \\ v(x,t) &= -\left(4k^2c_0\left(\beta+k^2\right)/3c_1^2\right) + \left(4k^2\left(\beta+k^2\right)/3c_1\right)\tanh\left(k\left(x+\beta t\right)\right), \\ w(x,t) &= c_0 + c_1\tanh\left(k\left(x+\beta t\right)\right). \end{aligned} \tag{4.2}$$

The g_i functions seen at the initial conditions of the Eq. (1.1) are obtained by taking $t = 0$ in exact solutions given in Eqs. (4.2). In our study, these boundary conditions are denoted as IC_1. Here β, c_0, c_1, and k are arbitrary constants. The solution interval of the problem is taken as $x \in [-40, 40]$ and the constants $c_0 = c_1 = \beta = 0.5$, $k = 0.1$. First of all, the numerical solutions of u, v and w for various space steps Δx and time steps Δt at final times $t =$ 1,2.5 and 5 are computed and the error norms of these numerical solutions for $u(x,t)$ are presented in Table 1. As it can be seen from the table, an increase in step size Δx results in a decrease in the error norms L_2 and L_∞ for $u(x,t)$. Also, with the decreasing values of time step Δt, the error norm L_2 has decreased but the error norm L_∞ has decreased generally. And the numerical solutions of $v(x,t)$ and $w(x,t)$ also exhibit the same behaviour with $u(x,t)$.

In Table 2, the numerical solutions of Ref. [13] and those obtained using Galerkin finite element method are compared. As it can be seen from the table, in the study of Ref. [13], more converging numerical solutions have been obtained by various selections of constant c. But in the Galerkin finite element method according to the selection of constant c, we have obtained more converging

TABLE 2. A comparison of the error norms $L_\infty \times 10^3$ for initial condition IC_1.

				[13]			Galerkin		
t	h	Δt	c	$u(x,t)$	$v(x,t)$	$w(x,t)$	$u(x,t)$	$v(x,t)$	$w(x,t)$
2.5	1	0.01	0.005	0.12723	0.07341	2.6989	0.1051626	0.0139917	0.0135262
5	1	0.01	0.005	0.27944	0.14712	5.4090	0.2335409	0.0391992	0.0334201
2.5	0.5	0.01	0.005	0.11790	0.06940	2.5516	0.1126641	0.0126608	0.0117784
5	0.5	0.01	0.005	0.25904	0.14029	5.1580	0.2479476	0.0321426	0.0291231
2.5	1	0.001	0.005	0.12748	0.07341	0.2699	0.1051623	0.0139931	0.0135949
5	1	0.001	0.005	0.27969	0.14713	5.4092	0.2335403	0.0392007	0.0334541
2.5	0.5	0.001	0.005	0.11814	0.06937	2.5504	0.1126634	0.0126596	0.0117979
5	0.5	0.001	0.005	0.25929	0.14023	5.1557	0.2479466	0.0321436	0.0291564
2.5	0.2	0.001	0.005	0.11559	0.06828	0.2510	0.1150315	0.0105834	0.0104451
5	0.2	0.001	0.005	0.25313	0.13767	5.0617	0.2519611	0.0274180	0.0265197
		0.01	0.005	0.25289	0.13772	5.0635	0.2519632	0.0274165	0.0264914

results than those in Ref. [13]. At the end, the simulation of numerical solutions of u, v and w for $\Delta x = 0.5, \Delta t = 0.01$ and $t = 5$ are shown in Fig.1.

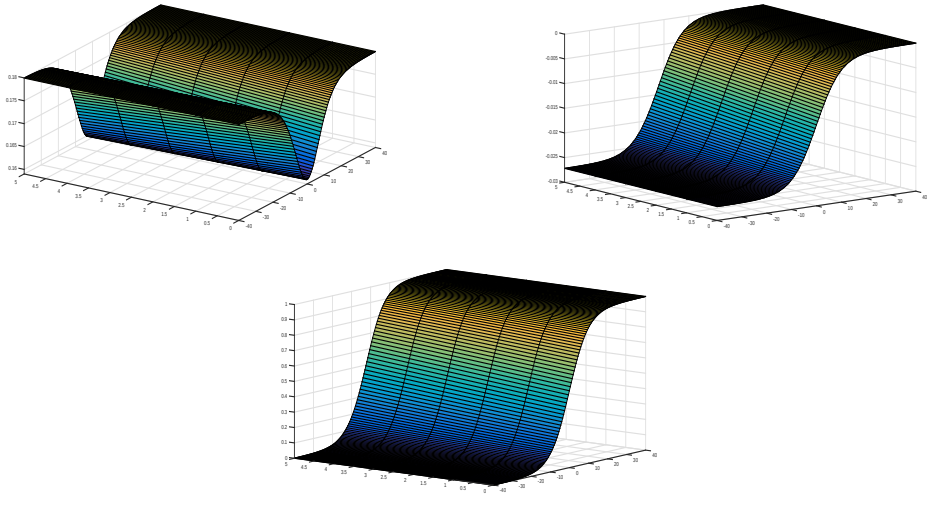


FIGURE 1. The simulation of numerical results for $u(x,t)$, $v(x,t)$ and $w(x,t)$.

TABLE 3. The error norms L_2 and L_∞ for $u(x,t)$ for $\lambda = 1.5, b = 0.1, k = 0.1$.

		$t = 0.5$		$t = 1$	
Δx	Δt	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
	0.001	0.1678883	0.1374688	0.3354814	0.1720950
0.2	0.01	0.1692813	0.1496138	0.3353346	0.1826120
	0.1	0.2002517	0.2777804	0.4252583	0.4857331
	0.001	0.1623373	0.0799344	0.3361460	0.1384881
0.5	0.01	0.1625489	0.0808101	0.3366527	0.1417296
	0.1	0.1651594	0.0901660	0.3435334	0.1752064
	0.001	0.1575962	0.0531536	0.3326967	0.1159666
0.75	0.01	0.1576365	0.0531556	0.3329109	0.1159597
	0.1	0.1582942	0.0561025	0.3355222	0.1191042
	0.001	0.0159183	0.0140368	0.3282989	0.1158673
1	0.01	0.1544672	0.0525622	0.3283975	0.1158817
	0.1	0.1545785	0.0525691	0.3294763	0.1159031

4.2 Numerical results for the generalized Hirota-Satsuma coupled modified KdV equation

The generalized Hirota-Satsuma coupled mKdV equation has been considered for $\lambda \neq 0$, as kink-2 type for u and v

$$\begin{aligned} u(x,t) &= \tfrac{b_1}{2k} + k \tanh\left(k\left(x + 0.25\left(-4k^2 - 6\lambda + \tfrac{6k\lambda}{b_1} + \tfrac{3b_1^2}{k^2}\right)t\right)\right), \\ v(x,t) &= \tfrac{\lambda}{2}\left(1 + \tfrac{k}{b_1}\right) + b_1 \tanh\left(k\left(x + \tfrac{1}{4}\left(-4k^2 - 6\lambda + \tfrac{6k\lambda}{b_1} + \tfrac{3b_1^2}{k^2}\right)t\right)\right). \end{aligned} \tag{4.3}$$

Here k and b_1 are arbitrary constants and $\lambda = 1.5$ is chosen for kink-2 type solution. The initial conditions for the problem are obtained by taking $t = 0$ at exact solutions in Eqs. (4.3) and denoted IC_2. In the solution process,first of all, the problem considered using various values of space and time steps for $t = 0.5$ and $t = 1$ over the interval $x \in [-30, 30]$. The numerical results for u and v by choosing $\lambda = 1.5$, $k = b_1 = 0.1$ and $\lambda = 1.5$, $k = b_1 = 0.05$ are obtained. Also the error norms L_2 and L_∞ for $u(x,t)$ are presented in Tables 3 and 4, respectively. As it is seen from the Tables 3 and 4, increasing time steps Δt result in increasing the error norms L_2 and L_∞ while increasing space steps Δx result in decreasing the error norms L_2 and L_∞ for $u(x,t)$. The numerical solutions of $v(x,t)$ exhibit the same behaviour with those of $u(x,t)$.

Finally, step size $\Delta x = 1, \Delta t = 0.001$ have been chosen and a comparison of the numerical solutions obtained using conditions IC_2 for u and v with those in Refs. [6, 12] and [13] for $\lambda = 1.5, b = 0.1, k = 0.1$ and $t = 0.5$ has been presented in Table 5 and 6. Moreover, for $x \in [-40, 40]$, a comparison of the present results with those given in Ref. [12] is presented in Table 7 for $\Delta t = 0.1$. It is seen from these tables that the present results are better at converging than those in References.

The simulation of numerical solutions for u and v with the initial conditions IC_2 is presented in Fig. 2 for $\Delta t = 0.01, \Delta x = 0.5$ and $t = 5$.

TABLE 4. The error norms L_2 and L_∞ for $u(x,t)$ for $\lambda = 1.5, b = 0.05, k = 0.05$.

		$t = 0.5$		$t = 1$	
Δx	Δt	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
	0.001	0.1678883	0.1374688	0.5520328	0.7130065
0.2	0.01	0.4230022	0.7915751	0.6216782	0.8481784
	0.1	0.6701965	1.3441290	1.5834072	2.8708830
	0.001	0.1623373	0.0799344	0.5704216	0.6775883
0.5	0.01	0.3356339	0.3743494	0.5775878	0.6913134
	0.1	0.3600518	0.4140113	0.6624911	0.8320532
	0.001	0.1575962	0.0531536	0.5230661	0.4972931
0.75	0.01	0.2825318	0.2433479	0.5259560	0.5013827
	0.1	0.2889314	0.2561951	0.5574975	0.5433902
	0.001	0.1544577	0.0525608	0.4744520	0.3763510
1	0.01	0.2473806	0.1781441	0.4756362	0.3780712
	0.1	0.2482638	0.1837114	0.4887715	0.3958066

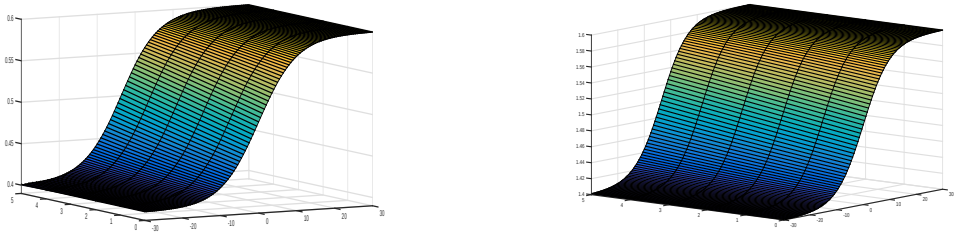


FIGURE 2. The numerical simulations for $u(x,t)$ and $v(x,t)$ with the initial conditions IC_2 for $\Delta t = 0.01, \Delta x = 0.5$ and $t = 5$.

TABLE 5. A Comparison of the numerical results for $u(x,t)$ with the initial conditions IC_2.

x		Exact [12]	VIM [6]	Galerkin
$u(x,t)$	-30	0.40053241	0.40419725	0.4005324
	-20	0.403868194	0.40731614	0.4038676
	-10	0.42543880	0.42762934	0.4254395
	0	0.50369831	0.5037	0.5036982
	10	0.57767006	0.57977066	0.5776695
	20	0.59665505	0.60008386	0.5966547
	30	0.59954067	0.60320275	0.5995407
$v(x,t)$	-30	1.40053241	1.40053103	1.4005324
	-20	1.40386819	1.40385865	1.4038685
	-10	1.42543880	1.42539449	1.4254387
	0	1.50369831	1.5037	1.5036987
	10	1.57767006	1.57771332	1.5776699
	20	1.59665505	1.59666417	1.5966551
	30	1.59954067	1.59954198	1.5995407

TABLE 6. A Comparison of the absolute error norms for $u(x,t)$ with the initial conditions IC_2.

x		$\lvert MQ - Exact\rvert$ [13]	$\lvert VIM - Exact\rvert$ [13]	$\lvert Galerkin - Exact\rvert$
$u(x,t)$	-30	0.0	3.665×10^{-3}	0.0
	-20	1.991×10^{-5}	3.448×10^{-3}	5.709913×10^{-7}
	-10	1.652×10^{-5}	2.191×10^{-3}	6.980766×10^{-7}
	0	1.024×10^{-5}	1.688×10^{-6}	1.363767×10^{-7}
	10	3.703×10^{-6}	2.101×10^{-3}	5.749434×10^{-7}
	20	4.321×10^{-6}	3.429×10^{-3}	3.915380×10^{-7}
	30	4.00×10^{-8}	3.662×10^{-3}	0.0
$v(x,t)$	-30	8.8787×10^{-8}	1.3774×10^{-6}	0.0
	-20	5.4158×10^{-6}	9.5409×10^{-5}	2.718416×10^{-7}
	-10	4.3534×10^{-7}	4.4307×10^{-6}	0.589249×10^{-7}
	0	3.17×10^{-6}	1.6875×10^{-6}	3.686134×10^{-7}
	10	2.2323×10^{-5}	4.3258×10^{-5}	1.178493×10^{-7}
	20	3.4096×10^{-5}	9.1142×10^{-6}	0.243889×10^{-7}
	30	2.5406×10^{-7}	1.3117×10^{-6}	0.0

TABLE 7. A Comparison of the error norm $L_\infty \times 10^3$ for $u(x,t)$ with the initial conditions IC_2.

			$t = 0.5$		$t = 1$	
N	b_1	k	[12]	Galerkin	[12]	Galerkin
5	0.1	0.1	1.12	0.7530296	2.23	1.5331290
10			0.501	0.1906547	0.995	0.3679419
5	0.05	0.05	0.0865	0.0210135	0.174	0.0437143
10			0.00739	0.0080300	0.0148	0.0157169
			$t = 1.5$		$t = 2$	
			[12]	Galerkin	[12]	Galerkin
5	0.1	0.1	2.34	2.3384480	4.43	3.1670790
10			1.48	0.5294479	1.94	0.6729349
5	0.05	0.05	0.262	0.0680763	0.351	0.0940675
10			0.0219	0.0241756	0.0290	0.0335586

5 Conclusion

In summary, we have derived Lumped Galerkin finite element model for the generalized Hirota-Satsuma coupled KdV and coupled mKdV equations with the aid of quadratic B-spline base functions. The numerical solutions of discretized equations have been obtained using the fourth-order Runge-Kutta method. In order to show how effective the method is the errors norms L_2 and L_∞ have been computed at different time and space step sizes. The newly computed error norms are reasonably small. Lumped Galerkin finite element method is a powerful and effective method which can be applied to a wide range of various nonlinear partial differential equation systems.

References

[1] S. Abbasbandy, The application of homotopy analysis method to solve a generalized Hirota-Satsuma coupled KdV equation, *Phys. Lett. A* **361.6** (2007) 478–483.

[2] A. S. Arife, S. K. Vanani and A. Yıldırım, Numerical solution of Hirota-Satsuma couple Kdv and a coupled MKdv equation by means of homotopy analysis method, *WASJ*, **13.11**(2011) 2271–2276.

[3] A. Esen and S. Kutluay, Application of a lumped Galerkin method to the regularized long wave equation,*Appl. Math. Comput.* **174.2** (2006) 833–845.

[4] A. Esen, A lumped Galerkin method for the numerical solution of the modified equal-width wave equation using quadratic B-splines,*Int. J. Comput. Math.*, **83.5-6**(2006), 449–459.

[5] E. Fan, Soliton solutions for a generalized Hirota-Satsuma coupled KdV equation and a coupled MKdV equation, *Phys. Lett. A*,**282.1-2**(2001),18–22.

[6] D. D. Ganji M. Jannatabadi and E. Mohseni, Application of Hes variational iteration method to nonlinearJaulentMiodek equations and comparing it with ADM, *Comput Appl Math*,**2007**(2007), 3545.

[7] R. Hirota and J. Satsuma, Soliton solution of a coupled KdV equation, *Phys. Lett. A*, **85.8-9**(1981), 407–408.

[8] S. Kutluay and A. Esen , A Lumped Galerkin method for solving the Burgers equation, *Int. J. Comput. Math.*, **81.11**(2004), 1433-1444.

[9] S. Kutluay and Y. Ucar , Numerical solution of a coupled modified Korteweg-de Vries equation by the Galerkin method using quadratic B-splines,*Int. J. Comput. Math.*, **90.11**(2013), 2353–2371.

[10] S. Kutluay and Y. Ucar , A quadratic B-spline Galerkin approach for solving a coupled KdV equation, *Math. Model. Anal.*, **18.1**(2013), 103–121.

[11] D. Kaya, Solitary wave solutions for a generalized Hirota-Satsuma coupled KdV equation, *Appl. Math. Comput.* , **147.1**(2004),69–78.

[12] A. H. Khater, R. S. Temsah and D. K. Callebaut, Numerical solutions for some coupled nonlinear evolution equations by using spectral collocation method, *Math. Comput Model.*,**48.7-8**(2008), 1237–1253.

[13] A. J. Khattak, S.I. A. Tirmizi and S. U. Islam, Application of meshfree collocation method to a class of nonlinear partial differential equations, *Eng. Anal. Bound. Elem.*,**33.5**(2009), 661–667.

[14] F. Kangalgil and F. Ayaz, Solitary wave solutions for Hirota-Satsuma coupled KdV equation and coupled mKdV equation by differential transform method, *Arab. J. Sci. Eng.*,**35.2D**(2010)203–213.

[15] A. Kumar and R. D. Pankaj, Laplace modified decomposition method for the generalized Hirota-Satsuma coupled KdV equation,*Can. J. Basic Appl. Sci*,**03**(2015),126–133.

[16] P. M. Prenter, Splines and variational methods, *John Wiley and Sons*, New York, 1975.

[17] Yongtang Wu, Xianguo Geng, Xingbiao Hu and Siming Zhu, A generalized Hirota-Satsuma coupled Korteweg-de Vries equation and miura transformation, *Phys. Lett. A* **255.4-6**(1999), 259-264.

[18] Yaxuan Yu, Qi Wang and Hongqing Zhang, The extended Jacobi elliptic function method to solve a generalized Hirota- Satsuma coupled KdV equations, *Chaos Solitons Fract.*,**26.5**(2005),1415–1421.

[19] Xue-lin Yong and Hong-Qing Zhang, New exact solutions to the generalized coupled Hirota-Satsuma KdV system, *Chaos Solitons Fract.*, **26.4**(2005), 1105–1110.

[20] E. M. E. Zayed, H. A. Zedan, K. A. Gepreel, On the solitary wave solutions for nonlinear Hirota-Satsuma coupled KdV of equations, *Chaos Solitons Fract.* **22.2**(2004), 285–303.

[21] Zhao Guo-Zhong, Yu Xi-Jun, Xu Yun, Zhu Jiang, Wu Di, Approximate analytic solutions for a generalized Hirota-Satsuma coupled KdV equation and a coupled mKdV equation, *Chinese Phys. B*,**19.8**(2010), 080204.