# $(\varepsilon - \delta)$ conditions and fixed point theorems*

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**Abstract**

In this paper, we prove some fixed point theorems under generalized $(\varepsilon - \delta)$ type rational contractions in which the fixed point may be a point of discontinuity. Our results generalize and improve a host of well-known fixed point theorems existing in the literature. In addition to it we give a fixed point theorem for $(\varepsilon - \delta)$ non-expansive mappings in metric spaces. Several examples are given to illustrate our results.

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## 1 Introduction

Fixed point theorems are essentially a set of sufficient conditions which ensure the existence of a fixed point of a mapping under suitable assumptions both on the space and the mapping. In addition to guaranteeing the existence of a fixed point, it may be necessary to prove its uniqueness. From a computational point of view, a constructive algorithm to approximate the value of a fixed point is desirable. Such algorithm uses iterates of the given mapping. The questions about the existence, uniqueness and approximation of a fixed point render three substantial key characteristics of a general fixed point theorem. Fixed point theorems for contractive definitions answer all the three questions convincingly.

The well-known Banach contraction principle (BCP) states that:

**Theorem 1.1.** Let $(X, d)$ be a complete metric space and let $T$ be a self-mapping on $X$ satisfying the condition; $d(Tx, Ty) \leq kd(x, y), 0 \leq k < 1$, for each $x, y \in X$. Then $T$ has a unique fixed point.

BCP has long been one of the most important tools in the study of nonlinear problems. This theorem is very useful in the existence and uniqueness theories in analysis. Strict contractive conditions comprise a very important class of mappings which include contraction mappings as their subclass. However, strict contractive conditions do not ensure the existence of fixed points unless some strong condition like compactness of the space or some sequence of iterates is assumed to be Cauchy sequence.

In 1962, Eldelstein [11] generalized the BCP and proved the following fixed point theorem by assuming compactness of the space.

**Theorem 1.2.** Let $(X, d)$ be a compact metric space and let $T$ be a self-mapping on $X$ satisfying the condition; $d(Tx, Ty) < d(x, y)$, whenever $x \neq y$ for each $x, y \in X$. Then $T$ has a unique fixed point.

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A mapping satisfying both the contractive conditions of Theorems 1.1 and 1.2 are uniformly continuous.

One of the noteworthy generalizations of the BCP was given by Meir and Keeler [19]. They proved the following theorem:

**Theorem 1.3.** Let $(X, d)$ be a complete metric space and let $T$ be a self-mapping on $X$ such that that for any $x, y \in X$;

$$\text{for a given } \varepsilon > 0 \text{ there exists a } \delta(\varepsilon) > 0 \text{ such that}$$
$$\varepsilon \leq d(x, y) < \varepsilon + \delta \text{ implies } d(Tx, Ty) < \varepsilon.$$

Then $T$ has a unique fixed point.

In his seminal paper, Matkowski [18] combined the approaches of Eldelstein [11] and Meir-Keeler[19] and proved the following fixed point theorem:

**Theorem 1.4.** Let $(X, d)$ be a complete metric space and let $T$ be a self-mapping on $X$ such that that for any $x, y \in X$;

(i) for a given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that
$\varepsilon < d(x, y) < \varepsilon + \delta$ implies $d(Tx, Ty) \leq \varepsilon$,

(ii) $d(Tx, Ty) < d(x, y)$, for all $x, y \in X, x \neq y$.

Then $T$ has a unique fixed point.

The result of Matkowski may also be seen as the first result which extends the famous Edelstein's result in the setting of complete metric space. In a recent work, Suzuki [29] averred that the condition (i) assumed in Theorem 1.4 is one of the weakest conditions needed to ensure the existence of a fixed point for strict contractive condition.

In [13] Jachymski extended Theorem 1.4 by replacing $d(x, y)$ by a convex combination of distances between the four points $x, y, Tx$ and $Ty$ which is a variant of quasi contraction introduced by Ćirić [7].

**Theorem 1.5.** Let $(X, d)$ be a complete metric space and let $T$ be a continuous self-mapping on $X$ such that that for any $x, y \in X$;

(i) for a given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that
$\varepsilon < m(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty), [d(x, Ty)+d(y, Tx)]/2\} < \varepsilon+\delta$ implies $d(Tx, Ty) \leq \varepsilon$,

(ii) $d(Tx, Ty) < m(x, y)$, for all $x, y \in X, x \neq y$.

Then $T$ has a unique fixed point.

In a recent work the first author and Pant [2] proved a weaker version of Theorem 1.5 by completely removing the requirement of continuity of the mapping. However, the first result in this direction is credited to Pant [24].

A large body of literature of fixed point theorems has grown around quasi contraction introduced by Ćirić [7] (see also [26]) and it has become an area of vigorous research activity. However, fixed point theory for contractive definitions involving rational expressions [1, 5, 8, 12, 14, 15, 16, 17, 23, 28] is equally fruitful and Dass and Gupta [8] initiated some work in this direction. The following fixed point theorem was given by Dass and Gupta:

**Theorem 1.6.** Let $(X, d)$ be a complete metric space and let $T$ be a continuous self-mapping on $X$ such that for any $x, y \in X$;

$$d(Tx, Ty) \leq \alpha \frac{d(y, Ty)(1 + d(x, Tx))}{1 + d(x, y)} + \beta d(x, y),$$

where $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$. Then $T$ has a unique fixed point.

It is of worth to note that contractive definition considered by Dass and Gupta [8] is not symmetric in their variables. In 2015, Almeida et al. [1] used the following variant of contractive definition involving rational expressions to prove the existence of a fixed point by using different kinds of control functions. Moreover, their definition was free from the issue of lack of symmetry in their variables.

$$d(Tx, Ty) \leq \max\left\{ d(x, y), \frac{d(x, Tx)(1 + d(y, Ty))}{1 + d(x, y)}, \frac{d(y, Ty)(1 + d(x, Tx))}{1 + d(x, y)} \right\}.$$

Fixed point theorems for discontinuous mappings have found wide applications in Biology, Economics, Artificial Intelligence and neural networks with discontinuous activation functions and the study of fixed point theorems for discontinuous mappings is presently a very active area of research (e.g., [9, 10, 20, 21, 22, 30, **?**]).

In this paper, we prove convergence theorems under generalized $(\varepsilon - \delta)$ type rational contractions in which the fixed point may be a point of discontinuity. As a by-product we provide few more affirmative answers to the open question regarding the existence of contractive conditions which assure a fixed point but do not force the mapping to be continuous at the fixed point (see Rhoades [26]).

## 2 Main results

In what follows we shall use the following notation:

$$M(x, y) = \max\left\{ d(x, y), \tfrac{d(x,Tx)(1+d(y,Ty))}{1+d(x,y)}, \tfrac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)}, \tfrac{d(x,Ty)+d(y,Tx)}{2} \right\}.$$

**Theorem 2.1.** Let $(X, d)$ be a complete metric space and let $T$ be a self-mapping on $X$ such that for any $x, y \in X$;

(i) for a given $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\varepsilon < M(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) \leq \varepsilon;$$

(ii) $d(Tx, Ty) \leq \varphi(M(x, y))$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\varphi(t) < t$ for each $t > 0$.

Then $T$ has a unique fixed point, say $z \in X$, and for each $x \in X$ the sequence of iterates $T^n x$ converges to the fixed point. Moreover, $T$ is continuous at $z$ if and only if $lim_{x \to z} M(x, z) = 0$.

**Proof** Let $x_0$ be any point in $X$. Define a sequence $\{x_n\}$ in $X$ recursively by $x_{n+1} = Tx_n$, i.e., $x_{n+1} = T^n x_0$ for some $n \in \mathbb{N} \bigcup \{0\}$ . If $x_n = x_{n+1}$ for some $n$, then $x_n = x_{n+1} = x_{n+2} = ...$, that is, $\{x_{n+1}\} = \{T^n x_0\}$ is a Cauchy sequence and $x_n$ is a fixed point of $T$. Therefore we assume

$x_n \neq x_{n+1}$ and set $c_n = d(x_n, x_{n+1})$ for all $n \in \mathbb{N}\bigcup\{0\}$. Let us first prove that for all $n \in \mathbb{N}$ we have

$$c_n < c_{n-1}. \tag{2.1}$$

To see this observe that

$$c_n = d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) < \varphi(M(x_{n-1}, x_n)) < M(x_{n-1}, x_n) =$$
$$\max\left\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, Tx_{n-1})(1+d(x_n, Tx_n))}{1+d(x_{n-1}, x_n)}, \frac{d(x_n, Tx_n)(1+d(x_{n-1}, Tx_{n-1}))}{1+d(x_{n-1}, x_n)}, \frac{d(x_{n-1}, Tx_n)+d(x_n, Tx_{n-1})}{2}\right\},$$

which implies that

$$c_n < M(x_{n-1}, x_n) = \max\left\{c_{n-1}, \frac{c_{n-1}(1+c_n)}{1+c_{n-1}}, c_n, \frac{d(x_{n-1}, x_{n+1})}{2}\right\}.$$

**Case I** If $M(x_{n-1}, x_n) = c_{n-1}$, then clearly (2.1) holds.

**Case II** If $M(x_{n-1}, x_n) = \frac{c_{n-1}(1+c_n)}{1+c_{n-1}}$, then we get $c_n < c_{n-1}$.

**Case III** If $M(x_{n-1}, x_n) = c_n$, then we get $c_n < c_n$, a contradiction.

**Case IV** Finally, if $M(x_{n-1}, x_n) = \frac{d(x_{n-1}, x_{n+1})}{2} \leq \frac{c_{n-1}+c_n}{2}$, then we get $c_n < c_{n-1}$.

Thus in all cases (2.1) is true and $\{c_n\}$ is a strictly decreasing sequence of positive real numbers and, hence, tends to a limit $c \geq 0$. If possible, suppose $c > 0$. Then there exists a positive integer $k \in \mathbb{N}$ such that $n \geq k$ implies

$$c < c_n < c + \delta(c). \tag{2.2}$$

It follows from (i) and $c_n < c_{n-1}$ that $c_n \leq c$, for $n \geq k$, which contradicts (2.2). Thus we have $c = 0$.

We shall now show that $\{x_n\}$ is a Cauchy sequence. Fix an $\varepsilon > 0$. Without loss of generality, we may assume that $\delta = \delta(\varepsilon) < \varepsilon$. Since $c_n \to 0$, there exists $k \in \mathbb{N}$ such that $c_n < \frac{\delta}{2}$, for $n \geq k$.

We shall use induction to show that, for any $n \in \mathbb{N}$,

$$d(x_k, x_{k+n}) < \varepsilon + \frac{\delta}{2}. \tag{2.3}$$

Inequality (2.3) holds for $n = 1$. Assuming (2.3) is true for some $n$ we shall prove it for $n + 1$. By the triangle inequality, we have

$$d(x_k, x_{k+n+1}) \leq d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+n+1}). \tag{2.4}$$

Observe that it suffices to show that

$$d(x_{k+1}, x_{k+n+1}) \leq \varepsilon. \tag{2.5}$$

To show it we shall prove that $M(x_k, x_{k+n}) \leq \varepsilon + \delta$, where

$$M(x_k, x_{k+n}) =$$
$$\max\left\{d(x_k, x_{k+n}), \frac{d(x_k,Tx_k)(1+d(x_{k+n},Tx_{n+k}))}{1+d(x_k,x_{k+n})}, \frac{d(x_{k+n},Tx_{k+n})(1+d(x_k,Tx_k))}{1+d(x_k,x_{k+n})}, \frac{d(x_k,Tx_{k+n})+d(x_{k+n},Tx_k)}{2}\right\}.$$

which implies that

$$M(x_k, x_{k+n}) =$$
$$\max\left\{d(x_k, x_{k+n}), \frac{d(x_k,x_{k+1})(1+d(x_{k+n},x_{k+n+1}))}{1+d(x_k,x_{k+n})}, \frac{d(x_{k+n},x_{k+n+1})(1+d(x_k,x_{k+1}))}{1+d(x_k,x_{k+n})}, \frac{d(x_k,x_{k+n+1})+d(x_{k+n},x_{k+1})}{2}\right\}$$

By the induction hypothesis, we get

$$d(x_k, x_{k+n}) < \varepsilon + \frac{\delta}{2},$$
$$\frac{d(x_k,x_{k+1})(1+d(x_{k+n},x_{n+k+1}))}{1+d(x_k,x_{k+n})} < \frac{\frac{\delta}{2}(1+\frac{\delta}{2})}{(1+\varepsilon+\frac{\delta}{2})} < \frac{\delta}{2},$$
$$\frac{d(x_{k+n},x_{k+n+1})(1+d(x_k,x_{k+1}))}{1+d(x_k,x_{k+n})} < \frac{\frac{\delta}{2}(1+\frac{\delta}{2})}{(1+\varepsilon+\frac{\delta}{2})} < \frac{\delta}{2},$$
$$\frac{d(x_k,x_{k+n+1})+d(x_{k+n},x_{k+1})}{2} < \frac{d(x_k,x_{k+n})+d(x_{k+n},x_{k+n+1})+d(x_{k+1},x_k)+d(x_k,x_{k+n})}{2} < \varepsilon + \delta.$$

Thus $M(x_k, x_{k+n}) < \varepsilon + \delta$ so by (i) $d(x_{k+1}, x_{k+n+1}) \leq \varepsilon$, completing the induction. Hence (2.3) implies that $\{x_n\}$ is a Cauchy sequence. Since $X$ is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. Also $Tx_n \to z$. We claim that $Tz = z$. For if $Tz \neq z$, we get

$$d(Tz, Tx_n) \leq \varphi\left(\max\left\{d(z, x_n), \frac{d(z,Tz)(1+d(x_n,Tx_n))}{1+d(z,x_n)}, \frac{d(x_n,Tx_n)(1+d(z,Tz))}{1+d(z,x_n)}, \frac{d(z,Tx_n)+d(x_n+Tz)}{2}\right\}\right).$$

On letting $n \to \infty$ this yields, $d(Tz, z) \leq \varphi(d(Tz, z)) < d(Tz, z)$, a contradiction. Thus $z$ is a fixed point of $T$. Uniqueness of the fixed point follows from (ii).

Now, let $T$ be continuous at the fixed point $z$ and $x_n \to z$ as $n \to \infty$. Then $Tx_n \to Tz = z$. Hence

$$\lim_{x_n \to z} M(x_n, z) = \max\left\{d(x_n, z), \frac{d(x_n,Tx_n)(1+d(z,Tz))}{1+d(x_n,z)}, \frac{d(z,Tz)(1+d(x_n,Tx_n))}{1+d(x_n,z)}, \frac{d(x_n,Tz)+d(z,Tx_n)}{2}\right\} = 0.$$

Conversely, if $\lim_{x_n \to z} M(x_n, z) = 0$, then $d(x_n, Tx_n) \to 0$ as $x_n \to z$. This implies that $Tx_n \to z = Tz$, i.e., $T$ is continuous at $z$. This concludes the theorem.

**Remark 1.** The last part of Theorem 2.1 can alternatively be stated as: $T$ is discontinuous at $z$ if and only if $lim_{x \to z} M(x, z) \neq 0$.

Some consequences of Theorem 2.1 are the following results.

**Corollary 2.2.** Let $(X, d)$ be a complete metric space and let $T$ be a self-mapping on $X$. Suppose there exists a continuous function $\varphi : (0, \infty) \to (0, \infty)$ satisfying $\varphi(t) < t$ for $t > 0$ such that for any $x, y$ in $X$ we have

$$d(Tx, Ty) \leq \varphi \max\left\{d(x, y), \frac{d(x,Tx)(1+d(y,Ty))}{1+d(x,y)}, \frac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)}, \frac{d(x,Ty)+d(y,Tx))}{2}\right\}.$$

Then $T$ has a unique fixed point.

**Corollary 2.3.** Let $(X, d)$ be a complete metric space and let $T$ be a self-mapping on $X$. Suppose there exists a non-decreasing function $\varphi : (0, \infty) \to (0, \infty)$ satisfying $\varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for $t > 0$ such that for any $x, y$ in $X$ we have

$$d(Tx, Ty) \leq \varphi \max\left\{ d(x, y), \frac{d(x,Tx)(1+d(y,Ty))}{1+d(x,y)}, \frac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)}, \frac{d(x,Ty)+d(y,Tx))}{2} \right\}.$$

Then $T$ has a unique fixed point.

**Corollary 2.4.** Let $(X, d)$ be a complete metric space and let $T$ be a self-mapping on $X$ such that for any $x, y \in X$

$$d(Tx, Ty) \leq k \max\left\{ d(x, y), \frac{d(x,Tx)(1+d(y,Ty))}{1+d(x,y)}, \frac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)}, \frac{d(x,Ty)+d(y,Tx))}{2} \right\},$$

where $k \in (0, 1)$. Then $T$ has a unique fixed point.

**Corollary 2.5.** Let $(X, d)$ be a complete metric space and let $T$ be a self-mapping on $X$ such that for any $x, y \in X$

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta \frac{d(x,Tx)(1+d(y,Ty))}{1+d(x,y)} + \gamma \frac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)} + \delta \frac{d(x,Ty)+d(y,Tx))}{2},$$

where $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + \beta + \gamma + 2\delta < 1$. Then $T$ has a unique fixed point.

**Definition 2.1.** A self-mapping $T$ of a metric space $(X, d)$ is called $k-$continuous [25], $k = 1, 2, 3, ...$, if $T^k x_n \to Tz$, whenever $\{x_n\}$ is a sequence in $X$ such that $T^{k-1} x_n \to z$.

**Remark 2.** It is important to note that for a self-mapping $T$ of a metric space $(X, d)$, the notion of $1-$continuity coincides with continuity. However,

$$1-\text{continuity} \Rightarrow 2-\text{continuity} \Rightarrow 3-\text{continuity} \Rightarrow ...,$$

but not conversely. The following example illustrates this very fact.

**Example 2.2.** Let $X = [0, 1]$ and $d$ be the usual metric on $X$. Define $T : X \to X$ by

$$T(x) = 1/2 \text{ if } x \in [0, 1/2],\ T(x) = 0 \text{ if } x \in (1/2, 1].$$

Then $Tx_n \to t \Rightarrow T^2 x_n \to Tt$ since $Tx_n \to t$ implies $t = 0$ or $t = 1/2$ and $T^2 x_n \to 1/2 = T1/2$ for all $n$. Hence $T$ is 2-continuous. However, $T$ is not 1-continuous at $x = 1/2$.

**Remark 3.** In [25] Pant and Pant shown that continuity of $T^k$ and $k-$continuity of $T$ are independent conditions when $k > 1$.

**Definition 2.3.** A self-mapping $T$ of a metric space $(X, d)$ is called orbitally continuous [6, 7] at a point $z \in X$ if for any sequence $\{x_n\} \subset O(x; T)$ (for some $x \in X$), where $O(x; T) = \{T^n x : n = 0, 1, 2, ...\}$ is called the orbit of the self-mapping $T$ at the point $x \in X$ $x_n \to z$ implies $Tx_n \to Tz$ as $n \to \infty$.

Every continuous self-mapping of a metric space is orbitally continuous, but the converse need not be true (see Example 2.4).

In the next theorem, we replace condition (ii) in Theorem 2.1 above by a weaker plain contractive condition. In order to ensure the existence of a fixed point for such mappings we assume $T$ to satisfy some weaker form of continuity.

**Theorem 2.6.** Let $(X, d)$ be a complete metric space. Let $T$ be a self-mapping on $X$ such that for any $x, y \in X$;

(i) for a given $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\varepsilon < M(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) \leq \varepsilon;$$

(ii) $d(Tx, Ty) < M(x, y)$, whenever $M(x, y) > 0$.

If $T$ is $k$-continuous or $T^k$ is continuous for some $k \geq 1$ or $T$ is orbitally continuous then $T$ has a unique fixed point in $X$.

**Proof** Let $x_0$ be any point in $X$. Define a sequence $\{x_n\}$ in $X$ recursively by $x_{n+1} = Tx_n$, i.e., $x_{n+1} = T^n x_0$ for some $n \in \mathbb{N} \bigcup \{0\}$ . Following the proof of Theorem 2.1 we conclude that $\{x_n\}$ is a Cauchy sequence. Since $X$ is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. Also for each $k \geq 1$ we have $T^k x_n \to z$.

Now suppose that $T$ is $k-$continuous. Since $T^{k-1}x_n \to z$, $k$-continuity of $T$ implies that $lim_{n\to\infty} T^k x_n = Tz$. This yields $z = Tz$, that is, $z$ is a fixed point of $T$.

Next suppose that $T^k$ is continuous for some positive integer $k$. Then $lim_{n\to\infty} T^k x_n = T^k z$. This yields $T^k z = z$ as $T^k x_n \to z$. If $z \neq Tz$ we get

$$d(z, Tz) = d(T^k z, T^{k+1} z) < \max\left\{ d(T^{k-1}z, T^k z), \frac{d(T^{k-1}z, T^k z)(1+d(T^k z, T^{k+1} z))}{1+d(T^{k-1}z, T^k z)}, \right.$$
$$\left. \frac{d(T^k z, T^{k+1} z)(1+d(T^{k-1}z, T^k z))}{1+d(T^{k-1}z, T^k z)}, \frac{d(T^{k-1}z, T^k z)+d(T^k z, T^{k+1} z)}{2} \right\} = d(T^{k-1}z, T^k z) < ... < d(z, Tz),$$

a contradiction. Thus $z$ is a fixed point of $T$.

Finally, suppose that $T$ is orbitally continuous. Since $lim_{n\to\infty} x_n = z$, orbital continuity implies that $lim_{n\to\infty} Tx_n = Tz$. This gives $Tz = z$, that is, $z$ is a fixed point of $T$. Uniqueness of the fixed point follows from (ii).

**Remark 4.** In view of the results of [2, 3] Theorem 2.6 is also true if we replace $M(x, y)$ by either

$$M_1(x, y) = \max\left\{ M(x, y), d(x, Tx), d(y, Ty) \right\}$$

or

$$M_2(x, y) = \max\left\{ M(x, y), ad(x, Tx) + (1 - a)d(y, Ty), (1 - a)d(x, Tx) + ad(y, Ty) \right\}, 0 < a < 1.$$

We now give examples to illustrate Theorem 2.6.

**Example 2.4.** Let $X = [0, 6]$ and $d$ be the usual metric on $X$. Define $T : X \to X$ by

$$T(x) = \begin{cases} 3, \text{ if } 0 \leq x \leq 3; \\ 0, \text{ if } 3 < x \leq 6. \end{cases}$$

Then $T$ satisfies all the conditions of Theorems 2.1 and 2.6 and has a unique fixed point $x = 3$ at which $T$ is discontinuous. The mapping $T$ satisfies condition (i) with $\delta(\varepsilon) = 3$ for $\varepsilon \geq 3$ and $\delta(\varepsilon) = 3 - \varepsilon$ for $\varepsilon < 3$. Also, $T$ satisfies the contractive condition (ii) with $\varphi(t) = 3$ for $t > 3$ and $\varphi(t) = \frac{t}{2}$ for $t \leq 3$. It can also be easily seen that $lim_{x\to 3} M(x, 3) \neq 0$.

**Example 2.5.** Let $X = [-1, 3]$ and $d$ be the usual metric on $X$. Define $T : X \to X$ by

$$T(x) = \begin{cases} 1/2, \text{ if } -1 \leq x < 0; \\ 1, \text{ if } 0 \leq x \leq 1; \\ 0, \text{ if } 1 < x \leq 2; \\ -1, \text{ if } 2 < x \leq 3. \end{cases}$$

Then $T$ satisfies all the conditions of Theorem 2.6 and has a unique fixed point $x = 1$ at which $T$ is discontinuous. It may be noted that $T$ is 3-continuous.

**Remark 5.** The above theorems generalize and improve the results due to Boyd and Wang [4], Dass and Gupta [8], Eldelstein[11], Matkowski [18], Meir and Keeler [19] and Rhoades [27].

**Remark 6.** In the above proved results we have not assumed the mapping to be continuous. In fact the mapping assumed by us is discontinuous at the fixed point. We demonstrate that $(\varepsilon - \delta)$ rational definitions are strong enough to generate a fixed point and at the same time do not force the mapping to be continuous at the fixed point. Thus we provide new and distinct answers to the problem posed by Rhoades [26].

## 3 Fixed points of non-expansive mapping in metric space

In Theorem 2.6 the $(\varepsilon, \delta)$ condition (i) has been employed with the contractive condition (ii) because condition (i) by itself does not imply a contractive condition. It is observed that mappings satisfying non-expansive type condition, i.e., $(d(Tx, Ty) \leq d(x, y)$ may also satisfy condition (i) of Theorem 2.1. Motivated by this we now prove fixed point theorem for generalized non-expansive type mapping satisfying $(\varepsilon, \delta)$ condition.

**Theorem 3.1.** Let $(X, d)$ be a complete metric space. Let $T$ be a continuous self-mapping on $X$ such that for any $x, y \in X$;

(i) for a given $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\varepsilon < M(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) \leq \varepsilon;$$

(ii) $d(Tx, Ty) \leq M(x, y)$.

Then $T$ has a fixed point.

**Theorem 3.2.** Let $x_0$ be any point in $X$. Define a sequence $\{x_n\}$ in $X$ recursively by $x_{n+1} = Tx_n$, i.e., $x_{n+1} = T^n x_0$ for some $n \in \mathbb{N}\bigcup\{0\}$ . Then following the proof of Theorem 2.1 we conclude that $\{x_n\}$ is a Cauchy sequence. Since $X$ is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. Also $Tx_n \to z$. Continuity of $T$ implies that $Tx_n \to Tz$ as $n \to \infty$. Hence $z = Tz$ and $z$ is a fixed point of $T$.

It may be observed that there exists a number of ($\varepsilon - \delta$) type non-expansive conditions which yield more than one fixed point. The following example illustrates this fact.

**Example 3.1.** Let $X = [0, 1]$ and $d$ be the usual metric on $X$. Define $T : X \to X$ by

$$Tx = [x] \text{ (the greatest integer function), i.e., } Tx = 0 \text{ if } x < 1 \text{ and } T1 = 1.$$

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