Operator splitting method for numerical solution of modified equal width equation

İhsan Çelikkaya

Department of Mathematics, Faculty of Science and Art, Inönü University, Malatya, 44280, Turkey
E-mail: ihsancelikkaya85@gmail.com

Abstract

In this manuscript, numerical solutions of the equations in the form of $u_t = Au + B(u)$ have been sought for, where A and B are linear and nonlinear operators, respectively. The modified equal width (MEW) equation has been converted into two sub problems. Then, the sub problems were solved according to the Strang splitting scheme by applying the cubic B-spline collocation finite element method. Thus, more accurate results of the equation MEW have been obtained than those non-splitting users. In order to test the accuracy and efficiency of the present method; single soliton, interaction of two solitons and Maxwellian initial condition pulse problems have been considered. Moreover, the stability analysis of each sub problem has been investigated by von-Neumann analysis method.

2010 Mathematics Subject Classification. **35Q51**. 74J35, 33F10
Keywords. operator splitting method, soliton, finite element method, B-splines, solitary waves.

1 Introduction

Nonlinear phenomena play a fundamental role in many fields of science such as plasma physics, fluid mechanics, solid state physics, optical fibers, chemical kinetics and geochemistry. Dispersion, dissipation, diffusion, reaction and convection phenomena are very important in nonlinear wave equations [1].

The famously known nonlinear wave model Korteweg de-Vries (KdV) [2] equation is given in the following form

$$u_t + 6uu_x + u_{xxx} = 0.$$

Another equation describing nonlinear dispersive wave phenomena is regularised long wave (RLW) equation

$$u_t + u_x + uu_x - u_{xxt} = 0,$$

and it is more widely known one as an alternative to KdV equation. Another less frequently used equation modelling these wave phenomena is the equal width (EW) equation given in the following form

$$u_t + uu_x - u_{xxt} = 0,$$

has been introduced by [3, 4]. In the present study, the modified equal width (MEW) equation related to RLW has been given for $x \to \pm\infty$ when $u \to 0$ together with the physical boundary conditions

$$u_t + 3u^2u_x - \mu u_{xxt} = 0, \tag{1}$$

where t is time, x is pace coordinate, μ is a positive parameter and $u(x,t)$ is as in the fluid problem, u is related to the vertical displacement of the water surface, whereas in the plasma applications u

Tbilisi Mathematical Journal 12(3) (2019), pp. 51–67.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 24 May 2018.
Accepted for publication: 25 June 2019.

is the negative of the electrostatic potential [8]. MEW (1) equation has the following solitary wave solution

$$u(x,t) = A \sec h \left[k \left(x - x_0 - ct\right)\right], \tag{2}$$

where $k = 1/\sqrt{\mu}$ and $c = A^2/2$. Solitary waves are travelling waves conserving their shapes and speeds due to the balance between nonlinearity and dispersive effects. Meanwhile, a soliton is a very special type of solitary wave, preserving their shapes and speeds even after colliding with another wave [4]. The three invariants satisfied by MEW equation (1) are given by Olwer [5] as follows

$$I_1 = \int_{-\infty}^{+\infty} u dx, \quad I_2 = \int_{-\infty}^{+\infty} \left(u^2 + \mu (u_x)^2\right) dx, \quad I_3 = \int_{-\infty}^{+\infty} u^4 dx. \tag{3}$$

There are several analytical and numerical solutions of the MEW equation under various initial and boundary conditions. Among others, Hamdi *et al.* [6] derived exact solitary wave solutions of the generalized equal width wave equation. Wazwaz [1] investigated the MEW equation and two of its variants by the tanh and the sine-cosine methods. Esen and Kutluay [7] use a linearized numerical scheme based on finite difference method to obtain solitary wave solutions of the one-dimensional modified equal width (MEW) equation. Evans and Raslan [8] studied the generalized equal width wave equation by using the collocation method based on quadratic B-splines to obtain the numerical solutions of MEW equation. Raslan [9] solved generalized EW equation numerically by collocation of cubic B-splines finite element method. Jin [10] propose an analytical approach based on the homotopy perturbation method to solve the modified equal width equation. Variational iteration method is introduced to solve the MEW equation by Lu [11]. Esen [12] obtained numerical solution of the one-dimensional modified equal width wave (MEW) equation by using a lumped Galerkin method based on quadratic B-spline finite elements. Zaki [13] considered the solitary wave interactions for the MEW equation by collocation method using quintic B-spline finite elements and obtained the numerical solution of the EW equation by using least-squares method [14]. S.G. Karakoç and H. Zeybek [15] solved the generalized equal width (GEW) wave equation by using lumped Galerkin approach with cubic B-spline functions and used quintic B-spline collocation algorithm with two different linearization techniques in [16]. Roshan has solved the equation by using the Petrov-Galerkin method [17]. Geyikli and Karakoç [18] obtained Numerical solutions of the modified equal width wave equation by using collocation method with septic B-spline finite elements with three different linearization techniques and subdomain finite element method with quartic B-splines in [19]. Saka [20] solved time split MEW equation and space split MEW equation using Quintic B-spline collocation method. Karakoç *at al.* [21] concerned with applying the cubic B-spline function to develop a numerical method for approximating the analytic solution of the MEW equation. Generalized equal width wave (GEW) equation is solved using moving least squares collocation (MLSC) method by Kaplan and Dereli [22]. The improved element-free Galerkin method applied to MEW equation by Cheng [23], and so on.

We have used the error norms L_2 and L_∞ in order to test the difference between analytical and

numerical solutions. The error norms L_2 and L_∞ are calculated using the following formulas

$$L_2 = \left\|u^{exact} - U\right\|_2 = \left[h\sum_{j=1}^{N}\left|u_j^{exact} - U_j\right|^2\right]^{1/2}, \tag{4}$$
$$L_\infty = \left\|u^{exact} - U\right\|_\infty = \max_j \left|u_j^{exact} - U_j\right|.$$

In this study, the MEW equation has been taken into consideration using two subproblems, one nonlinear and the other one linear. Each sub problem has been numerically solved successively using finite element cubic B-spline collocation method. After that, the solution of the main equation has been obtained using Strang splitting [24] schema.

2 Operator splitting methods

Generally, in splitting methods, the original equation is splitted into several equations, and each splitted equation is separately solved independently of the original one over the time interval $[t_n, t_{n+1}]$. These methods, in general, are called as *time splitting* or *fractional step* methods. In case of splitting the equation in such a way that it includes different pysical phenomena, the method is known as "*operator splitting*" [25].

Let us consider a Cauchy problem given as follows

$$\frac{du(t)}{du} = Cu(t), \quad u(0) = u_0, \quad t \in [0, T]. \tag{5}$$

In (5), it is assumed that the function $u(x,t)$ is semi-discretized along spatial direction. We are going to concentrate on cases where the operator $C = \hat{A} + \hat{B}$ can be written as summation of two linear (and/or nonlinear) operators. That is it can be written as follows

$$\frac{du(t)}{du} = \hat{A}u(t) + \hat{B}u(t), \quad u(0) = u_0, \quad t \in [0, T]. \tag{6}$$

Here, the vector $u(x,t)$ is the solution vector obtained from the $u_0 \in X$ initial condition, and the operators C, $\hat{A}$, $\hat{B}$ are bounded or unbounded operators in a finite or infinite X Banach space. With the aid of the Lie operator formulation, the expression (6) in general (may be nonlinear) can be written as follows

$$\frac{du(t)}{du} = Au(t) + Bu(t) \tag{7}$$

Here, the operators A and B are Lie operators applied to the function $u(t)$ as follows

$$A = \hat{A}\left(u(t)\right)\frac{\partial}{\partial u}, \quad B = \hat{B}\left(u(t)\right)\frac{\partial}{\partial u} \tag{8}$$

The formal solution of the problem (7) is $u(t_{n+1}) = e^{\Delta t(A+B)}u(t_n)$. This solution can also be written as follows using the Taylor series expansion of exponential function

$$u(t_{n+1}) = e^{\Delta t(A+B)}u(t_n) = \sum_{k=0}^{\infty}\frac{t^k}{k!}\left(\hat{A}\left(u(t)\right)\frac{\partial}{\partial u} + \hat{B}\left(u(t)\right)\frac{\partial}{\partial u}\right)^k u(t_n).$$

In order to solve Eq. (7) numerically, the splitting technique splits the problem into two sub problems as follows

$$\frac{du(t)}{dt} = Au(t) \quad \text{and} \quad \frac{du(t)}{dt} = Bu(t) \tag{9}$$

and tries to find the solution numerically or analitically [26]. Let us assume that $\varphi_{\Delta t}^{[A]}$ and $\varphi_{\Delta t}^{[B]}$ are the exact or numerical solutions of the equations in (9) involving the operators A and B, respevtively, then the most basic first order splitting technique is defined as follows

$$L_{\Delta t} = \varphi_{\Delta t}^{[A]} \circ \varphi_{\Delta t}^{[B]} \equiv e^{\Delta t A} e^{\Delta t B} \quad \text{or} \quad L_{\Delta t}^{*} = \varphi_{\Delta t}^{[B]} \circ \varphi_{\Delta t}^{[A]} \equiv e^{\Delta t B} e^{\Delta t A}$$

and known as Lie-Trotter splitting [27] techniques. If we change the place of operators A and B and take the combination for half time step as follows

$$S_{\Delta t} = e^{\frac{\Delta t}{2} A} e^{\Delta t B} e^{\frac{\Delta t}{2} A} \quad \text{or } S_{\Delta t}^{*} = e^{\frac{\Delta t}{2} B} e^{\Delta t A} e^{\frac{\Delta t}{2} B} \tag{10}$$

we obtain so-called symmetric Marchuk [28] or more widely known Strang splitting [24] techniques having the schemes "$A - B - A$" and "$B - A - B$". The algorithm for Strang splitting technique as follows

$$\begin{aligned}
&\frac{du^{*}(t)}{dt} = Au^{*}(t), \quad u^{*}(t_n) = u_n^0, && t \in \left[t_n, t_{n+\frac{1}{2}}\right] \\
&\frac{du^{**}(t)}{dt} = Bu^{**}(t), \quad u^{**}(t_n) = u^{*}(t_{n+\frac{1}{2}}), && t \in [t_n, t_{n+1}] \\
&\frac{du^{***}(t)}{dt} = Au^{***}(t), \quad u^{***}(t_{n+\frac{1}{2}}) = u^{**}(t_{n+1}), && t \in \left[t_{n+\frac{1}{2}}, t_{n+1}\right].
\end{aligned}$$

Here $t_{n+\frac{1}{2}} = t_n + \frac{\Delta t}{2}$ and the desired solutions are obtained from equation $u(t_{n+1}) = u^{***}(t_{n+1})$. Replacing the original problem with sub-problems naturally leads to a fault called splitting error [29]. E_n being the local truncation error, then the following equation

$$u(t_{n+1}) = e^{\Delta t(A+B)} u(t_n) \approx e^{\frac{\Delta t}{2} A} e^{\Delta t B} e^{\frac{\Delta t}{2} A} u(t_n) + \Delta t E_n$$

can be written. Using the Taylor series of exponential functions E_n is obtained as follows

$$E_n = \frac{\Delta t^2}{24} \left([A,[B,A]] - 2\,[B,[A,B]]\right) u(t_n) + O(\Delta t^3). \tag{11}$$

3 Cubic B-spline collocation

Let us assume a solution domain $[a, b]$ be a closed interval, and a smooth finite fragment x_m, $m = 0, 1, ..., N$, of this region is $a = x_0 < x_1 < ... < x_N = b$. If we define the distance between two consecutive nodal points as $h = x_{m+1} - x_m$, and take $\Phi_m(x)$, $m = -1(1)N + 1$, then cubic B-spline base functions on the closed domain $[a, b]$ at nodal point x_m can be defined as follows

$$\Phi_m(x) = \frac{1}{h^3} \begin{cases} (x - x_{m-2})^3, & x \in [x_{m-2}, x_{m-1}] \\ h^3 + 3h^2(x - x_{m-1}) + 3h(x - x_{m-1})^2 - 3(x - x_{m-1})^3, & x \in [x_{m-1}, x_m] \\ h^3 + 3h^2(x_{m+1} - x) + 3h(x_{m+1} - x)^2 - 3(x_{m+1} - x)^3, & x \in [x_m, x_{m+1}] \\ (x_{m+2} - x)^3, & x \in [x_{m+1}, x_{m+2}] \\ 0, & \text{otherwise} \end{cases} \tag{12}$$

as stated in Prenter [30]. If we assume that the function $u(x,t)$ is defined on the interval $[a,b]$ then function $u(x,t)$ can be approximated as follows in terms of cubic B-spline base functions and time dependent parameters $\delta_m(t)$ as follows

$$U(x,t) \cong \sum_{m=-1}^{N+1} \delta_m(t)\Phi_m(x). \tag{13}$$

Here $\delta_m(t)$ are the time dependent parameters that can be obtained by using Eq. (1) and auxiliary conditions. For the solution of Eq. (1), the values of u and its and the second order derivatives at the nodal points using the approximations in Eqs. (12) and (13) are obtained as follows in terms of time parameters $\delta_m(t)$

$$\begin{aligned} U_m &= U(x_m) = \delta_{m-1} + 4\delta_m + \delta_{m+1} \\ U_m^{'} &= U'(x_m) = \frac{3}{h}\left(\delta_{m+1} - \delta_{m-1}\right) \\ U_m'' &= U''(x_m) = \frac{6}{h^2}\left(\delta_{m-1} - 2\delta_m + \delta_{m+1}\right) \end{aligned} \tag{14}$$

Here $'$ and $''$ denotes the first order and second order derivatives with respect to space variable x, respectively.

4 Operator splitting solution of MEW equation

The time-split form of MEW (1) equation is as follows

$$u_t - \mu u_{xxt} + 3u^2 u_x = 0, \tag{15}$$

$$u_t - \mu u_{xxt} = 0. \tag{16}$$

If the values of U, $U^{'}$ and $U^{''}$in Eq. (14) are replaced in Eqs. (15) and (16), the following system of ordinary differential equations is obtained

$$\overset{\bullet}{\delta}_{m-1} + 4\overset{\bullet}{\delta}_m + \overset{\bullet}{\delta}_{m+1} - \frac{6\mu}{h^2}\left(\overset{\bullet}{\delta}_{m-1} - 2\overset{\bullet}{\delta}_m + \overset{\bullet}{\delta}_{m+1}\right) + \frac{9z_m}{h}\left(\delta_{m+1} - \delta_{m-1}\right) = 0, \tag{17}$$

$$\overset{\bullet}{\delta}_{m-1} + 4\overset{\bullet}{\delta}_m + \overset{\bullet}{\delta}_{m+1} - \frac{6\mu}{h^2}\left(\overset{\bullet}{\delta}_{m-1} - 2\overset{\bullet}{\delta}_m + \overset{\bullet}{\delta}_{m+1}\right) = 0. \tag{18}$$

Here the symbol "$\bullet$" denotes the first order derivative with respect to t and the linearization is made as follows

$$z_m = \left(\delta_{m-1} + 4\delta_m + \delta_{m+1}\right)^2.$$

In Eqs. (17) and (18), if Crank-Nicolson approximation $\frac{\delta_m^{n+1}+\delta_m^n}{2}$ is written in place of parameters δ_m and forward difference scheme $\frac{\delta_m^{n+1}-\delta_m^n}{\Delta t}$ is written in place of parameters $\overset{\bullet}{\delta}_m$, the following system of equations are obtained, respectively

$$\alpha_1\delta_{m-1}^{n+1} + \alpha_2\delta_m^{n+1} + \alpha_3\delta_{m+1}^{n+1} = \alpha_3\delta_{m-1}^n + \alpha_2\delta_m^n + \alpha_1\delta_{m+1}^n \tag{19}$$

$$\beta_1\delta_{m-1}^{n+1}+\alpha_2\delta_m^{n+1}+\beta_1\delta_{m+1}^{n+1}=\beta_1\delta_{m-1}^{n}+\alpha_2\delta_m^{n}+\beta_1\delta_{m+1}^{n} \tag{20}$$

$$\alpha_1=1-\frac{6\mu}{h^2}-\frac{9z_m\Delta t}{2h},\ \alpha_2=4+\frac{12\mu}{h^2},\ \alpha_3=1-\frac{6\mu}{h^2}+\frac{9z_m\Delta t}{2h},\beta_1=1-\frac{6\mu}{h^2}. \tag{21}$$

The systems (19) and (20) consist of $(N+1)$ equations and $(N+3)$ unknowns, namely δ_m, $m=0(1)N$, time dependent parameters. Using the boundary conditions $u(a,0)=u(b,0)=0$, we obtain a solvable $(N+1)\times(N+1)$ band matrix for solving the parameters δ_m. The initial vector can be obtained using the initial condition $u(x_m,0)=f(x_m)$ given together with the problem.

4.1 Stability analysis

(19) and (20) numerical schemes have been considered by the Fourier von Neumann [31] method. In this method $\delta_m^n=\xi^n e^{i\beta mh}$ is taken, where $i=\sqrt{-1}$, β is mode number, ξ is amplification factor and h is the space step in the method. In Eq. (1) in the term u^2u_x, since we take $u^2=z_m$ for linearization purpose, it will behave as a local constant. Let us assume that the amplification factors related to the schemes in (19) and (20) are ρ_A and ρ_B, respectively. If we write $\delta_m^n=\xi^n e^{i\beta mh}$ in Eq.(19), we obtain

$$\rho_A\left(\frac{\xi^{n+1/2}}{\xi^n}\right)=\frac{X-iY}{X+iY},$$

$$X=\alpha_2+(\alpha_1+\alpha_3)\cos\beta h,\quad Y=(\alpha_3-\alpha_1)\sin\beta h.$$

Thus, since $\left|\rho_A\left(\frac{\xi^{n+1/2}}{\xi^n}\right)\right|\leq 1$ is valid, the linearized schme is unconditionally stable. In a similar way, if we take $\delta_m^n=\xi_B^n e^{i\beta mh}$ in Eq. (20), we obtain $\left|\rho_B\left(\frac{\xi^{n+1}}{\xi^n}\right)\right|\leq 1$. Thus,

$$\rho_L(\xi)=\rho_A^{n+1}\rho_B^{n+1}\qquad\text{or}\qquad\rho_L(\xi)=\rho_B^{n+1}\rho_A^{n+1}$$

$$|\rho_L(\xi)|\leq\left|\rho_A\left(\frac{\xi^{n+1}}{\xi^n}\right)\right|\left|\rho_B\left(\frac{\xi^{n+1}}{\xi^n}\right)\right|\leq 1$$

the Lie-Trotter splitting schemes are unconditionally stable. In a similar way,

$$\rho_S(\xi)=\rho_A^{n+1/2}\rho_B^{n+1}\rho_A^{n+1/2}\qquad\text{or}\qquad\rho_S(\xi)=\rho_B^{n+1/2}\rho_A^{n+1}\rho_B^{n+1/2}$$

$$|\rho_S(\xi)|\leq\left|\rho_A\left(\frac{\xi^{n+1/2}}{\xi^n}\right)\right|\left|\rho_B\left(\frac{\xi^{n+1}}{\xi^n}\right)\right|\left|\rho_A\left(\frac{\xi^{n+1/2}}{\xi^n}\right)\right|\leq 1$$

so that the Strang splitting schemes are unconditionally stable.

5 The motion of single solitary wave

For this problem, we are going to consider MEW (1) equation for $x\to\pm\infty$ when $u\to 0$ together with the physical boundary conditions and initial condition

$$u(x,0)=A\sec h\left[k\left(x-x_0\right)\right].$$

The solitary wave type solution (2) of Eq. (1) is not only on a unbounded region, but also at the same time it has a solitary wave solution on the closed interval $a\leq x\leq b$. The three invariants I_1,

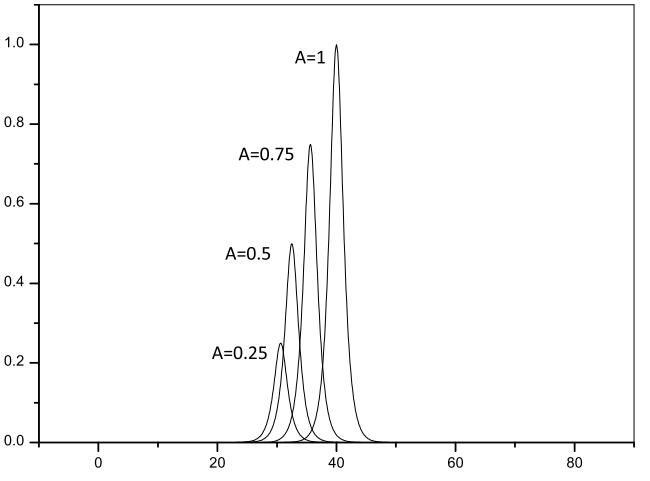


Figure 1: The movement of single solitary wave at time $t = 20$ for various values of amplitude and $h = 0.1$, $\Delta t = 0.05$.

I_2 and I_3 satisfied by MEW (1) equation are computed as follows by taking U_j and $U_j^{'}$ as the mesh values calculated from numerical solution

$$I_1 = h\sum_{j=0}^{N} U_j, \quad I_2 = h\sum_{j=0}^{N}\left[U_j^2 + \mu(U_j^{'})^2\right], \quad I_3 = h\sum_{j=0}^{N} U_j$$

Tha analytical values of the invariants I_1, I_2 and I_3 are as follows

$$I_1 = \frac{A\pi}{k}, \quad I_2 = \frac{2A^2}{k} + \frac{2\mu\pi A^2}{3}, \quad I_3 = \frac{4A^4}{3k}$$

as given by [13]. In order to be able to make a comparison with some of the studies in the literature, we take the parameters as $t = 20$, $\mu = 1$, $x_0 = 30$, $A = 0.25$ and $\Delta t = 0.05$. In Fig 1, the movement of solitary wave has been given for various values of amplitudes $A = 0.25$, 0.5, 0.75 and 1. As can be seen from this figure, the larger wave, which has a large amplitude, has traveled a lot because its velocity is bigger.In Table 1, in order to be able to obtain the peak value, $h = 0.0125$ and $\Delta t = 0.05$ are taken. As you can see from this chart, the numerical solution and the exact solution are at their highest values at the same position point. In Table 2, the invariants I_1, I_2 and I_3 together with the error norms L_2 and L_∞ are presented. As you can see from this table, the conservation constants are preserved quite well as time progresses. Moreover, the values of the error norms L_2 and L_∞ are better than those using cubic B-spline collocation without splitting in Ref. [21]. In Tables 3 and 4, some pointwise comparisons have been made with those in Ref. [8] and Ref. [23], respectively. As it is seen from these tables, the calculated pointwise values are very good. In Tablo 5, for values of $A = 0.25$, $h = 0.1$, $\Delta t = 0.05$, $x_0 = 30$ the invariants I_1, I_2, I_3 and the error norms L_2, L_∞ have been compared with those in Ref. [7], Ref. [8] and Ref. [12]. As it is seen from this table,

Table 1: Location and peak estimate for single solitary wave movement.

	t	Numerical		Exact			
A		Position	Amplitude	Position	Amplitude	$L_2 \times 10^3$	$L_\infty \times 10^3$
0.25		30.3125	0.250000	30.3125	0.250000	0.000001	0.000001
0.5	10	31.25	0.499996	31.25	0.500000	0.000008	0.000006
0.75		32.8125	0.749993	32.8125	0.750000	0.000093	0.000062
1		35.0	0.999991	35.0	1	0.000790	0.000492
0.25		30.625	0.249999	30.625	0.250000	0.000003	0.000003
0.5	20	32.5	0.499996	32.5	0.500000	0.000012	0.000007
0.75		35.625	0.749993	35.625	0.750000	0.000193	0.000125
1		40.0	0.999990	40.0	1	0.001594	0.000985

as time progresses, our invariants almost remain the same, and our error norms L_2, L_∞ are better than those in Ref. [7], Ref. [8] while worse than those in Ref. [12]. For values of $h = 0.1$ and Δt, a comparison with Ref. [22] has been presented in Table 6. Except for value of $\Delta t = 0.05$, for other chosen values of Δt our error norms L_2 and L_∞ are better.

6 Interaction of two solitary waves

In this problem, the physical initial and boundary conditions are taken as follows for $x \to \pm\infty$ when $u \to 0$ and $k = 1/\sqrt{\mu}$

$$u(x,0) = A_1 \sec h\left[k\left(x - x_1\right)\right] + A_2 \sec h\left[k\left(x - x_2\right)\right].$$

It has been observed that in interaction simulations the waves advance after preserving their shape and speed after interaction. However, after the interaction, it was observed that the waves produced a small or even disturbance behind them. In order that the interaction occurs, the solution domain is taken as $0 \le x \le 80$ for values of $h = 0.1$, $\Delta t = 0.2$, $\mu = 1$, $A_1 = 1$, $A_2 = 0.5$, $x_1 = 15$, $x_2 = 30$. While the wave having an amplitude of 1 is at the left of the wave having an amplitude of 0.5 at the time $t = 0$, as time progresses, it is seen from Fig. 2 that the larger wave leaves the smaller one its behind. At time $t = 80$, the calculated amplitude of the larger wave is 0.99925 at $x = 57.0$, whereas the calculated amplitude of the smaller wave is 0.49849 at $x = 37.7$. In addition, there was no elastic interaction because the waves after the interaction left small tail waves behind them. For this reason, these two solitary waves are not solitons [13]. In Fig. 3, the initial and after interaction positions of the waves are presented at time $t = 0$. The locations and amplitudes of the waves after interaction are calculated as follows: a) smaller wave at $x = 55$ having an amplitude of -0.99374, larger wave at $x = 127$ having an amplitude of -1.99887. b) smaller wave at $x = 61.2$ having an amplitude of -1.49057, larger wave at $x = 152.5$ having an amplitude of -2.99902. c) smaller wave at $x = 52.5$ having an amplitude of -0.97396, larger wave at $x = 123.7$ having an amplitude of 2.00181. d) smaller wave at $x = 52.5$ having an amplitude of 0.97396, larger wave at $x = 123.7$ having an amplitude of -2.00181.

7 Birth of solitary waves

For this problem, the initial and boundary conditions are taken as follows

$$u(x,0) = e^{-x^2} \tag{22}$$

$$u(-20,0) = u(20,0) = 0, \;\; t > 0. \tag{23}$$

Table 2: The calculated values of I_1, I_2 and I_3 on the domain $0 \leq x \leq 80$ for values of $h = 0.1$, $\Delta t = 0.05$, $x_0 = 30$.

A	t	I_1	I_2	I_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
0.25	0	0.7853982	0.1666665	0.0052083	0.000000	0.000000
	5	0.7853982	0.1666665	0.0052083	0.044681	0.042301
	10	0.7853982	0.1666665	0.0052083	0.088993	0.086630
	15	0.7853982	0.1666665	0.0052083	0.132574	0.131553
	20	0.7853982	0.1666666	0.0052083	0.175081	0.176288
	20[21]	0.7853967	0.1666662	0.0052083	0.17523269	0.17642205
	Exact	0.7853982	0.1666667	0.0052083		
0.5	0	1.5707963	0.6666658	0.0833333	0.000000	0.000000
	5	1.5707963	0.6666662	0.0833333	0.348442	0.350862
	10	1.5707963	0.6666668	0.0833333	0.653700	0.646159
	15	1.5707963	0.6666672	0.0833333	0.890311	0.796858
	20	1.5707963	0.6666673	0.0833333	1.058555	0.860246
	20[21]	1.5707931	0.6666660	0.0833330	1.06979673	0.86864227
	Exact	1.5707963	0.6666667	0.0833333		
0.75	0	2.3561945	1.4999981	0.4218750	0.000000	0.000000
	5	2.3561944	1.5000006	0.4218750	1.053237	1.018811
	10	2.3561944	1.5000013	0.4218750	1.623353	1.267303
	15	2.3561944	1.5000012	0.4218750	1.872928	1.360292
	20	2.3561944	1.5000012	0.4218750	2.038191	1.469106
	20[21]	2.3561896	1.4999982	0.4218733	2.24293300	1.62010840
	Exact	2.3561945	1.5000000	0.4218750		
1	0	3.1415927	2.6666634	1.3333333	0.000000	0.000000
	5	3.1415923	2.6666689	1.3333330	1.877658	1.493092
	10	3.1415923	2.6666686	1.3333330	2.230961	1.563564
	15	3.1415922	2.6666685	1.3333330	2.360266	1.669304
	20	3.1415922	2.6666685	1.3333329	2.494352	1.781858
	20[21]	3.1415842	2.6666609	1.3333253	3.98833508	2.84859636
	Exact	3.1415927	2.6666667	1.3333333		

Table 3: A comparison of numerical solution with those in Ref. [8] on the domain $0 \leq x \leq 80$ for values of $h = 0.1$, $\Delta t = 0.2$, $t = 20$.

x	Present	Ref. [8]	Exact
23.05	0.0002566078	0.0002565747	0.0002565601
24.05	0.0006974681	0.0006973820	0.0006974016
25.05	0.0018957202	0.001895497	0.0018957105
26.05	0.0051521677	0.005151595	0.0051526023
27.05	0.0139943827	0.01399303	0.0139967274
28.05	0.0378494632	0.03784846	0.0378573612
29.05	0.0992470286	0.09927855	0.0992506739
30.05	0.2137323069	0.2138382	0.2136902454
31.05	0.2289741658	0.2288491	0.2290048151
32.05	0.1136813105	0.1136122	0.1136785718
33.05	0.0438977981	0.04386965	0.0438954295
34.05	0.0162577024	0.01624580	0.0162574165
35.05	0.0059860725	0.005981089	0.0059862473
36.05	0.0022023199	0.002200265	0.0022024903
37.05	0.0008101626	0.0008093246	0.0008102645
38.05	0.0002980284	0.0002976900	0.0002980803
39.05	0.0001096332	0.0001094977	0.0001096577
40.05	0.0000403299	0.00004027593	0.0000403408

Table 4: A comparison of some point-wise values at time $t = 1$ with those in Ref. [23].

	Present	Ref. [23]	Ref. [23]	Ref. [23]	Ref. [23]	Exact
x	$\Delta t = 10^{-2}$	$\Delta t = 10^{-2}$ (EFG)	$\Delta t = 10^{-2}$ (IEFG)	$\Delta t = 10^{-3}$ (EFG)	$\Delta t = 10^{-3}$ (IEFG)	
33.0	0.1631	0.1431	0.1457	0.1597	0.1622	0.1631
32.5	0.2659	0.2437	0.2541	0.2576	0.2587	0.2658
32.0	0.4251	0.4005	0.4104	0.4153	0.4198	0.4251
31.5	0.6479	0.6158	0.6189	0.6275	0.6348	0.6481
31.0	0.8864	0.8391	0.8472	0.8527	0.8677	0.8868
30.5	0.9999	0.9824	0.9879	0.9876	0.9912	1.0000
30.0	0.8873	0.9217	0.9063	0.9004	0.8945	0.8868
29.5	0.6484	0.6930	0.6831	0.6574	0.6512	0.6481
29.0	0.4251	0.4682	0.4574	0.4426	0.4319	0.4251
28.5	0.2657	0.2959	0.2876	0.2797	0.2693	0.2658
28.0	0.1630	0.1844	0.1789	0.1754	0.1656	0.1631

Table 5: A comparison of I_1, I_2 and I_3 on the domain $0 \leq x \leq 80$ for values of $h = 0.1$, $\Delta t = 0.05$, $x_0 = 30$, $A = 0.25$.

	t	I_1	I_2	I_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
Present	0	0.7853982	0.1666665	0.0052083	0.000000	0.000000
	5	0.7853982	0.1666665	0.0052083	0.044681	0.042301
	10	0.7853982	0.1666665	0.0052083	0.088993	0.086630
	15	0.7853982	0.1666665	0.0052083	0.132574	0.131553
	20	0.7853982	0.1666666	0.0052083	0.175081	0.176288
	Exact	0.7853982	0.1666667	0.0052083		
Ref.[7]	0	0.7853978	0.1664730	0.0052083	0.0000121	0.0000106
	5	0.7853974	0.1664731	0.0052083	0.0680823	0.0608773
	10	0.7853981	0.1664731	0.0052083	0.1358524	0.1252660
	15	0.7853977	0.1664733	0.0052083	0.2030194	0.1912210
	20	0.7853977	0.1664735	0.0052083	0.2692812	0.2569972
Ref.[8]	0	0.7853979	0.1666178	0.005208333	0.0000173	0.0000149
	5	0.7852870	0.1665823	0.005206116	0.0657442	0.0627339
	10	0.7851756	0.1665467	0.005203893	0.1360450	0.1252592
	15	0.7850653	0.1665117	0.005201699	0.2107168	0.1877248
	20	0.7849545	0.1664765	0.005199504	0.2905166	0.2498925
Ref.[12]	0	0.7853982	0.1667643	0.0052083	0.0	0.0
	5	0.7853941	0.1667631	0.0052083	0.0202557	0.0118506
	10	0.7853913	0.1667621	0.0052082	0.0406275	0.0239157
	15	0.7853913	0.1667621	0.0052082	0.0603728	0.0353568
	20	0.7853898	0.1667614	0.0052082	0.0796940	0.0465523

Table 6: A comparison of Problem 1 with Ref. [22] for values of $h = 0.1$, $\Delta t = 1$, 0.5, 0.25, 0.1, 0.05 and $A = 1$.

		Present		Ref. [22]	
Δt	$h = 0.1$	L_2	L_∞	L_2	L_∞
1		0.124960	0.098944	0.82353	0.51054
0.5		0.140529	0.087377	0.20019	0.12222
0.25		0.037108	0.023197	0.048679	0.030607
0.1		0.003553	0.002558	0.0078578	0.0049308
0.05		0.002494	0.001782	0.0019616	0.0012224

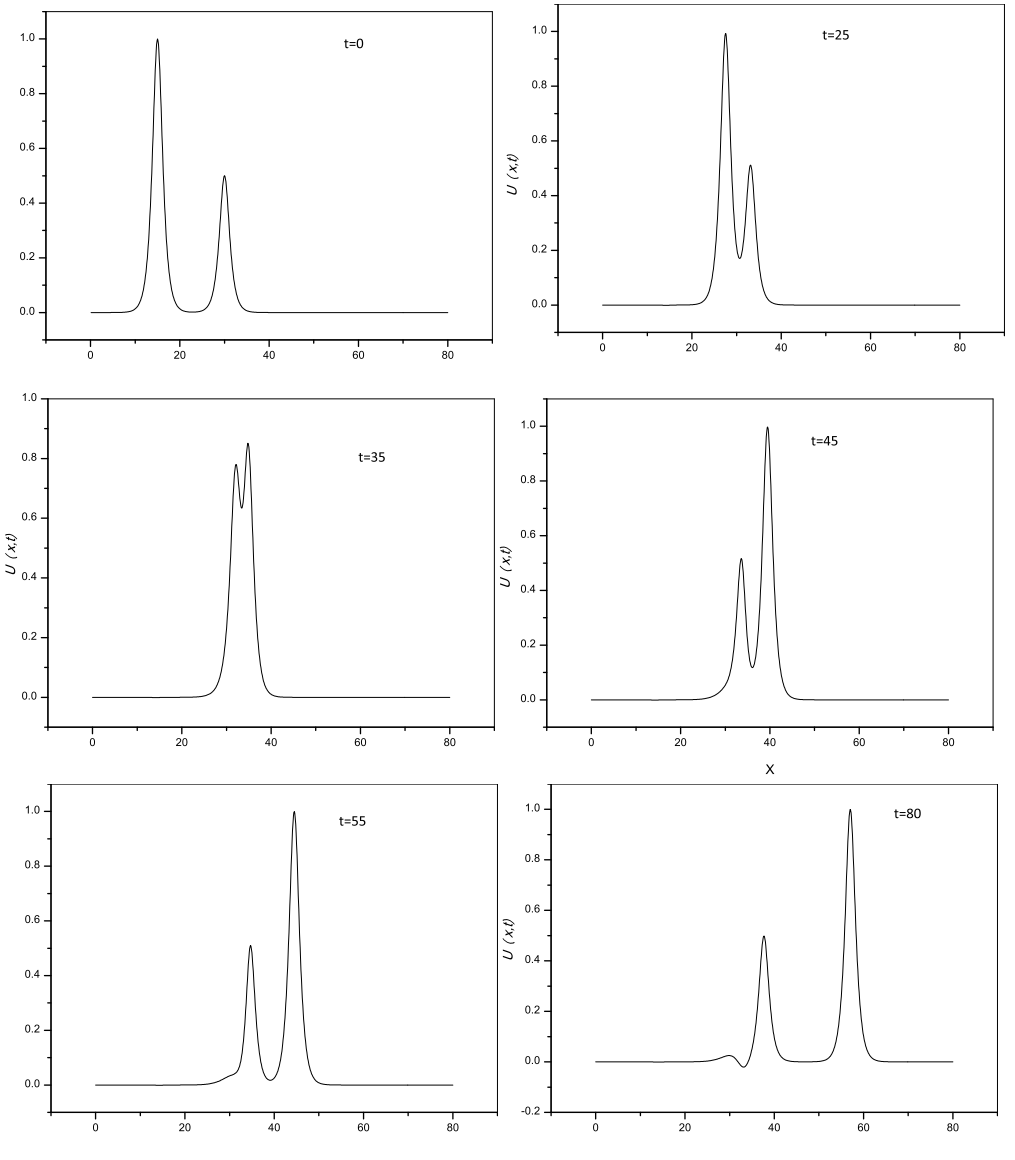


Figure 2: The interaction of two solitary waves during the time interval $0 \leq t \leq 80$.

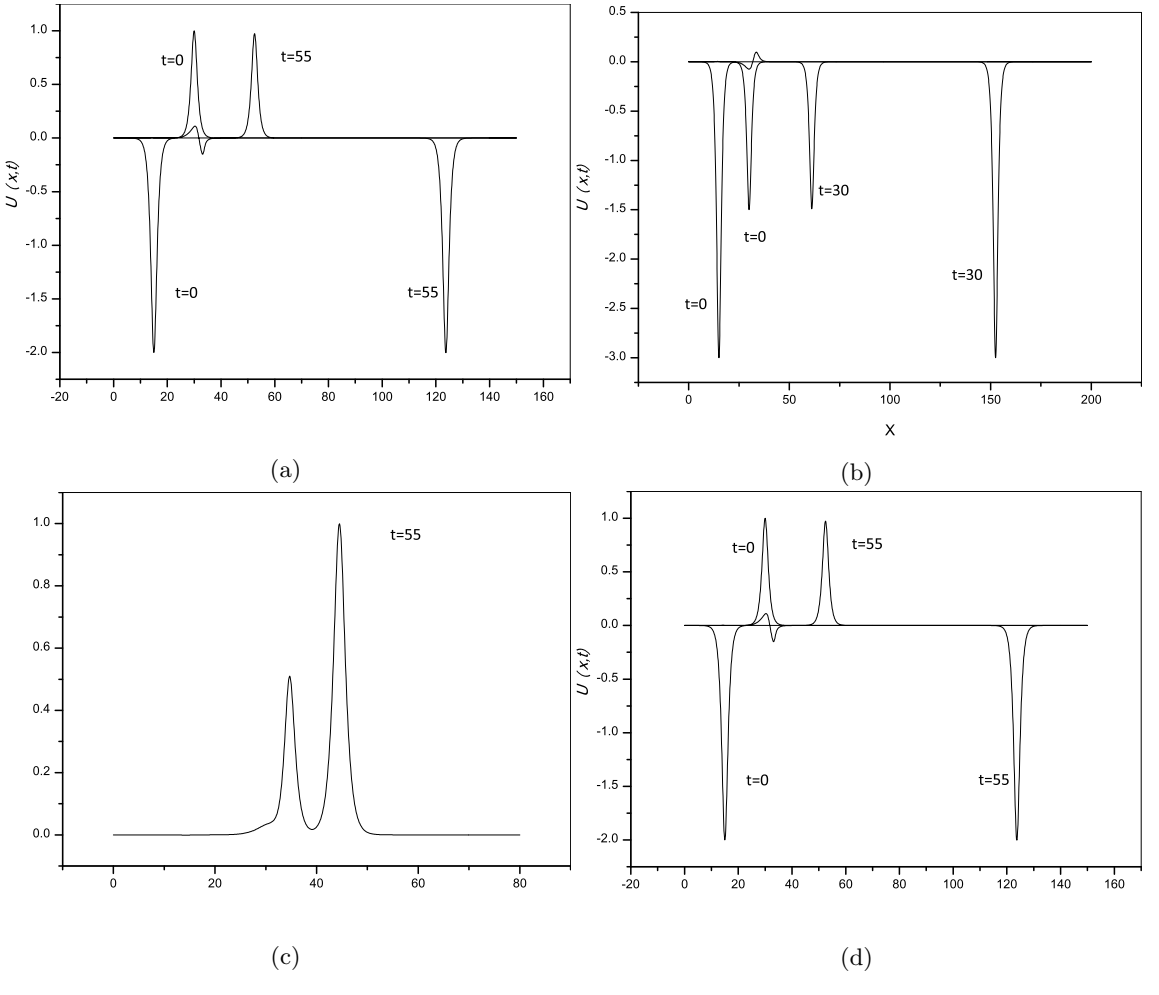


Figure 3: The interaction of waves having negative amplitudes *a*) $A_1 = -2$, $A_2 = -1$, $t = 55$, $0 \leq x \leq 150$. *b*) $A_1 = -3$, $A_2 = -1.5$, $t = 30$, $0 \leq x \leq 200$. *c*) $A_1 = 2$, $A_2 = -1$, $t = 55$, $0 \leq x \leq 150$. *d*) $A_1 = -2$, $A_2 = 1$, $t = 55$, $0 \leq x \leq 150$.

Table 7: The invariants I_1, I_2, I_3 for various values of μ for Maxwellian solution.

Time	I_1	I_2	I_3	I_1	I_2	I_3
	$\mu = 0.5$			$\mu = 0.1$		
0.0	1.77245	1.87997	0.88623	1.77245	1.37865	0.88623
2.5	1.77246	1.87979	0.88623	1.77248	1.37783	0.88627
5.0	1.77245	1.88004	0.88623	1.77248	1.37784	0.88627
7.5	1.77245	1.88008	0.88623	1.77248	1.37781	0.88627
10.0	1.77245	1.88008	0.88623	1.77249	1.37777	0.88627
12.5	1.77245	1.88008	0.88623	1.77249	1.37774	0.88627
	$\mu = 0.05$			$\mu = 0.02$		
0.0	1.77245	1.31598	.88623	1.77245	1.27838	0.88623
2.5	1.77253	1.31461	.88638	1.77270	1.27538	0.88712
5.0	1.77254	1.31439	.88639	1.77273	1.27491	0.88716
7.5	1.77254	1.31432	.88639	1.77273	1.27473	0.88717
10.0	1.77254	1.31433	.88639	1.77274	1.27466	0.88717
12.5	1.77254	1.31431	.88639	1.77275	1.27458	0.88717
	$\mu = 0.005$			$\mu = 0.0025$		
0.0	1.77245	1.25958	0.88623	1.77245	1.25645	0.88623
2.5	1.77404	1.25230	0.89776	1.77670	1.25018	0.92425
5.0	1.77436	1.25104	0.89857	1.77783	1.25039	0.93345
7.5	1.77450	1.25063	0.89877	1.77826	1.24949	0.92806
10.0	1.77459	1.25043	0.89894	1.77864	1.25039	0.93453
12.5	1.77465	1.25032	0.89902	1.77868	1.24930	0.92893

The behavior of the solution in the Maxwellian (22) initial condition varies depending on the parameter μ. The calculations are carried out for values of $\mu = 0.5$, 0.1, 0.05, 0.02, 0.005, 0.0025, $h = 0.05$, $\Delta t = 0.01$ and $t = 12.5$. Fig 4 (a) makes a oscillation at $\mu = 0.5$ and no soliton exists. In Fig 4 (b) for $\mu = 0.1$, it is seen that the solution produces a neat solitary wave at location $x = 9.30$ having an amplitude of 1.19498. In Fig 4 (c), (d), (e) and (f) the solution have produced $2, 3, 7, 10$ solitary waves, respectively. The locations and amplitudes of the solitary wave having larger amplitude have been found as (c) $x = 10.85$ and amplitude 1.29011. (d) $x = 12.85$ and amplitude 1.39672. (e) $x = 15.35$ and amplitude 1.50896. (f) $x = 16.20$ and amplitude 1.46348. Table 7 also shows that the invariants I_1, I_2, I_3 for Maxwellian are well preserved and are in agreement with those in the literature.

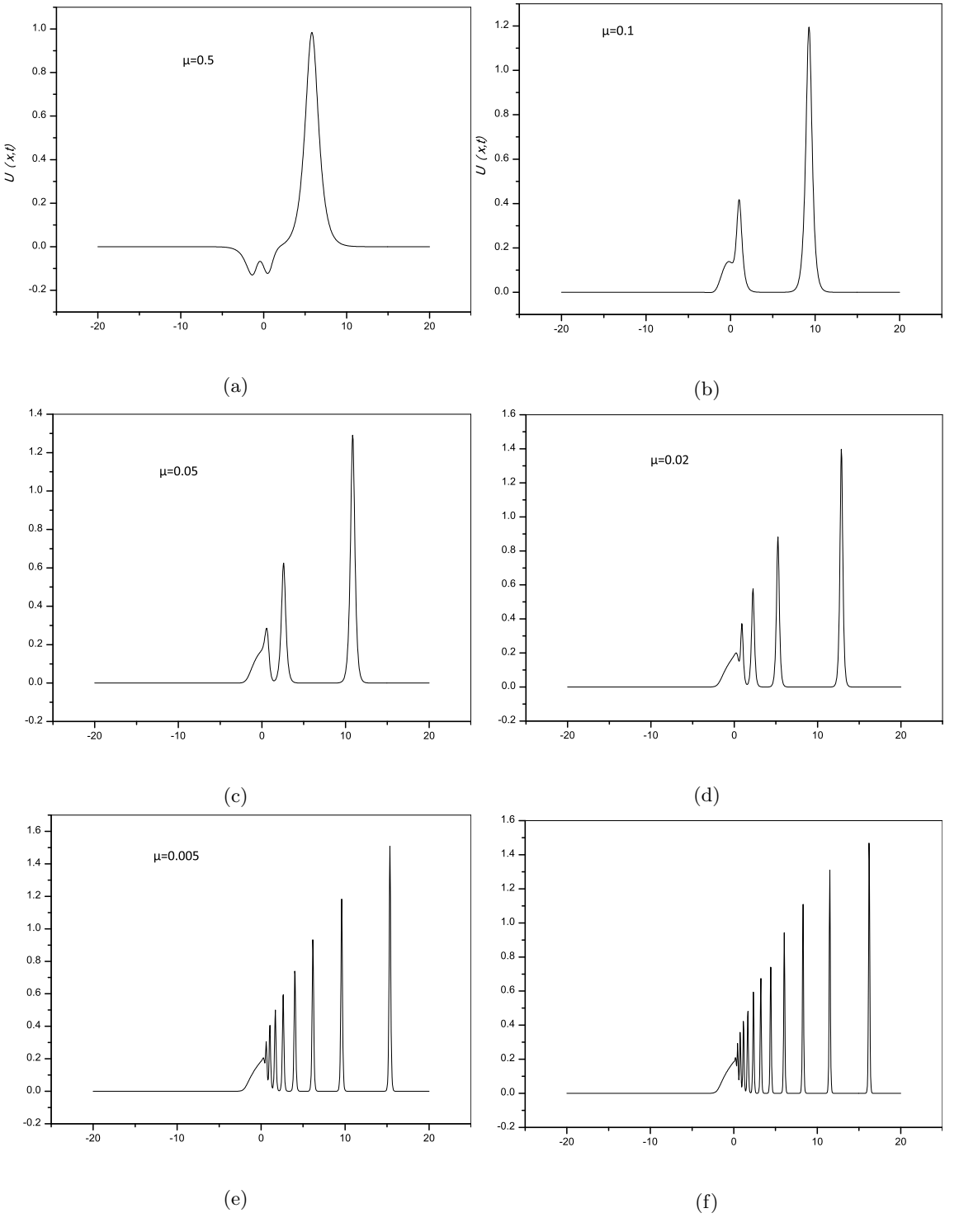


Figure 4: Maxwellian initial condition $t = 12.5$.

8 Conclusion

The operator splitting method has been successfully applied to the MEW equation. The MEW equation was split into two simpler sub-equations, one nonlinear and the other linear. Each sub-equation was solved by finite element cubic B-spline collocation method and Strang splitting method was applied. It has been found that the results obtained are in accordance with the results in the literature, and that the operator splitting method is an easy method to apply to the computer. In addition, this method can be easily applied to different kinds of non-linear equations.

Bibliography

[1] A. M. Wazwaz, The tanh and sine cosine methods for a reliable treatment of the modified equal width equation and its variants, Commun. Nonlinear Sci. Numer. Simul. 11 (2) (2006), 148–160.

[2] D.J. Korteweg, G. de Vries, On the change in form of long waves advancing in a rectangular canal, and on a new type of long stationary wave, Phil. Mag. 39 (5) (1895), 422–443.

[3] P.J. Morrison, J.D. Meiss, J.R. Carey, Scattering of RLW solitary waves, Physica 11 D (1984), 324–336.

[4] L.R.T. Gardner, G.A. Gardner, Solitary waves of the EWE equation, J. Comput. Phys. 101 (1992), 218–223.

[5] P.J. Olver, Euler operators and conservation laws of the BBM equation, Math. Proc. Cambridge Philos. Soc. 85 (1979), 143–159.

[6] S. Hamdi, W. H. Enright, W. E. Schiesserand, J. J. Gottlieb, Exact solutions of the generalized equal width wave equation. In Proceedings of the International Conference on Computational Science and its Applications, LNCS 2668, pp. 725–734, Berlin: Springer-Verlag, 2003.

[7] A. Esen and S. Kutluay, "Solitary wave solutions of the modified equal width wave equation," Commun. Non linear Sci. Numer. Simul. 13 (3) (2008), 1538–1546.

[8] D. J. Evans and K. R. Raslan, Solitary waves for the generalized equal width (GEW) equation, Int. J. Comput. Math. 82 (4) (2005), 445–455.

[9] K. R. Raslan, Collocation method using cubic B-spline for the generalised equal width equation, Int. J. Simulat. Process Model, 2 (2006), 37–44.

[10] L. Jin, Analytical Approach to the Modified Equal Width Equation, Int. J. Contemp. Math. Sci., 4 (23) (2009), 1113 -1119.

[11] J. Lu, He's variational iteration method for the modified equal width equation, Chaos Solitons Fractals, 39 (2009), 2102–2109.

[12] A. Esen, A lumped Galerkin method for the numerical solution of the modified equal-width wave equation using quadratic B-splines, Int. J. Comput. Math., 83 (5-6) (2006), 449–459.

[13] S. I. Zaki, "Solitary wave interactions for the modified equal width equation," Comput. Phys. Comm., 126 (3) (2000), 219–231.

[14] S.I. Zaki, A least-squares ®nite element scheme for the EW equation, Comput. Methods Appl. Mech. Engrg. 189 (2000), 587-594.

[15] S. B. Gazi Karakoç, H. Zeybek, A cubic B-spline Galerkin approach for the numerical simulation of the GEW equation, Stat. Optim. Inf. Comput., 4 (2016), 30-41.

[16] H. Zeybek, S. B. Gazi Karakoç, Application of The Collocation Method With B-Splines To The GEW Equation, Electron. Trans. Numer. Anal., 46 (2017), 71-88.

[17] T. Roshan, A Petrov-Galerkin method for solving the generalized equal width (GEW) equation, J. Comput. Appl. Math., 235 (2011), 1641–1652.

[18] T. Geyikli, S. B. Gazi Karakoç, Septic B-Spline Collocation Method for the Numerical Solution of the Modified Equal Width Wave Equation, Applied Mathematics, 2 (2011), 739-749.

[19] T. Geyikli, S. B. Gazi Karakoç, Subdomain Finite Element Method with Quartic B Splines for the Modified Equal Width Wave Equation, Comput. Math. Math. Phys., 55 (3) (2015), 410-421.

[20] B. Saka, Algorithms for numerical solution of the modified equal width wave equation using collocation method, Math. Comput. Modelling, 45 (2007), 1096–1117.

[21] S. B. Gazi Karakoç, Y. Ucar, N. Murat Yagmurlu, Different Linearization Techniques for the Numerical Solution of the MEW Equation, Selçuk J. Appl. Math., 13 (2) (2012), 43-62.

[22] A. Gül Kaplan, Y. Dereli, Numerical solutions of the GEW equation using MLS collocation method, Internat. J. Modern Phys. C, 28 (1) (2017), 1750011 (23 pages).

[23] R. J. Cheng, K. M. Liew, Analyzing modified equal width (MEW) wave equation using the improved element-free Galerkin method, Eng. Anal. Bound. Elem., 36 (2012), 1322–1330.

[24] G. Strang, On The Construction And Comparison Of Difference Schemes, SIAM J. Numer. Anal. 5 (3) (1968), 506-517.

[25] W. Hundsdorfer, Numerical Solution of Advection-Diffüsion-Reaction Equations, Lecture notes for PH.D. course, Thomas Stieltjes İnstitute, Amsterdam, 2000.

[26] M. Seydaoğlu, U. Erdoğan, T. Öziş, Numerical solution of Burgers' equation with high order splitting methods, J. Comput. Appl. Math., 291 (2016), 410–421.

[27] H. F. Trotter, On the product of semi-groups of operators, Proc. Amer. Math. Soc., 10 (1959), 545-551.

[28] G.I. Marchuk, Some application of splitting-up methods to the solution of mathematical physics problems, Apl. mat., 13 (1968), 103–132.

[29] W. Hundsdorfer, J. Verwer, Numerical Solution of Time-Dependent Advection-Diffusion-Reaction Equations (First Edition), Springer-Verlag Berlin Heidelberg, 2003.

[30] P.M. Prenter, Splines and Variatinoal Methods, John Wiley, New York (1975).

[31] J. Von Neumann, R. D. Richymyer, A Method for the Numerical Calculation of Hydrodynamic Shocks, J. Appl. Phys. 21 (1950), 232-237.