On $(4, 5)$-regular bipartitions with odd parts distinct

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Abstract

In his work, K. Alladi considered the partition function $pod(n)$, the number of partitions of an integer n with odd parts distinct (the even parts are unrestricted). He obtained a series expansion for the product generating function of these partitions. Later Hirschhorn and Sellers obtained some internal congruences involving the infinite families and Ramanujan's type congruences for $pod(n)$. Let $B_{4,5}(n)$ denote the number of $(4, 5)$-regular bipartitions of a positive integer n with odd parts distinct. In this paper, we establish many infinite families of congruences modulo powers of 2 for $B_{4,5}(n)$.

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1 Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. An ℓ-regular partition is a partition in which none of the parts is divisible by ℓ. Let $b_\ell(n)$ denote the number of ℓ-regular partitions of n with $b_\ell(0) = 1$. The generating function for $b_\ell(n)$ is

$$\sum_{n=0}^{\infty} b_\ell(n)\, q^n = \frac{f_\ell}{f_1},$$

where $f_\ell := (q^\ell; q^\ell)_\infty = \prod_{n=1}^{\infty} (1 - q^{n\ell})$.

Recently, arithmetic properties of ℓ-regular partition functions have been studied by a number of mathematicians. Calkin et al. [7] established congruences for 5-regular partitions modulo 2 and for 13-regular partitions modulo 2 and 3 using the theory of modular forms. For more details, one can see [2], [3], [8], [12], [14], [15] and [19].

K. Alladi [4] considered the partition function $pod(n)$, the number of partitions of an integer n with odd parts distinct (the even parts are unrestricted). He obtained a series expansion for the product generating function of these partitions. Subsequently Hirschhorn and Sellers [10] obtained some internal congruences involving the infinite families and Ramanujan's type congruences for $pod(n)$. For more details, one can see [9], [13], [17] and [20]. Suppose $\ell, m > 0$. A partition is an (ℓ, m)-regular partition of the positive integer n if none of the parts is divisible by ℓ and m. For more details, we can see [1, 16]. Let $B_{4,5}(n)$ denote the number of $(4, 5)$-regular bipartitions of n with odd parts distinct with $B_{4,5}(0) = 1$.

The generating function is given by

$$\sum_{n=0}^{\infty} B_{4,5}(n)\, q^n = \frac{f_2^2 f_5^2 f_{16}^2 f_{40}^2}{f_1^2 f_8^2 f_{10}^2 f_{80}^2}. \tag{1.1}$$

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For example, there are 16 bipartitions for $B_{4,5}(5)$, namely

$$(0,3+2),\ (3+2,0),\ (0,2+2+1),\ (2+2+1,0),\ (2+2,1),\ (1,2+2),\ (3,2),\ (2,3),\ (1+3,1),\ (1,3+1),\ (2,2+1),\ (2+1,2).$$

We prove many congruences of the following form. For $\alpha, \beta \geq 0$,

$$\sum_{n=0}^{\infty} B_{4,5}\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta}n + 28\cdot 3^{2\alpha}\cdot 5^{2\beta}+3\right)q^n \equiv 8f_2^3f_5^3 \pmod{16}.$$

2 Preliminary results

Lemma 2.1. The following 2-dissections hold :

$$f_1^4 = \frac{f_4^{10}}{f_2^2f_8^4} - 4q\frac{f_2^2f_8^4}{f_4^2} \tag{2.1}$$

and

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}. \tag{2.2}$$

For proofs, see [6, p. 40].

Lemma 2.2. The following 3-dissection holds :

$$f_1^3 = \frac{f_6f_9^6}{f_3f_{18}^3} - 3qf_9^3 + 4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}. \tag{2.3}$$

For a proof, see [6, p. 345].

Lemma 2.3. The following 2-dissections hold :

$$\frac{f_5}{f_1} = \frac{f_8f_{20}^2}{f_2^2f_{40}} + q\frac{f_4^3f_{10}f_{40}}{f_2^3f_8f_{20}} \tag{2.4}$$

and

$$\frac{f_1}{f_5} = \frac{f_2f_8f_{20}^3}{f_4f_{10}^3f_{40}} - q\frac{f_4^2f_{40}}{f_8f_{10}^2}. \tag{2.5}$$

The equation (2.4) was proved by Hirschhorn and Sellers [12] ; see also [18]. Replacing q by $-q$ in (2.4) and using the fact that

$$(-q;-q)_\infty = \frac{f_2^3}{f_1f_4},$$

we obtain (2.5).

Lemma 2.4. We have

$$f_1f_5^3 = f_2^3f_{10} - q\frac{f_2^2f_{10}^2f_{20}}{f_4} + 2q^2f_4f_{20}^3 - 2q^3\frac{f_4^4f_{10}f_{40}^2}{f_2f_8^2} \tag{2.6}$$

and

$$f_1^3f_5 = \frac{f_2^2f_4f_{10}^2}{f_{20}} + 2qf_4^3f_{20} - 5qf_2f_{10}^3 + 2q^2\frac{f_4^6f_{10}f_{40}^2}{f_2f_8^2f_{20}^2}. \tag{2.7}$$

For proofs, see [15].

Lemma 2.5. [6, p.303, Entry 17(v)] We have

$$f_1 = f_{49}\left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right), \tag{2.8}$$

where $A(q) = f(-q^3, -q^4)$, $B(q) = f(-q^2, -q^5)$ and $C(q) = f(-q, -q^6)$.

We proved the following Theorems:

Theorem 2.6. Let $s_1 \in \{124, 156\}$, $s_2 \in \{124, 316\}$, $s_3 \in \{332, 428\}$, $s_4 \in \{284, 476\}$, $s_5 \in \{172, 268\}$, $s_6 \in \{20, 52\}$, $s_7 \in \{56, 184, 248, 312\}$, $s_8 \in \{56, 248, 632, 824\}$, $s_9 \in \{184, 376, 568, 952\}$, $s_{10} \in \{248, 312\}$, $s_{11} \in \{112, 368, 496, 624\}$, $s_{12} \in \{112, 496, 1264, 1648\}$, $s_{13} \in \{368, 752, 1136, 1904\}$, $s_{14} \in \{496, 624\}$, then for all $\alpha, \beta, \gamma \geq 0$, we have, modulo 16,

$$\sum_{n=0}^{\infty} B_{4,5}\left(32 \cdot 5^{2\beta}n + 12 \cdot 5^{2\beta} + 3\right) q^n \equiv 8f_1^9 + 8f_4f_5, \tag{2.9}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(32 \cdot 5^{2\beta+1}n + 28 \cdot 5^{2\beta+1} + 3\right) q^n \equiv 8f_1f_{20} + 8qf_5^9, \tag{2.10}$$

$$B_{4,5}\left(32 \cdot 5^{2\beta+2}n + s_1 \cdot 5^{2\beta+1} + 3\right) \equiv 0, \tag{2.11}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(32 \cdot 3^{2\alpha} \cdot 5^{2\beta}n + 28 \cdot 3^{2\alpha} \cdot 5^{2\beta} + 3\right) q^n \equiv 8f_2^3f_5^3, \tag{2.12}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(32 \cdot 3^{2\alpha} \cdot 5^{2\beta+1}n + 4 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} + 3\right) q^n \equiv 8qf_1^3f_{10}^3, \tag{2.13}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(32 \cdot 3^{2\alpha+1} \cdot 5^{2\beta}n + 28 \cdot 3^{2\alpha} \cdot 5^{2\beta} + 3\right) q^n \equiv 8f_2f_5, \tag{2.14}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(32 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1}n + 44 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} + 3\right) q^n \equiv 8f_1f_{10}, \tag{2.15}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(32 \cdot 3^{2\alpha+1} \cdot 5^{2\beta}n + 4 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} + 3\right) q^n \equiv 8q^2f_6^3f_{15}^3, \tag{2.16}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(32 \cdot 3^{2\alpha+1} \cdot 5^{2\beta}n + 92 \cdot 3^{2\alpha} \cdot 5^{2\beta} + 3\right) q^n \equiv 8f_5f_6^3 + 8qf_2f_{15}^3, \tag{2.17}$$

$$\begin{aligned}&\sum_{n=0}^{\infty} B_{4,5}\left(32 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1}n + 76 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} + 3\right) q^n \\ &\equiv 8f_3^3f_{10} + 8q^3f_1f_{30}^3,\end{aligned} \tag{2.18}$$

$$B_{4,5}\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}n+s_2\cdot 3^{2\alpha}\cdot 5^{2\beta}+3\right)\equiv 0, \tag{2.19}$$

$$B_{4,5}\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta+2}n+s_3\cdot 3^{2\alpha}\cdot 5^{2\beta+1}+3\right)\equiv 0, \tag{2.20}$$

$$B_{4,5}\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}n+s_4\cdot 3^{2\alpha}\cdot 5^{2\beta}+3\right)\equiv 0, \tag{2.21}$$

$$B_{4,5}\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta+2}n+s_5\cdot 3^{2\alpha}\cdot 5^{2\beta+1}+3\right)\equiv 0, \tag{2.22}$$

$$B_{4,5}\left(32\cdot 3^{2\alpha+2}\cdot 5^{2\beta}n+s_6\cdot 3^{2\alpha+1}\cdot 5^{2\beta}+3\right)\equiv 0, \tag{2.23}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(64\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+24\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)q^n\equiv 8f_1^9, \tag{2.24}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(64\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+8\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}+3\right)q^n\equiv 8q^2f_7^9, \tag{2.25}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(64\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+56\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)q^n\equiv 8qf_5^9, \tag{2.26}$$

$$B_{4,5}\left(64\cdot 3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+s_7\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.27}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(64\cdot 3^{4\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+88\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)q^n\equiv 8f_2f_3^3, \tag{2.28}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(64\cdot 3^{4\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+152\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)q^n\equiv 8f_1f_6^3, \tag{2.29}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(64\cdot 3^{4\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+8\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}+3\right)q^n \equiv 8q^2f_{10}f_{15}^3, \tag{2.30}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(64\cdot 3^{4\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+184\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)q^n \equiv 8q^3f_5f_{30}^3, \tag{2.31}$$

$$B_{4,5}\left(64\cdot 3^{4\alpha+1}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+s_8\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.32}$$

$$B_{4,5}\left(64\cdot 3^{4\alpha+1}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+s_9\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.33}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(64\cdot 5^{2\beta}n+56\cdot 5^{2\beta}+3\right)q^n\equiv 8f_1f_{20}+8f_2^3f_5^3, \tag{2.34}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(64\cdot 5^{2\beta+1}n+24\cdot 5^{2\beta+1}+3\right)q^n\equiv 8f_4f_5+8qf_1^3f_{10}^3, \tag{2.35}$$

$$B_{4,5}\left(64\cdot 5^{2\beta+1}n+s_{10}\cdot 5^{2\beta}+3\right)\equiv 0, \tag{2.36}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(128\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+48\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)q^n\equiv 8f_1^9, \tag{2.37}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(128\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+16\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}+3\right)q^n\equiv 8q^2f_7^9, \tag{2.38}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(128\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+112\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)q^n\equiv 8qf_5^9, \tag{2.39}$$

$$B_{4,5}\left(128\cdot 3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+s_{11}\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.40}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(128\cdot 3^{4\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+176\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)q^n\equiv 8f_2f_3^3, \tag{2.41}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(128\cdot 3^{4\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+304\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)q^n\equiv 8f_1f_6^3, \tag{2.42}$$

$$\begin{aligned}&\sum_{n=0}^{\infty}B_{4,5}\left(128\cdot 3^{4\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+112\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)q^n\\&\equiv 8q^2f_{10}f_{15}^3,\end{aligned} \tag{2.43}$$

$$\begin{aligned}&\sum_{n=0}^{\infty}B_{4,5}\left(128\cdot 3^{4\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+368\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)q^n\\&\equiv 8q^3f_5f_{30}^3,\end{aligned} \tag{2.44}$$

$$B_{4,5}\left(128\cdot 3^{4\alpha+1}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+s_{12}\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.45}$$

$$B_{4,5}\left(128\cdot 3^{4\alpha+1}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+s_{13}\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.46}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(128\cdot 5^{2\beta}n+112\cdot 5^{2\beta}+3\right)q^n\equiv 8f_1f_{20}+8f_2^3f_5^3, \tag{2.47}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(128\cdot 5^{2\beta+1}n+48\cdot 5^{2\beta+1}+3\right)q^n\equiv 8f_4f_5+8qf_1^3f_{10}^3, \tag{2.48}$$

$$B_{4,5}\left(128\cdot 5^{2\beta+1}n+s_{14}\cdot 5^{2\beta}+3\right)\equiv 0. \tag{2.49}$$

Theorem 2.7. Let $t_1\in\{22,38\}$, $t_2\in\{34,66\}$ and $t_3\in\{26,42,58,74\}$, then for all $\alpha,\beta,\gamma\geq 0$, we have, modulo 8,

$$\sum_{n=0}^{\infty}B_{4,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)q^n\equiv 4f_1^3, \tag{2.50}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}+3\right)q^n \equiv 4f_7^3, \tag{2.51}$$

$$B_{4,5}\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right) \equiv \begin{cases} 4 & \text{if } n=k(3k+1)/2 \text{ for some } k\in\mathbb{Z}, \\ 0 & \text{otherwise.}\end{cases} \tag{2.52}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)q^n \equiv 4f_3^3, \tag{2.53}$$

$$B_{4,5}\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+34\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.54}$$

$$B_{4,5}\left(16\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+t_1\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.55}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}+3\right)q^n \equiv 4f_5^3, \tag{2.56}$$

$$B_{4,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+t_2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.57}$$

$$B_{4,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+t_3\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)\equiv 0. \tag{2.58}$$

Theorem 2.8. Let $r_1 \in \{44, 76\}$, $r_2 \in \{68, 132\}$, $r_3 \in \{52, 84, 116, 148\}$, $r_4 \in \{88, 152\}$, $r_5 \in \{136, 264\}$, $r_6 \in \{104, 168, 232, 296\}$, $r_7 \in \{176, 304\}$, $r_8 \in \{272, 528\}$ and $r_9 \in \{208, 336, 464, 592\}$, then for all $\alpha, \beta, \gamma \geq 0$, we have, modulo 4,

$$\sum_{n=0}^{\infty} B_{4,5}\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)q^n \equiv 2f_1^3, \tag{2.59}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}+3\right)q^n \equiv 2f_7^3, \tag{2.60}$$

$$B_{4,5}\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right) \equiv \begin{cases} 2 & \text{if } n=k(3k+1)/2 \text{ for some } k\in\mathbb{Z}, \\ 0 & \text{otherwise.}\end{cases} \tag{2.61}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)q^n \equiv 2f_3^3, \tag{2.62}$$

$$B_{4,5}\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+68\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.63}$$

$$B_{4,5}\left(32\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+r_1\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.64}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}+3\right)q^n \equiv 2f_5^3, \tag{2.65}$$

$$B_{4,5}\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+r_2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.66}$$

$$B_{4,5}\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+r_3\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.67}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)q^n\equiv 2f_1^3, \tag{2.68}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+2}+3\right)q^n\equiv 2f_7^3, \tag{2.69}$$

$$\begin{aligned}&B_{4,5}\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)\\ &\equiv\begin{cases}2 & \text{if } n=k(3k+1)/2 \text{ for some } k\in\mathbb{Z},\\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{2.70}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha+2}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)q^n\equiv 2f_3^3, \tag{2.71}$$

$$B_{4,5}\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+68\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.72}$$

$$B_{4,5}\left(32\cdot 3^{2\alpha+2}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+r_1\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.73}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}+3\right)q^n\equiv 2f_5^3, \tag{2.74}$$

$$B_{4,5}\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+r_2\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.75}$$

$$B_{4,5}\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}n+r_3\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.76}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(64\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+8\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)q^n\equiv 2f_1^3, \tag{2.77}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(64\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+8\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}+3\right)q^n\equiv 2f_7^3, \tag{2.78}$$

$$\begin{aligned}&B_{4,5}\left(64\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+8\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)\\ &\equiv\begin{cases}2 & \text{if } n=k(3k+1)/2 \text{ for some } k\in\mathbb{Z},\\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{2.79}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(64\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+24\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)q^n\equiv 2f_3^3, \tag{2.80}$$

$$B_{4,5}\left(64\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+136\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.81}$$

$$B_{4,5}\left(64\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+r_4\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.82}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(64\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+8\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}+3\right)q^n\equiv 2f_5^3, \tag{2.83}$$

$$B_{4,5}\left(64\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+r_5\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.84}$$

$$B_{4,5}\left(64\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+r_6\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.85}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(64\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+8\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)q^n\equiv 2f_1^3, \tag{2.86}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(64\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}n+8\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+2}+3\right)q^n\equiv 2f_7^3, \tag{2.87}$$

$$B_{4,5}\left(64\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+8\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)$$
$$\equiv \begin{cases} 2 & \text{if } n=k(3k+1)/2 \text{ for some } k\in\mathbb{Z},\\ 0 & \text{otherwise.}\end{cases} \tag{2.88}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(64\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+24\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)q^n\equiv 2f_3^3, \tag{2.89}$$

$$B_{4,5}\left(64\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+136\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.90}$$

$$B_{4,5}\left(64\cdot 3^{2\alpha+2}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+r_4\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.91}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(64\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+8\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}+3\right)q^n\equiv 2f_5^3, \tag{2.92}$$

$$B_{4,5}\left(64\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+r_5\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.93}$$

$$B_{4,5}\left(64\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}n+r_6\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.94}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(128\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+16\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)q^n\equiv 2f_1^3, \tag{2.95}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(128\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+16\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}+3\right)q^n\equiv 2f_7^3, \tag{2.96}$$

$$B_{4,5}\left(128\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+16\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)$$
$$\equiv \begin{cases} 2 & \text{if } n=k(3k+1)/2 \text{ for some } k\in\mathbb{Z},\\ 0 & \text{otherwise.}\end{cases} \tag{2.97}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(128\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+16\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)q^n\equiv 2f_3^3, \tag{2.98}$$

$$B_{4,5}\left(128\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+272\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.99}$$

$$B_{4,5}\left(128\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+r_7\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.100}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(128\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+16\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}+3\right)q^n\equiv 2f_5^3, \tag{2.101}$$

$$B_{4,5}\left(128\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+r_8\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.102}$$

$$B_{4,5}\left(128\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+r_9\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.103}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(128\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)q^n\equiv 2f_1^3, \tag{2.104}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(128\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}n+16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+2}+3\right)q^n \equiv 2f_7^3, \tag{2.105}$$

$$B_{4,5}\left(128\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right) \equiv \begin{cases} 2 & \text{if } n=k(3k+1)/2 \text{ for some } k\in\mathbb{Z},\\ 0 & \text{otherwise.}\end{cases} \tag{2.106}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(128\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+16\cdot 3^{2\alpha+2}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)q^n \equiv 2f_3^3, \tag{2.107}$$

$$B_{4,5}\left(128\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+272\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.108}$$

$$B_{4,5}\left(128\cdot 3^{2\alpha+2}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+r_7\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.109}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(128\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+16\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}+3\right)q^n\equiv 2f_5^3, \tag{2.110}$$

$$B_{4,5}\left(128\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+r_8\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.111}$$

$$B_{4,5}\left(128\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}n+r_9\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}+3\right)\equiv 0, \tag{2.112}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(16n+1\right)q^n\equiv 2\frac{f_1^7}{f_{10}}+2qf_1f_{20}, \tag{2.113}$$

$$B_{4,5}\left(16n+9\right)\equiv 0. \tag{2.114}$$

3 Proof of the Theorem (2.6)

Using (2.4) in (1.1) and extracting the terms involving q^{2n+1} from both sides, we get

$$\sum_{n=0}^{\infty} B_{4,5}\,(2n+1)\,q^n = 2\frac{f_2^3 f_8^2 f_{10} f_{20}^2}{f_1^3 f_4^2 f_5 f_{40}^2}. \tag{3.1}$$

From the binomial theorem, it is easy to see that for any positive integers k and m,

$$f_k^{2m} \equiv f_{2k}^m \pmod 2, \tag{3.2}$$
$$f_k^{4m} \equiv f_{2k}^{2m} \pmod{2^2}, \tag{3.3}$$
$$f_k^{8m} \equiv f_{2k}^{4m} \pmod{2^3}. \tag{3.4}$$

Employing (2.2) and (2.5) in (3.1) along with (3.2) and (3.4), we obtain, modulo 16,

$$\sum_{n=0}^{\infty} B_{4,5}\,(4n+1)\,q^n \equiv 2\frac{f_2^7 f_{10}^5}{f_1^2 f_4 f_5^2 f_{20}^3} + 8q\frac{f_1 f_2^8}{f_5} \tag{3.5}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}\,(4n+3)\,q^n \equiv 8\frac{f_2^{10}}{f_{20}} - 2\frac{f_2^2 f_4 f_{10}^2}{f_1^3 f_5 f_{20}}. \tag{3.6}$$

Using (2.2) and (2.5) in (3.6), we arrive at

$$\sum_{n=0}^{\infty} B_{4,5}\,(8n+3)\,q^n \equiv 8\frac{f_2^5}{f_{10}} - 2\frac{f_2^2 f_4 f_{10}^2}{f_1^3 f_5 f_{20}} + 8q f_2^7 f_{10} \tag{3.7}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}\,(8n+7)\,q^n \equiv 2\frac{f_2^5 f_{20}}{f_1^4 f_4 f_{10}} + 8\frac{f_1 f_2^8}{f_5}. \tag{3.8}$$

Employing (2.2) and (2.5) in (3.8), we get

$$\sum_{n=0}^{\infty} B_{4,5}\,(16n+7)\,q^n \equiv 2\frac{f_2 f_{10}}{f_1 f_5} + 8\frac{f_1 f_2^5}{f_5} \tag{3.9}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}\,(16n+15)\,q^n \equiv 8 f_1^3 f_2^5 f_5 + 8 f_8 f_{10}. \tag{3.10}$$

Using (2.7) in (3.10), we arrive at

$$\sum_{n=0}^{\infty} B_{4,5}\,(32n+15)\,q^n \equiv 8 f_1^9 + 8 f_4 f_5 \tag{3.11}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}\,(32n+31)\,q^n \equiv 8 f_2^3 f_5^3. \tag{3.12}$$

The congruence (3.11) is $\beta = 0$ case of the congruence (2.9). Suppose that the congruence (2.9) is true for some integer $\beta \geq 0$.
Ramanujan recorded the following identity in his notebooks without proof:

$$f_1 = f_{25}(R(q^5)^{-1} - q - q^2R(q^5)), \tag{3.13}$$

where $R(q) = \dfrac{f(-q,-q^4)}{f(-q^2,-q^3)}$.
For a proof of (3.13), one can see [11], [21].
Using (3.13) in (2.9) and then comparing the coefficients of q^{5n+4}, we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(32\cdot 5^{2\beta+1}n + 28\cdot 5^{2\beta+1} + 3\right) q^n \equiv 8f_1f_{20} + 8qf_5^9, \tag{3.14}$$

which is (2.10). Again using (3.13) in (3.14) and then comparing the coefficients of q^{5n+1} in the resultant equation, we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(32\cdot 5^{2\alpha+2}n + 12\cdot 5^{2\alpha+2} + 3\right) q^n \equiv 8f_1^9 + 8f_4f_5, \tag{3.15}$$

which implies that the congruence (2.9) is true for $\beta+1$. By mathematical induction, the congruence (2.9) is true for all integers β.
Comparing the coefficients of q^{5n+i} for $i = 3, 4$ in (3.14), we obtain (2.11).
The equation (3.12) is the $\alpha = \beta = 0$ case of (2.12). From (2.3), the congruence (2.12) with $\beta = 0$ becomes

$$\begin{aligned}&\sum_{n=0}^{\infty} B_{4,5}\left(32\cdot 3^{2\alpha}n + 28\cdot 3^{2\alpha} + 3\right) q^n\\ &\equiv 8f_6f_{15} + 8q^2f_{15}f_{18}^3 + 8q^5f_6f_{45}^3 + 8q^7f_{18}^3f_{45}^3,\end{aligned} \tag{3.16}$$

which implies

$$\sum_{n=0}^{\infty} B_{4,5}\left(32\cdot 3^{2\alpha+1}n + 20\cdot 3^{2\alpha+1} + 3\right) q^n \equiv 8q^2f_6^3f_{15}^3, \tag{3.17}$$

which implies

$$\sum_{n=0}^{\infty} B_{4,5}\left(32\cdot 3^{2\alpha+2}n + 28\cdot 3^{2\alpha+2} + 3\right) q^n \equiv 8f_2^3f_5^3, \tag{3.18}$$

which implies that the congruence (2.12) is true for $\alpha + 1$ with $\beta = 0$. Hence, by induction, the congruence (2.12) is true for any integer α with $\beta = 0$. Suppose that the congruence (2.12) holds for some integers $\alpha, \beta > 0$. Employing the equation (3.13) in (2.12) and then comparing the coefficients of q^{5n+1}, we find that

$$\sum_{n=0}^{\infty} B_{4,5}\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+1}n + 4\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1} + 3\right) q^n \equiv 8qf_1^3f_{10}^3, \tag{3.19}$$

which proves (2.13). Extracting the coefficients of q^{5n+4} in (3.19), we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+2}n+28\cdot 3^{2\alpha}\cdot 5^{2\beta+2}+3\right)q^n \equiv 8f_2^3f_5^3, \tag{3.20}$$

which implies that the congruence (2.12) is true for $\beta+1$. Hence, by induction, the congruence (2.12) is true for any non-negative integers $\alpha,\beta>0$.
Employing (2.3) in (2.12) and then comparing the coefficients of q^{3n}, q^{3n+1} and q^{3n+2}, we get (2.14), (2.16) and (2.17) respectively.
Utilizing (3.13) in (2.14) and (2.17), we get (2.15) and (2.18) respectively.
Using the equations (2.14) and (2.15) along with the equation (3.13), we obtain (2.19) and (2.20) respectively.
Using the equations (2.17) and (2.18) along with the equation (3.13), we obtain (2.21) and (2.22) respectively.
Comparing the coefficients of q^{3n+i} for $i=1,3$ in the equation (2.16), we get (2.23).
Employing (2.2) and (2.5) in (3.7), we arrive at

$$\sum_{n=0}^{\infty} B_{4,5}\left(16n+3\right)q^n \equiv 8\frac{f_1^5}{f_5}-2\frac{f_2^2f_4f_{10}^2}{f_1^3f_5f_{20}}+8qf_2^7f_{10} \tag{3.21}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}\left(16n+11\right)q^n \equiv 8\frac{f_1^3f_2^7f_5}{f_{10}}+2\frac{f_2^5f_{20}}{f_1^4f_4f_{10}}+8f_1^3f_2^2f_5. \tag{3.22}$$

Utilizing (2.2) and (2.7) in (3.22), we obtain

$$\sum_{n=0}^{\infty} B_{4,5}\left(32n+11\right)q^n \equiv 2\frac{f_2f_{10}}{f_1f_5}+8\frac{f_1f_2^5}{f_5}+8f_2^3 \tag{3.23}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}\left(32n+27\right)q^n \equiv 8f_1^3f_5f_{10}+8f_8f_{10}+8f_1^3f_2^5f_5. \tag{3.24}$$

Using (2.7) in (3.24), we have

$$\sum_{n=0}^{\infty} B_{4,5}\left(64n+27\right)q^n \equiv 8f_1^9 \tag{3.25}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}\left(64n+59\right)q^n \equiv 8f_1f_{20}+8f_2^3f_5^3. \tag{3.26}$$

The equation (3.25) is $\alpha=\beta=\gamma=0$ case of (2.24). Suppose that the congruence (2.24) is true for $\alpha\geq 0$ with $\beta=\gamma=0$. Employing (2.3) in (2.24) with $\beta=\gamma=0$ and then comparing the coefficients of q^{3n}, we get

$$\begin{aligned}&\sum_{n=0}^{\infty} B_{4,5}\left(64\cdot 3^{4\alpha+1}n+24\cdot 3^{4\alpha}+3\right)q^n\\ &\equiv 8f_1^3+8qf_3^9 \equiv 8f_3+8qf_9^3+8qf_3^9,\end{aligned} \tag{3.27}$$

which implies

$$\sum_{n=0}^{\infty} B_{4,5}\left(64\cdot 3^{4\alpha+2}n+24\cdot 3^{4\alpha+2}+3\right)q^n$$
$$\equiv 8f_3^3+8f_1^9\equiv 8qf_6f_9^3+8q^2f_3f_9^6+8q^3f_9^9, \tag{3.28}$$

which implies

$$\sum_{n=0}^{\infty} B_{4,5}\left(64\cdot 3^{4\alpha+3}n+24\cdot 3^{4\alpha+2}+3\right)q^n\equiv 8qf_3^9, \tag{3.29}$$

which implies

$$\sum_{n=0}^{\infty} B_{4,5}\left(64\cdot 3^{4\alpha+4}n+24\cdot 3^{4\alpha+4}+3\right)q^n\equiv 8f_1^9, \tag{3.30}$$

which implies that the congruence (2.24) is true for $\alpha+1$ with $\beta=\gamma=0$. Hence, by mathematical induction, the congruence (2.24) is true for all $\alpha\geq 0$. Suppose that the congruence (2.24) holds for $\alpha,\beta\geq 0$ with $\gamma=0$. By using (3.13) in (2.24) and then comparing the coefficients of q^{5n+4}, we arrive at

$$\sum_{n=0}^{\infty} B_{4,5}\left(64\cdot 3^{4\alpha}\cdot 5^{2\beta+1}n+56\cdot 3^{4\alpha}\cdot 5^{2\beta+1}+3\right)q^n\equiv 8qf_5^9, \tag{3.31}$$

which implies

$$\sum_{n=0}^{\infty} B_{4,5}\left(64\cdot 3^{4\alpha}\cdot 5^{2\beta+2}n+24\cdot 3^{4\alpha}\cdot 5^{2\beta+2}+3\right)q^n\equiv 8f_1^9, \tag{3.32}$$

which implies that the congruence (2.24) is true for $\beta+1$ with $\gamma=0$. By mathematical induction, the congruence (2.24) holds for $\alpha,\beta\geq 0$ with $\gamma=0$. Suppose the congruence (2.24) is true for $\alpha,\beta,\gamma\geq 0$ and then utilizing (2.8), we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(64\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+8\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}+3\right)q^n\equiv 8q^2f_7^9, \tag{3.33}$$

which proves (2.25) and which implies that

$$\sum_{n=0}^{\infty} B_{4,5}\left(64\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+24\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}+3\right)q^n\equiv 8f_1^9, \tag{3.34}$$

which implies that the congruence (2.24) is true for $\gamma+1$. Hence, by mathematical induction, the congruence (2.24) is true for all integers $\alpha,\beta,\gamma\geq 0$.
Using (3.13) in (2.24), we obtain (2.26).
From the congruence (2.26), we get (2.27).
Employing (2.3) in (2.24) and then comparing the coefficients of q^{3n+1} and q^{3n+2} in the resultant equation, we obtain (2.28) and (2.29) respectively.

From the congruences (2.28) and (2.29) along with (3.13), we get (2.30) and (2.31) respectively.
From the congruences (2.30) and (2.31), we arrive at (2.32) and (2.33) respectively.
The congruence (3.26) is $\beta = 0$ case of (2.34). Suppose the congruence (2.34) holds for $\beta \geq 0$ then by using (3.13) in (2.34) and comparing the coefficients of q^{5n+1}, we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(64 \cdot 5^{2\beta+1}n + 24 \cdot 5^{2\beta+1} + 3\right) q^n \equiv 8f_4f_5 + 8qf_1^3f_{10}^3, \tag{3.35}$$

which is (2.35). Again by using (3.13) in (3.35), we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(64 \cdot 5^{2\beta+2}n + 56 \cdot 5^{2\beta+2} + 3\right) q^n \equiv 8f_1f_{20} + 8f_2^3f_5^3, \tag{3.36}$$

which implies that the congruence (2.34) is true for $\beta + 1$. By mathematical induction, the congruence (2.34) is true for all integers β.
Comparing the coefficients of q^{5n+3} and q^{5n+4} in (2.34) along with (3.13), we get (2.36).
Using (2.2) and (2.5) in (3.21), we arrive at

$$\sum_{n=0}^{\infty} B_{4,5}\left(32n + 3\right) q^n \equiv 8\frac{f_1^5}{f_5} - 2\frac{f_2^2f_4f_{10}^2}{f_1^3f_5f_{20}} + 8qf_2^7f_{10} \tag{3.37}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}\left(32n + 19\right) q^n \equiv 8f_2f_{10} + 8\frac{f_1^3f_2^7f_5}{f_{10}} + 2\frac{f_2^5f_{20}}{f_1^4f_4f_{10}} + 8f_1^3f_2^2f_5. \tag{3.38}$$

From the congruences (3.21) and (3.37), we get

$$B_{4,5}\left(32n + 3\right) \equiv B_{4,5}\left(16n + 3\right).$$

Using (2.2) and (2.7) in (3.38), we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(64n + 19\right) q^n \equiv 2\frac{f_2f_{10}}{f_1f_5} + 8f_1f_5 + 8\frac{f_1f_2^5}{f_5} + 8f_2^3 \tag{3.39}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}\left(64n + 51\right) q^n \equiv 8f_8f_{10} + 8f_1^3f_2^5f_5 + 8f_1^3f_5f_{10}. \tag{3.40}$$

Employing (2.7) in (3.40), we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(128n + 51\right) q^n \equiv 8f_1^9 \tag{3.41}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}\left(128n + 115\right) q^n \equiv 8f_1f_{20} + 8f_2^3f_5^3. \tag{3.42}$$

The rest of the proofs of the identities (2.37)-(2.49) are similar to the proofs of the identities (2.24)-(2.36). So, we omit the details.

4 Proof of the Theorem (2.7)

Employing (2.2) and (2.5) in (3.5), we get, modulo 8,

$$\sum_{n=0}^{\infty} B_{4,5}\,(8n+1)\,q^n \equiv 2\frac{f_2^5 f_{10}}{f_1 f_5^5} + 2q\frac{f_2^3 f_5 f_{10}}{f_1^3} \tag{4.1}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}\,(8n+5)\,q^n \equiv 4f_2^3, \tag{4.2}$$

which implies that

$$\sum_{n=0}^{\infty} B_{4,5}\,(16n+5)\,q^n \equiv 4f_1^3 \tag{4.3}$$

and

$$B_{4,5}\,(16n+13) \equiv 0. \tag{4.4}$$

The equation (4.3) is the $\alpha=\beta=\gamma=0$ case of (2.50). Suppose that the congruence (2.50) is true for $\alpha \geq 0$ with $\beta=\gamma=0$. Utilizing (2.3) in the equation (2.50) with $\beta=\gamma=0$ and then collecting the coefficients of q^{3n+1}, we arrive at

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{2\alpha+1}n + 2\cdot 3^{2\alpha+2}+3\right) q^n \equiv 4f_3^3, \tag{4.5}$$

which implies

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{2\alpha+2}n + 2\cdot 3^{2\alpha+2}+3\right) q^n \equiv 4f_1^3, \tag{4.6}$$

which implies that the congruence (2.50) is true for $\alpha+1$ with $\beta=\gamma=0$. By mathematical induction, the congruence (2.50) is true for all α. Suppose that the congruence (2.50) holds for $\alpha,\beta \geq 0$ with $\gamma=0$. Employing (3.13) in (2.50) with $\gamma=0$, we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}n + 2\cdot 3^{2\alpha}\cdot 5^{2\beta+2}+3\right) q^n \equiv 4f_5^3, \tag{4.7}$$

which yields

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+2}n + 2\cdot 3^{2\alpha}\cdot 5^{2\beta+2}+3\right) q^n \equiv 4f_1^3, \tag{4.8}$$

which implies that the congruence (2.50) is true for $\beta+1$ with $\gamma=0$. By mathematical induction, the congruence (2.50) is true for all non-negative integers α,β with $\gamma=0$. Suppose that the congruence (2.50) holds for $\alpha,\beta,\gamma \geq 0$. Employing (2.8) in (2.50), we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n + 2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}+3\right) q^n \equiv 4f_7^3, \tag{4.9}$$

which proves (2.51). The congruence (4.9) reduces to

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}+3\right)q^n \equiv 4f_1^3, \tag{4.10}$$

which implies that the congruence (2.50) is true for $\gamma+1$. By mathematical induction, the congruence (2.50) is true for all integers α, β and γ.
Employing (2.3) in (2.50) and then comparing the coefficients of q^{3n}, q^{3n+1} and q^{3n+2}, we obtain (2.52), (2.53) and (2.54) respectively.
Comparing the coefficients of q^{3n+1} and q^{3n+2} in (2.53), we get (2.55).
Using (3.13) in (2.50) and then comparing the coefficients of q^{5n+3}, we get (2.56).
Using (2.50) along with (3.13), we obtain (2.57).
Comparing the coefficients of q^{5n+i} for $i = 1, 2, 3, 4$ in the equation (2.56), we arrive at (2.58).

5 Proof of the Theorem (2.8)

From the equation (3.9), we have, modulo 4,

$$\sum_{n=0}^{\infty} B_{4,5}\left(16n+7\right)q^n \equiv 2\frac{f_1 f_5^3}{f_{10}}. \tag{5.1}$$

Using (2.6) in (5.1), we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(32n+7\right)q^n \equiv 2f_1^3 \tag{5.2}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}\left(32n+23\right)q^n \equiv 2f_5^3. \tag{5.3}$$

The rest of the proofs the identities (2.59)-(2.76) are similar to the proofs of the identities (2.50)-(2.58). So, we omit the details.
From the equation (3.23), we arrive at

$$\sum_{n=0}^{\infty} B_{4,5}\left(32n+11\right)q^n \equiv 2\frac{f_1 f_5^3}{f_{10}}. \tag{5.4}$$

Using (2.6) in (5.4), we arrive at

$$\sum_{n=0}^{\infty} B_{4,5}\left(64n+11\right)q^n \equiv 2f_1^3 \tag{5.5}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}\left(64n+43\right)q^n \equiv 2f_5^3. \tag{5.6}$$

The rest of the proofs of the identities (2.77)-(2.94) are similar to the proofs of the identities (2.50)-(2.58). So, we omit the details.
From the equation (3.39), we have

$$\sum_{n=0}^{\infty} B_{4,5}\left(64n+19\right)q^n \equiv 2\frac{f_1 f_{10}}{f_5}. \tag{5.7}$$

Employing (2.5) in (5.7), we obtain

$$\sum_{n=0}^{\infty} B_{4,5}\left(128n+19\right)q^n \equiv 2f_1^3 \tag{5.8}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}\left(128n+83\right)q^n \equiv 2f_5^3. \tag{5.9}$$

The rest of the proofs of the identities (2.95)-(2.112) are similar to the proofs of the identities (2.50)-(2.58). So, we omit the details.
From the equation (4.1), we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(8n+1\right)q^n \equiv 2\frac{f_1^3 f_2^3 f_5}{f_{20}} + 2q f_1 f_2 f_5^3. \tag{5.10}$$

Employing (2.6) and (2.7) in (5.10) and collecting the terms involving q^{2n} and q^{2n+1} from the resultant equation, we get (2.113) and (2.114) respectively.

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