Solutions to some systems of adjointable operator equations over Hilbert C^*-modules

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Abstract

In this paper, by using operator matrix techniques, we present necessary and sufficient conditions for the existence of a solution to the system of equations $AXD + FX^*B = C$, $GXF^* + FX^*G^* = H$ for adjointable operators between Hilbert C^*-modules, and derive an expression for the general solution to the system. We establish necessary and sufficient conditions for the existence of a solution to the system of adjointable operator equations $AXF = H_1$, $CXD = H_2$, $BXD = H_3$ over Hilbert C^*-modules. Some of the findings of this paper extend some known results in the literature.

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1 Introduction

It is a very active research topic to investigate solvability of system of operator equations over Hilbert spaces and Hilbert C^*-modules. For finite matrices, Khatri and Mitra [7] got the necessary and sufficient conditions for existence and expressions of general Hermitian solutions to the system of linear equations

$$AX = C, XB = D. \tag{1.1}$$

Thereafter, using the generalized inverse of operators, Dajić and Koliha [2] proposed the common Hermitian and positive solutions to (1.1) for bounded linear operators acting on a Hilbert space. Moreover, Xu [17], Fang et al. [4] studied system (1.1) in general setting of Hilbert C^*-module. Recently, Wang and Wu [16] investigated the Hermitian solution to the system of equations

$$A_1X = C_1, \qquad XB_1 = C_2, \qquad A_3XA_3^* = C_3, \qquad A_4XA_4^* = C_4 \tag{1.2}$$

for operators between Hilbert C^*-modules and presented an expression for the common Hermitian solution to the equations in terms of Moore-Penrose inverse of operators when the solvability conditions are satisfied. Several general operator equations and systems in the framework of Hilbert space and Hilbert C^*-module operators have been studied by some mathematicians; see, e.g., [1, 5, 9, 10, 11, 12, 13, 14, 15].

In this paper, by using operator matrix techniques, we aim to provide some necessary and sufficient conditions for the existence of solutions to the systems of operator equations

$$AXD + FX^*B = C,\ GXF^* + FX^*G^* = H \tag{1.3}$$

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and

$$AXF = H_1,\ CXD = H_2,\ BXD = H_3 \tag{1.4}$$

between Hilbert C^*-modules, as well as, present expressions for the general solutions to the systems when the solvability conditions are satisfied.
The paper is organized as follows. In Section 2, we begin with some basic notions and facts about the solvability of some operator equations over Hilbert C^*-modules. We establish in Section 3 some sufficient conditions for the existence of a solution to (1.3) for adjointable operators between Hilbert C^*-modules. An example is given that shows these conditions are not necessary in general. We also give some necessary and sufficient conditions for the existence of a unique solution to this system and when the solvability conditions are met, we present an expression for the general solution to (1.3). In Section 4 we provide the necessary and sufficient conditions for existence and expression of general solution to the system of operator equations (1.4).

2 Preliminaries

Definition 2.1. [8] Let $\mathcal{A}$ be a C^*-algebra. A Hilbert C^*-module is a right $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ such that the induced norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ is complete.

Throughout this paper $\mathcal{X}$ and $\mathcal{Y}$ denote two Hilbert C^*-modules and $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is the set of all maps $T : \mathcal{X} \to \mathcal{Y}$ for which there is a map $T^* : \mathcal{Y} \to \mathcal{X}$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$, for any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. We know that any element T of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a bounded $\mathcal{A}$-linear operator. We call $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the set of adjointable operators from $\mathcal{X}$ into $\mathcal{Y}$. In case $\mathcal{X} = \mathcal{Y}$, $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ which we abbreviate to $\mathcal{L}(\mathcal{X})$, is a C^*-algebra. The identity operator on a Hilbert C^*-module $\mathcal{X}$ is denoted by $1_{\mathcal{X}}$ or 1 if there is no ambiguity. We write $\mathcal{R}(T)$ and $\ker(T)$ for the range and null space of T, respectively.
Recall that if $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range, then the map $T^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ has closed range, too [8]. The Moore-Penrose inverse [18] of a closed range operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is the unique operator $T^\dagger \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ satisfying the Penrose equations:

$$TT^\dagger T = T,\ \ T^\dagger TT^\dagger = T^\dagger,\quad (TT^\dagger)^* = TT^\dagger,\quad (T^\dagger T)^* = T^\dagger T.$$

Then, $(T^\dagger)^* = (T^*)^\dagger$, $TT^\dagger$ and $T^\dagger T$ are orthogonal projections, in the sense that they are self-adjoint idempotent operators. Furthermore, we have the following properties:

$$\mathcal{R}(T) = \mathcal{R}(TT^\dagger), \qquad \mathcal{R}(T^\dagger) = \mathcal{R}(T^\dagger T) = \mathcal{R}(T^*),$$

$$\ker(T) = \ker(T^\dagger T), \qquad \ker(T^\dagger) = \ker(TT^\dagger) = \ker(T^*).$$

For simplicity, we use E_T and P_T to stand for $1 - TT^\dagger$ and $1 - T^\dagger T$, respectively.

Remark 2.2. Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules, we use the notation $\mathcal{X} \oplus \mathcal{Y}$ to denote the direct sum of $\mathcal{X}$ and $\mathcal{Y}$, which is also a Hilbert $\mathcal{A}$-module whose $\mathcal{A}$-valued inner product is given by

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = \langle x_1, x_2\rangle + \langle y_1, y_2\rangle,$$

for $x_i \in \mathcal{X}$ and $y_i \in \mathcal{Y}$, $i = 1, 2$.

Lemma 2.3. [6] If $A \in \mathcal{L}(\mathcal{X})$ and $B \in \mathcal{L}(\mathcal{Y})$ are invertible operators, then $M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ is invertible for every $C \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$.

Proof. The inverse of M_C is $\begin{bmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{bmatrix}$. Q.E.D.

Lemma 2.4. [3] Let $A \in \mathcal{L}(\mathcal{X})$, $B \in \mathcal{L}(\mathcal{Y})$, $C \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, $\mathcal{R}(A)$ and $\mathcal{R}(B)$ be closed. Then, the 2-by-2 block operator valued matrix $M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ is Moore-Penrose invertible with $M^\dagger = \begin{bmatrix} A^\dagger & -A^\dagger CB^\dagger \\ 0 & B^\dagger \end{bmatrix}$ if and only if $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subseteq \mathcal{R}(B^*)$.

Theorem 2.5. [10] Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be Hilbert $\mathcal{A}$-modules, $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $T \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ be invertible operators and $A \in \mathcal{L}(\mathcal{Y})$. Then the following statements are equivalent:
(a) There exists a solution $X \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$ to the operator equation $TXS^* + SX^*T^* = A$.
(b) $A = A^*$.
If (a) or (b) is satisfied, then any solution to

$$TXS^* + SX^*T^* = A,\ X \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$$

has the form

$$X = \frac{1}{2}T^{-1}A(S^*)^{-1} - T^{-1}Z(S^*)^{-1},$$

where $Z \in \mathcal{L}(\mathcal{Y})$ satisfies $Z^* = -Z$.

Theorem 2.6. [9] Suppose that $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ are Hilbert $\mathcal{A}$-modules, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $S \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ have closed ranges, and $A \in \mathcal{L}(\mathcal{Y})$. Then the equation

$$TXS = A, \qquad X \in \mathcal{L}(\mathcal{Z}, \mathcal{X}), \tag{2.1}$$

has a solution if and only if

$$TT^\dagger AS^\dagger S = A.$$

In which case, any solution of (2.1) has the form

$$X = T^\dagger AS^\dagger + V - T^\dagger TVSS^\dagger.$$

where $V \in \mathcal{L}(\mathcal{Z}, \mathcal{X})$ is arbitrary.

Remark 2.7. [10] Let $T \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ and $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ have closed ranges, and $A \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$. Then the equation

$$TXS = A, \qquad X \in \mathcal{L}(\mathcal{Y}), \tag{2.2}$$

has a solution if and only if

$$TT^\dagger AS^\dagger S = A.$$

In which case, any solution of (2.2) has the form

$$X = T^\dagger AS^\dagger.$$

We conclude this section recalling the following version of [10, Theorem 2.2.].

Theorem 2.8. [10] Suppose $S \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ is an invertible operator, $T \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ has closed range and $A \in \mathcal{L}(\mathcal{Y})$. Then the following statements are equivalent:
(a) There exists a solution $X \in \mathcal{L}(\mathcal{Z})$ to the operator equation $TXS^* + SX^*T^* = A$.
(b) $A = A^*$ and $(1 - TT^\dagger)A(1 - TT^\dagger) = 0$.
If (a) or (b) is satisfied, then any solution to $TXS^* + SX^*T^* = A$ has the form

$$X = \frac{1}{2}T^\dagger ATT^\dagger (S^*)^{-1} + T^\dagger ZTT^\dagger (S^*)^{-1} + T^\dagger A(1 - TT^\dagger)(S^*)^{-1} + (1 - T^\dagger T)Y(S^*)^{-1},$$

where $Z \in \mathcal{L}(\mathcal{Y})$ satisfies $T^*(Z + Z^*)T = 0$ and $Y \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ is arbitrary.

3 The system of operator equations $AXD + FX^*B = C$, $GXF^* + FX^*G^* = H$

In this section, in the general setting of Hilbert C^*-module operators, we establish some approaches for solving system of operator equations (1.3).

Theorem 3.1. Let $\mathcal{X}, \mathcal{Y}$ be Hilbert $\mathcal{A}$-modules, $A, F, G \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $B, D \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and $C \in \mathcal{L}(\mathcal{Y})$ such that A, B, D, F, G be invertible. Suppose that $H \in \mathcal{L}(\mathcal{Y})$ is a self-adjoint operator such that $A^{-1}CD^{-1} = G^{-1}H(F^*)^{-1}$. If

$$A(B^*)^{-1}C^*(F^*)^{-1}D + F(D^*)^{-1}C^*(A^*)^{-1}B = 2C, \tag{3.1}$$

then the system of operator equations (1.3) has a solution $X \in \mathcal{L}(\mathcal{X})$. In the case any solution of the system (1.3) is represented by

$$X = \frac{1}{4}A^{-1}CD^{-1} + \frac{1}{4}(B^*)^{-1}C^*(F^*)^{-1} + (B^*)^{-1}K^*(F^*)^{-1} - A^{-1}KD^{-1}, \tag{3.2}$$

where $K \in \mathcal{L}(\mathcal{Y})$ satisfies $A(B^*)^{-1}K^*(F^*)^{-1}D = F(D^*)^{-1}K^*(A^*)^{-1}B$ and
$GA^{-1}KD^{-1}F^* = -(GA^{-1}KD^{-1}F^*)^*$.

Proof. Suppose that (3.1) is satisfied. Taking $K = 0$ in (3.2), we have that the operator X defined by (3.2) is the solution of the system of operator equations (1.3). We will show that if the system (1.3) has a solution $X \in \mathcal{L}(\mathcal{X})$, then it must be of the form (3.2). For this purpose, let us assume that $T = \begin{bmatrix} A & G \\ 0 & B^* \end{bmatrix} : \mathcal{X} \oplus \mathcal{X} \to \mathcal{Y} \oplus \mathcal{Y}$, $Y = \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} : \mathcal{X} \oplus \mathcal{X} \to \mathcal{X} \oplus \mathcal{X}$, $S = \begin{bmatrix} F & 0 \\ 0 & D^* \end{bmatrix} : \mathcal{X} \oplus \mathcal{X} \to \mathcal{Y} \oplus \mathcal{Y}$ and $N = \begin{bmatrix} H & C \\ C^* & 0 \end{bmatrix} : \mathcal{Y} \oplus \mathcal{Y} \to \mathcal{Y} \oplus \mathcal{Y}$. Regarding to Lemma 2.3 T and S are invertible.
Consider the operator equation $TYS^* + SY^*T^* = N$, we have

$$\begin{aligned} TYS^* + SY^*T^* &= \begin{bmatrix} A & G \\ 0 & B^* \end{bmatrix}\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}\begin{bmatrix} F^* & 0 \\ 0 & D \end{bmatrix} \\ &+ \begin{bmatrix} F & 0 \\ 0 & D^* \end{bmatrix}\begin{bmatrix} 0 & X^* \\ X^* & 0 \end{bmatrix}\begin{bmatrix} A^* & 0 \\ G^* & B \end{bmatrix} \\ &= \begin{bmatrix} H & C \\ C^* & 0 \end{bmatrix}, \end{aligned}$$

or equivalently

$$\begin{bmatrix} GXF^* + FX^*G^* & AXD + FX^*B \\ B^*XF^* + D^*X^*A^* & 0 \end{bmatrix} = \begin{bmatrix} H & C \\ C^* & 0 \end{bmatrix}.$$

Therefore

$$\begin{cases} AXD + FX^*B = C, \\ GXF^* + FX^*G^* = H. \end{cases}$$

By Theorem 2.5, it follows that Y has the following representation

$$Y = \frac{1}{2}T^{-1}N(S^*)^{-1} - T^{-1}Z(S^*)^{-1}, \tag{3.3}$$

where $Z \in \mathcal{L}(\mathcal{Y}\oplus\mathcal{Y})$ satisfies $Z^* = -Z$. Let us take $Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}$. Hence $Z_1^* = -Z_1$, $Z_3^* = -Z_2$ and $Z_4^* = -Z_4$. From (3.3), we have

$$\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} A^{-1} & -A^{-1}G(B^*)^{-1} \\ 0 & (B^*)^{-1} \end{bmatrix}\begin{bmatrix} H & C \\ C^* & 0 \end{bmatrix}\begin{bmatrix} (F^*)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix}$$
$$- \begin{bmatrix} A^{-1} & -A^{-1}G(B^*)^{-1} \\ 0 & (B^*)^{-1} \end{bmatrix}\begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}\begin{bmatrix} (F^*)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix}.$$

It enforces that

$$\frac{1}{2}A^{-1}H(F^*)^{-1} - \frac{1}{2}A^{-1}G(B^*)^{-1}C^*(F^*)^{-1} - A^{-1}Z_1(F^*)^{-1} + A^{-1}G(B^*)^{-1}Z_3(F^*)^{-1} = 0, \tag{3.4}$$

$$X = \frac{1}{2}A^{-1}CD^{-1} - A^{-1}Z_2D^{-1} + A^{-1}G(B^*)^{-1}Z_4D^{-1}, \tag{3.5}$$

$$X = \frac{1}{2}(B^*)^{-1}C^*(F^*)^{-1} - (B^*)^{-1}Z_3(F^*)^{-1}, \tag{3.6}$$

$$(B^*)^{-1}Z_4D^{-1} = 0. \tag{3.7}$$

From (3.7) it follows that $Z_4 = 0$. Put $K = \frac{1}{2}Z_2$ and $W = Z_1$ in (3.5) and (3.6) we can write

$$X = \frac{1}{2}A^{-1}CD^{-1} - 2A^{-1}KD^{-1} = \frac{1}{2}(B^*)^{-1}C^*(F^*)^{-1} + 2(B^*)^{-1}K^*(F^*)^{-1}, \tag{3.8}$$

which implies that

$$\frac{1}{2}A^{-1}CD^{-1} = \frac{1}{2}(B^*)^{-1}C^*(F^*)^{-1} + 2(B^*)^{-1}K^*(F^*)^{-1} + 2A^{-1}KD^{-1}. \tag{3.9}$$

By multiplication A on the left and F^* on the right to (3.4), we get

$$G(B^*)^{-1}K^* = -\frac{1}{2}W - \frac{1}{4}G(B^*)^{-1}C^* + \frac{1}{4}H. \tag{3.10}$$

Now, by replacing the second part of (3.8) in FX^*B and the third part of (3.8) in AXD, it can be derived that

$A(B^*)^{-1}K^*(F^*)^{-1}D - F(D^*)^{-1}K^*(A^*)^{-1}B =$

$$\frac{1}{2}(AXD + FX^*B) - \frac{1}{4}[A(B^*)^{-1}C^*(F^*)^{-1}D + F(D^*)^{-1}C^*(A^*)^{-1}B].$$

Due to this fact, and according to the condition (3.1), it can be concluded that

$$A(B^*)^{-1}K^*(F^*)^{-1}D = F(D^*)^{-1}K^*(A^*)^{-1}B.$$

In view of (3.8) and (3.9), we deduce that

$$\begin{aligned} X &= A^{-1}CD^{-1} - A^{-1}FX^*BD^{-1} \\ &= A^{-1}CD^{-1} - A^{-1}F(\frac{1}{2}F^{-1}CB^{-1} + 2F^{-1}KB^{-1})BD^{-1} \\ &= \frac{1}{2}A^{-1}CD^{-1} - 2A^{-1}KD^{-1} \\ &= \frac{1}{4}A^{-1}CD^{-1} + \frac{1}{4}A^{-1}CD^{-1} - 2A^{-1}KD^{-1} \\ &= \frac{1}{4}A^{-1}CD^{-1} + \frac{1}{4}(B^*)^{-1}C^*(F^*)^{-1} + (B^*)^{-1}K^*(F^*)^{-1} - A^{-1}KD^{-1}. \end{aligned}$$

Applying (3.10) and our assumptions we observe that the above solution satisfying $GXF^* + FX^*G^* = H$ if and only if

$$GA^{-1}KD^{-1}F^* = -(GA^{-1}KD^{-1}F^*)^*.$$

Indeed, we have

$GXF^* + FX^*G^*$

$$\begin{aligned} &= \frac{1}{4}GA^{-1}CD^{-1}F^* + \frac{1}{4}G(B^*)^{-1}C^* \\ &\quad + G(B^*)^{-1}K^* - GA^{-1}KD^{-1}F^* + \frac{1}{4}F(D^*)^{-1}C^*(A^*)^{-1}G^* \\ &\quad + \frac{1}{4}CB^{-1}G^* + KB^{-1}G^* - F(D^*)^{-1}K^*(A^*)^{-1}G^* \\ &= \frac{1}{4}GA^{-1}CD^{-1}F^* + \frac{1}{4}G(B^*)^{-1}C^* - \frac{1}{2}W - \frac{1}{4}G(B^*)^{-1}C^* \\ &\quad + \frac{1}{4}H - GA^{-1}KD^{-1}F^* + \frac{1}{4}F(D^*)^{-1}C^*(A^*)^{-1}G^* + \frac{1}{4}CB^{-1}G^* \\ &\quad - \frac{1}{2}W^* - \frac{1}{4}CB^{-1}G^* + \frac{1}{4}H^* - F(D^*)^{-1}K^*(A^*)^{-1}G^* \\ &= H. \end{aligned}$$

Q.E.D.

As a consequence of the above theorem we have the following result.

Corollary 3.2. Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module, $A, B, C, G, H \in \mathcal{L}(\mathcal{X})$ such that A, B, G are invertible, H is a self-adjoint operator and $GA^{-1}C = H$. If

$$A(B^*)^{-1}C^* + C^*(A^*)^{-1}B = 2C,$$

then the system of operator equations

$$\begin{cases} AX + X^*B = C, \\ GX + X^*G^* = H, \end{cases} \tag{3.11}$$

has a solution $X \in \mathcal{L}(\mathcal{X})$. In the case any solution to the system (3.11) is represented by

$$X = \frac{1}{4}A^{-1}C + \frac{1}{4}(B^*)^{-1}C^* + (B^*)^{-1}K^* - A^{-1}K,$$

where $K \in \mathcal{L}(\mathcal{X})$ satisfies $(B^*)^{-1}K^*B^{-1} = A^{-1}K^*(A^*)^{-1}$, $GA^{-1}K = -(GA^{-1}K)^*$.

In Theorem 3.1, we gave sufficient conditions for the existence of a solution to the system of operator equations (1.3). These conditions are not necessary as the following example shows.

Example 3.3. Let us consider the set of 2×2 matrices and let

$$A = F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & -2 \\ -\frac{9}{2} & 4 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, G = \begin{bmatrix} 0 & 4 \\ 4 & -8 \end{bmatrix}, H = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}.$$

It is easy to check that $A^{-1}CD^{-1} = G^{-1}H(F^*)^{-1}$ and $X = \begin{bmatrix} -1 & 4 \\ 0 & 2 \end{bmatrix}$ is a solution of (1.3). So the system (1.3) is solvable but the condition (3.1) is not satisfied.

It would be especially interesting to find conditions that are necessary and sufficient. In the following we present necessary and sufficient conditions for the existence of a solution to the system (1.3) when the operators F and D have closed ranges.

Theorem 3.4. Suppose $A, F, G \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $B, D \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ such that A, B are invertible and F, D have closed ranges. Let $C \in \mathcal{L}(\mathcal{Y})$ and $H \in \mathcal{L}(\mathcal{Y})$ be a self-adjoint operator such that $GA^{-1}FF^{\dagger}CD^{\dagger}F^* = H$. Also, suppose that

$$\begin{aligned} GA^{-1}E_FCD^{\dagger}F^* &= -(GA^{-1}E_FCD^{\dagger}F^*)^*, \\ A(B^*)^{-1}E_{D^*}C^*(F^*)^{\dagger}D &= -F(D^*)^{\dagger}C^*E_F(A^*)^{-1}B, \\ A(B^*)^{-1}D^{\dagger}DC^*(F^*)^{\dagger}D &+ F(D^*)^{\dagger}C^*FF^{\dagger}(A^*)^{-1}B = 2C, \end{aligned}$$

Then the following statements are equivalent:

(a) There exists a solution $X \in \mathcal{L}(\mathcal{X})$ to the system of operator equations (1.3).

(b) $E_F C E_{D^*} = E_F H E_F = 0$.

If (a) or (b) is satisfied, then any solution to the system (1.3) has the form

$$\begin{aligned} X = &\frac{1}{2}A^{-1}CD^\dagger + \frac{1}{2}(B^*)^{-1}C^*(F^*)^\dagger - \frac{1}{4}A^{-1}FF^\dagger CD^\dagger - \frac{1}{4}(B^*)^{-1}D^\dagger DC^*(F^*)^\dagger \\ &-\frac{1}{2}A^{-1}FF^\dagger KD^\dagger + \frac{1}{2}(B^*)^{-1}D^\dagger DK^*(F^*)^\dagger \\ &+\frac{1}{2}(B^*)^{-1}V^*P_F - \frac{1}{2}A^{-1}W^*P_{D^*}, \end{aligned} \tag{3.12}$$

where $K \in \mathcal{L}(\mathcal{Y})$ and $V, W \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ satisfy

$$\begin{aligned} A(B^*)^{-1}V^*P_F D &= -FP_{D^*}W(A^*)^{-1}B, \\ GA^{-1}W^*P_{D^*}F^* &= -(GA^{-1}W^*P_{D^*}F^*)^*, \\ GA^{-1}FF^\dagger KD^\dagger F^* &= -(GA^{-1}FF^\dagger KD^\dagger F^*)^*, \\ A(B^*)^{-1}D^\dagger DK^*(F^*)^\dagger D &= F(D^*)^\dagger K^*FF^\dagger(A^*)^{-1}B. \end{aligned}$$

Proof. $(a) \Rightarrow (b)$: Suppose that the system (1.3) has a solution $X \in \mathcal{L}(\mathcal{X})$. Then

$$\begin{aligned} E_F C E_{D^*} &= (1 - FF^\dagger)(AXD + FX^*B)(1 - D^\dagger D) \\ &= (1 - FF^\dagger)AXD(1 - D^\dagger D) + (1 - FF^\dagger)FX^*B(1 - D^\dagger D) \\ &= 0 \end{aligned}$$

and a similar argument shows that $E_F H E_F = 0$.
$(b) \Rightarrow (a)$: Suppose that $E_F C E_{D^*} = E_F H E_F = 0$. We will show that system (1.3) has a solution of the form (3.12). To this end, let us take $T = \begin{bmatrix} F & 0 \\ 0 & D^* \end{bmatrix} : \mathcal{X} \oplus \mathcal{X} \to \mathcal{Y} \oplus \mathcal{Y}$, $\hat{X} = \begin{bmatrix} 0 & X^* \\ X^* & 0 \end{bmatrix} : \mathcal{X} \oplus \mathcal{X} \to \mathcal{X} \oplus \mathcal{X}$, $S = \begin{bmatrix} A & G \\ 0 & B^* \end{bmatrix} : \mathcal{X} \oplus \mathcal{X} \to \mathcal{Y} \oplus \mathcal{Y}$ and $N = \begin{bmatrix} H & C \\ C^* & 0 \end{bmatrix} : \mathcal{Y} \oplus \mathcal{Y} \to \mathcal{Y} \oplus \mathcal{Y}$. Further, consider the operator equation $T\hat{X}S^* + S(\hat{X})^*T^* = N$, where it can be written in the following form

$$\begin{aligned} &T\hat{X}S^* + S(\hat{X})^*T^* \\ &= \begin{bmatrix} F & 0 \\ 0 & D^* \end{bmatrix}\begin{bmatrix} 0 & X^* \\ X^* & 0 \end{bmatrix}\begin{bmatrix} A^* & 0 \\ G^* & B \end{bmatrix} + \begin{bmatrix} A & G \\ 0 & B^* \end{bmatrix}\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}\begin{bmatrix} F^* & 0 \\ 0 & D \end{bmatrix} \\ &= \begin{bmatrix} FX^*G^* + GXF^* & FX^*B + AXD \\ D^*X^*A^* + B^*XF^* & 0 \end{bmatrix} \\ &= \begin{bmatrix} H & C \\ C^* & 0 \end{bmatrix}. \end{aligned}$$

Hence, we find that

$$\begin{cases} AXD + FX^*B = C, \\ GXF^* + FX^*G^* = H. \end{cases}$$

We know that $N = N^*$ and also using condition (b) we have

$$\begin{aligned}(1 - TT^\dagger)N(1 - TT^\dagger) &= \begin{bmatrix} E_F H E_F & E_F C E_{D^*} \\ E_{D^*} C^* E_F & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.\end{aligned}$$

Therefore, all conditions of Theorem 2.8 hold and so the operator equation $T\hat{X}S^* + S(\hat{X})^*T^* = N$, has the solution

$$\hat{X} = \frac{1}{2}T^\dagger NTT^\dagger(S^*)^{-1} + T^\dagger ZTT^\dagger(S^*)^{-1} + T^\dagger N(1 - TT^\dagger)(S^*)^{-1} + (1 - T^\dagger T)Y(S^*)^{-1},$$

where $Z \in \mathcal{L}(\mathcal{Y} \oplus \mathcal{Y})$ satisfies $T^*(Z + Z^*)T = 0$ and $Y \in \mathcal{L}(\mathcal{Y} \oplus \mathcal{Y}, \mathcal{X} \oplus \mathcal{X})$ is arbitrary. Let us assume that $Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}$ and $Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}$. Then from the above solution we obtain

$$\begin{aligned}&\begin{bmatrix} 0 & X^* \\ X^* & 0 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} F^\dagger HFF^\dagger(A^*)^{-1} - F^\dagger CD^*(D^*)^\dagger B^{-1}G^*(A^*)^{-1} & F^\dagger CD^*(D^*)^\dagger B^{-1} \\ (D^*)^\dagger C^* FF^\dagger(A^*)^{-1} & 0 \end{bmatrix} \\ &+ \begin{bmatrix} F^\dagger Z_1 FF^\dagger(A^*)^{-1} - F^\dagger Z_2 D^*(D^*)^\dagger B^{-1}G^*(A^*)^{-1} & F^\dagger Z_2 D^*(D^*)^\dagger B^{-1} \\ (D^*)^\dagger Z_3 FF^\dagger(A^*)^{-1} - (D^*)^\dagger Z_4 D^*(D^*)^\dagger B^{-1}G^*(A^*)^{-1} & (D^*)^\dagger Z_4 D^*(D^*)^\dagger B^{-1} \end{bmatrix} \\ &+ \begin{bmatrix} F^\dagger HE_F(A^*)^{-1} - F^\dagger CE_{D^*}B^{-1}G^*(A^*)^{-1} & F^\dagger CE_{D^*}B^{-1} \\ (D^*)^\dagger C^* E_F(A^*)^{-1} & 0 \end{bmatrix} \\ &+ \begin{bmatrix} P_F Y_1(A^*)^{-1} - P_F Y_2 B^{-1}G^*(A^*)^{-1} & P_F Y_2 B^{-1} \\ P_{D^*} Y_3(A^*)^{-1} - P_{D^*} Y_4 B^{-1}G^*(A^*)^{-1} & P_{D^*} Y_4 B^{-1} \end{bmatrix}.\end{aligned}$$

It in turns implies that

$$\begin{aligned}&\frac{1}{2}F^\dagger HFF^\dagger(A^*)^{-1} - \frac{1}{2}F^\dagger CD^*(D^*)^\dagger B^{-1}G^*(A^*)^{-1} + F^\dagger Z_1 FF^\dagger(A^*)^{-1} \\ &-F^\dagger Z_2 D^*(D^*)^\dagger B^{-1}G^*(A^*)^{-1} + F^\dagger HE_F(A^*)^{-1} - F^\dagger CE_{D^*}B^{-1}G^*(A^*)^{-1} \\ &+P_F Y_1(A^*)^{-1} - P_F Y_2 B^{-1}G^*(A^*)^{-1} = 0, \qquad (3.13) \\ &(D^*)^\dagger Z_4 D^*(D^*)^\dagger B^{-1} + P_{D^*} Y_4 B^{-1} = 0, \qquad (3.14)\end{aligned}$$

$$X^* = \frac{1}{2}F^\dagger CD^*(D^*)^\dagger B^{-1} + F^\dagger Z_2 D^*(D^*)^\dagger B^{-1}$$

$$+F^{\dagger}CE_{D^*}B^{-1} + P_F Y_2 B^{-1}, \tag{3.15}$$

$$\begin{aligned} X^* = {} & \frac{1}{2}(D^*)^{\dagger}C^*FF^{\dagger}(A^*)^{-1} + (D^*)^{\dagger}Z_3FF^{\dagger}(A^*)^{-1} \\ & -(D^*)^{\dagger}Z_4D^*(D^*)^{\dagger}B^{-1}G^*(A^*)^{-1} + (D^*)^{\dagger}C^*E_F(A^*)^{-1} \\ & +P_{D^*}Y_3(A^*)^{-1} - P_{D^*}Y_4B^{-1}G^*(A^*)^{-1}. \end{aligned} \tag{3.16}$$

Utilizing the relation $T^*(Z + Z^*)T = 0$, together with $\ker(T^*) = \ker(T^{\dagger})$ it can be deduced that

$$\begin{aligned} F^{\dagger}Z_1F &= -F^{\dagger}Z_1^*F \\ F^{\dagger}Z_2D^* &= -F^{\dagger}Z_3^*D^*, \\ (D^*)^{\dagger}Z_3F &= -(D^*)^{\dagger}Z_2^*F, \\ (D^*)^{\dagger}Z_4D^* &= -(D^*)^{\dagger}Z_4^*D^*. \end{aligned} \tag{3.17}$$

On combining (3.14) and (3.16), we get

$$\begin{aligned} X^* = {} & \frac{1}{2}(D^*)^{\dagger}C^*FF^{\dagger}(A^*)^{-1} + (D^*)^{\dagger}Z_3FF^{\dagger}(A^*)^{-1} \\ & +(D^*)^{\dagger}C^*E_F(A^*)^{-1} + P_{D^*}Y_3(A^*)^{-1}. \end{aligned} \tag{3.18}$$

By taking adjoint from (3.15) and (3.18) and setting the two equal to each other and also applying (3.17) we derive that

$$\begin{aligned} & \frac{1}{2}(B^*)^{-1}D^{\dagger}DC^*(F^*)^{\dagger} + (B^*)^{-1}D^{\dagger}DZ_2^*(F^*)^{\dagger} + (B^*)^{-1}E_{D^*}C^*(F^*)^{\dagger} + (B^*)^{-1}Y_2^*P_F \\ & = -\frac{1}{2}A^{-1}(F^*)^{\dagger}F^*CD^{\dagger} - A^{-1}(F^*)^{\dagger}F^*Z_2D^{\dagger} + A^{-1}CD^{\dagger} + A^{-1}Y_3^*P_{D^*}. \end{aligned}$$

Put $K = Z_2$, $V = Y_2$ and $W = Y_3$ in the above relation and deduce

$$\begin{aligned} A^{-1}CD^{\dagger} = {} & \frac{1}{2}(B^*)^{-1}D^{\dagger}DC^*(F^*)^{\dagger} + (B^*)^{-1}D^{\dagger}DK^*(F^*)^{\dagger} \\ & +(B^*)^{-1}E_{D^*}C^*(F^*)^{\dagger} + (B^*)^{-1}V^*P_F + \frac{1}{2}A^{-1}(F^*)^{\dagger}F^*CD^{\dagger} \\ & +A^{-1}(F^*)^{\dagger}F^*KD^{\dagger} - A^{-1}W^*P_{D^*}. \end{aligned} \tag{3.19}$$

By multiplying F on the left and A^* on the right to (3.13), we get

$$\begin{aligned} FF^{\dagger}KD^*(D^*)^{\dagger}B^{-1}G^* = {} & \frac{1}{2}FF^{\dagger}HFF^{\dagger} - \frac{1}{2}FF^{\dagger}CD^*(D^*)^{\dagger}B^{-1}G^* \\ & +FF^{\dagger}Z_1FF^{\dagger} + FF^{\dagger}HE_F - FF^{\dagger}CE_{D^*}B^{-1}G^*. \end{aligned} \tag{3.20}$$

Now if, (1.3) has a solution X, then $AXD = C - FX^*B$ and by replacing (3.15) in the right side of this equation we get

$$\begin{aligned} AXD &= C - \frac{1}{2}FF^\dagger CD^*(D^*)^\dagger - FF^\dagger KD^*(D^*)^\dagger - FF^\dagger CE_{D^*} \\ &= C - \frac{1}{2}FF^\dagger CD^\dagger D - FF^\dagger KD^\dagger D - FF^\dagger CE_{D^*}. \end{aligned}$$

Set $M = C - \frac{1}{2}FF^\dagger CD^\dagger D - FF^\dagger KD^\dagger D - FF^\dagger CE_{D^*}$. Applying the condition (b) we have

$$\begin{aligned} AA^{-1}MD^\dagger D &= CD^\dagger D - \frac{1}{2}FF^\dagger CD^\dagger D - FF^\dagger KD^\dagger D \\ &= C - FF^\dagger CE_{D^*} - \frac{1}{2}FF^\dagger CD^\dagger D - FF^\dagger KD^\dagger D \\ &= M. \end{aligned}$$

Therefore, by Remark 2.7 and using (3.15) and (3.19) we observe that

$$\begin{aligned} X &= A^{-1}(C - FX^*B)D^\dagger \\ &= A^{-1}CD^\dagger - A^{-1}F\Big(\frac{1}{2}F^\dagger CD^*(D^*)^\dagger B^{-1} + F^\dagger KD^*(D^*)^\dagger B^{-1} \\ &\quad + F^\dagger CE_{D^*}B^{-1} + P_F VB^{-1}\Big)BD^\dagger \\ &= A^{-1}CD^\dagger - \frac{1}{2}A^{-1}FF^\dagger CD^\dagger - A^{-1}FF^\dagger KD^\dagger \\ &= \frac{1}{2}A^{-1}CD^\dagger + \frac{1}{4}(B^*)^{-1}D^\dagger DC^*(F^*)^\dagger + \frac{1}{2}(B^*)^{-1}D^\dagger DK^*(F^*)^\dagger \\ &\quad + \frac{1}{2}(B^*)^{-1}E_{D^*}C^*(F^*)^\dagger + \frac{1}{2}(B^*)^{-1}V^*P_F - \frac{1}{4}A^{-1}FF^\dagger CD^\dagger \\ &\quad + \frac{1}{2}A^{-1}FF^\dagger KD^\dagger - \frac{1}{2}A^{-1}W^*P_{D^*} - \frac{1}{2}A^{-1}FF^\dagger CD^\dagger - A^{-1}FF^\dagger KD^\dagger \\ &= \frac{1}{2}A^{-1}CD^\dagger + \frac{1}{2}(B^*)^{-1}C^*(F^*)^\dagger - \frac{1}{4}A^{-1}FF^\dagger CD^\dagger \\ &\quad - \frac{1}{4}(B^*)^{-1}D^\dagger DC^*(F^*)^\dagger - \frac{1}{2}A^{-1}FF^\dagger KD^\dagger \\ &\quad + \frac{1}{2}(B^*)^{-1}D^\dagger DK^*(F^*)^\dagger + \frac{1}{2}(B^*)^{-1}V^*P_F - \frac{1}{2}A^{-1}W^*P_{D^*}. \end{aligned}$$

Next, applying (3.20) and the above solution, we infer that $GXF^* + FX^*G^* = H$ if and only if

$$\begin{aligned} GA^{-1}FF^\dagger KD^\dagger F^* &= -(GA^{-1}FF^\dagger KD^\dagger F^*)^*, \\ GA^{-1}W^*P_{D^*}F^* &= -(GA^{-1}W^*P_{D^*}F^*)^*. \end{aligned}$$

Indeed, by our assumptions we have

$$\begin{aligned}
GXF^*+FX^*G^* &= \frac{1}{2}GA^{-1}CD^\dagger F^*+\frac{1}{2}G(B^*)^{-1}C^*(F^*)^\dagger F^*\\
&\quad -\frac{1}{4}GA^{-1}FF^\dagger CD^\dagger F^*-\frac{1}{4}G(B^*)^{-1}D^\dagger DC^*(F^*)^\dagger F^*\\
&\quad -\frac{1}{2}GA^{-1}FF^\dagger KD^\dagger F^*+\frac{1}{4}(F^*)^\dagger F^*H^*(F^*)^\dagger F^*\\
&\quad -\frac{1}{4}G(B^*)^{-1}D^\dagger DC^*(F^*)^\dagger F^*+\frac{1}{2}(F^*)^\dagger F^*U^*(F^*)^\dagger F^*\\
&\quad +\frac{1}{2}E_FH^*(F^*)^\dagger F^*-\frac{1}{2}G(B^*)^{-1}E_{D^*}C^*(F^*)^\dagger F^*\\
&\quad -\frac{1}{2}GA^{-1}W^*P_{D^*}F^*+\frac{1}{2}F(D^*)^\dagger C^*(A^*)^{-1}G^*\\
&\quad +\frac{1}{2}FF^\dagger CB^{-1}G^*-\frac{1}{4}F(D^*)^\dagger C^*(F^*)^\dagger F^*(A^*)^{-1}G^*\\
&\quad -\frac{1}{4}FF^\dagger CD^*(D^*)^\dagger B^{-1}G^*-\frac{1}{2}F(D^*)^\dagger K^*(F^*)^\dagger F^*(A^*)^{-1}G^*\\
&\quad +\frac{1}{4}FF^\dagger HFF^\dagger-\frac{1}{4}FF^\dagger CD^*(D^*)^\dagger B^{-1}G^*+\frac{1}{2}FF^\dagger UFF^\dagger\\
&\quad +FF^\dagger HE_F-\frac{1}{2}FF^\dagger CE_{D^*}B^{-1}G^*-\frac{1}{2}FP_{D^*}W(A^*)^{-1}G^*\\
&= \frac{1}{4}H+\frac{1}{4}FF^\dagger HFF^\dagger+\frac{1}{2}E_FHFF^\dagger\\
&\quad +\frac{1}{4}H+\frac{1}{4}FF^\dagger HFF^\dagger+\frac{1}{2}FF^\dagger HE_F\\
&= H.
\end{aligned}$$

Q.E.D.

4 The system of operator equations $AXF = H_1$, $CXD = H_2$, $BXD = H_3$

In this section, we find explicit solution to the system of operator equations (1.4) in the general setting of the adjointable operators between Hilbert C^*-modules.

Theorem 4.1. Suppose $A, B, C, D, F, H_1, H_2, H_3 \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ such that A, B, D, F have closed ranges. Also, suppose that $\mathcal{R}(C) \subseteq \mathcal{R}(A)$, $\mathcal{R}(C^*) \subseteq \mathcal{R}(A^*) = \mathcal{R}(B^*)$, $\mathcal{R}(F) = \mathcal{R}(D)$ and $AB^\dagger H_3 D^\dagger F = H_1$.
Then the following statements are equivalent:

(a) There exists a solution $X \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ to the system of operator equations (1.4),

(b) $AA^\dagger H_1 F^\dagger F = H_1$, $AA^\dagger H_2 D^\dagger D = H_2$ and $BB^\dagger H_3 D^\dagger D = H_3$.

If (a) or (b) is satisfied, then any solution of (1.4) can be expressed as

$$X = \frac{1}{2}A^\dagger H_1 F^\dagger + \frac{1}{2}B^\dagger H_3 D^\dagger + \frac{1}{2}U - \frac{1}{2}B^\dagger BUDD^\dagger - \frac{1}{2}W + \frac{1}{2}A^\dagger AWFF^\dagger, \tag{4.1}$$

where U and W are arbitrary operators in $\mathcal{L}(\mathcal{Y}, \mathcal{X})$.

Proof. (a)⇒(b): Suppose the system (1.4) has a solution $X \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$. Obviously $AA^{\dagger}H_1F^{\dagger}F = H_1$ and $BB^{\dagger}H_3D^{\dagger}D = H_3$. Now, since $\mathcal{R}(C) \subseteq \mathcal{R}(A)$, then $AA^{\dagger}C = C$. Therefore

$$\begin{aligned} AA^{\dagger}H_2D^{\dagger}D &= AA^{\dagger}CXDD^{\dagger}D \\ &= CXD = H_2. \end{aligned}$$

(b)⇒(a): Suppose that condition (b) holds. We will show that system (1.4) has a solution $X \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ of the form (4.1). To do this, let us take $T = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} : \mathcal{X} \oplus \mathcal{X} \to \mathcal{Y} \oplus \mathcal{Y}$, $Y = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} : \mathcal{Y} \oplus \mathcal{Y} \to \mathcal{X} \oplus \mathcal{X}$, $S = \begin{bmatrix} F & 0 \\ 0 & D \end{bmatrix} : \mathcal{X} \oplus \mathcal{X} \to \mathcal{Y} \oplus \mathcal{Y}$ and $N = \begin{bmatrix} H_1 & H_2 \\ 0 & H_3 \end{bmatrix} : \mathcal{X} \oplus \mathcal{X} \to \mathcal{Y} \oplus \mathcal{Y}$.

Regarding Lemma 2.4 T has Moore-Penrose inverse $T^{\dagger} = \begin{bmatrix} A^{\dagger} & -A^{\dagger}CB^{\dagger} \\ 0 & B^{\dagger} \end{bmatrix}$.

Consider the operator equation $TYS = N$, we have

$$\begin{aligned} TYS &= \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} F & 0 \\ 0 & D \end{bmatrix} \\ &= \begin{bmatrix} H_1 & H_2 \\ 0 & H_3 \end{bmatrix}, \end{aligned}$$

or equivalently

$$\begin{bmatrix} AXF & CXD \\ 0 & BXD \end{bmatrix} = \begin{bmatrix} H_1 & H_2 \\ 0 & H_3 \end{bmatrix}.$$

Therefore

$$\begin{cases} AXF = H_1, \\ CXD = H_2, \\ BXD = H_3. \end{cases}$$

Applying $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and condition (b), we obtain $TT^{\dagger}NS^{\dagger}S = N$. By Theorem 2.6, it follows that the operator equation $TYS = N$ has a solution Y and it has the following representation

$$Y = T^{\dagger}NS^{\dagger} + V - T^{\dagger}TVSS^{\dagger}, \tag{4.2}$$

where $V \in \mathcal{L}(\mathcal{Y} \oplus \mathcal{Y}, \mathcal{X} \oplus \mathcal{X})$ is arbitrary.

Let $V = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix}$. From (4.2), we have

$$\begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} = \begin{bmatrix} A^{\dagger} & -A^{\dagger}CB^{\dagger} \\ 0 & B^{\dagger} \end{bmatrix} \begin{bmatrix} H_1 & H_2 \\ 0 & H_3 \end{bmatrix} \begin{bmatrix} F^{\dagger} & 0 \\ 0 & D^{\dagger} \end{bmatrix} + \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix} -$$

$$\begin{bmatrix} A^{\dagger}AV_1FF^{\dagger} + A^{\dagger}CV_3FF^{\dagger} - A^{\dagger}CB^{\dagger}BV_3FF^{\dagger} & A^{\dagger}AV_2DD^{\dagger} + A^{\dagger}CV_4DD^{\dagger} - A^{\dagger}CB^{\dagger}BV_4DD^{\dagger} \\ B^{\dagger}BV_3FF^{\dagger} & B^{\dagger}BV_4DD^{\dagger} \end{bmatrix}.$$

It in turn enforces that

$$X = A^{\dagger}H_1F^{\dagger} + V_1 - A^{\dagger}AV_1FF^{\dagger} - A^{\dagger}CV_3FF^{\dagger} + A^{\dagger}CB^{\dagger}BV_3FF^{\dagger}, \tag{4.3}$$

$$X = B^\dagger H_3 D^\dagger + V_4 - B^\dagger B V_4 D D^\dagger, \tag{4.4}$$

$$V_3 - B^\dagger B V_3 F F^\dagger = 0, \tag{4.5}$$

$$A^\dagger H_2 D^\dagger - A^\dagger C B^\dagger H_3 D^\dagger + V_2 - A^\dagger A V_2 D D^\dagger - A^\dagger C V_4 D D^\dagger + A^\dagger C B^\dagger B V_4 D D^\dagger = 0. \tag{4.6}$$

On combining (4.3) and (4.5), we get

$$X = A^\dagger H_1 F^\dagger + V_1 - A^\dagger A V_1 F F^\dagger. \tag{4.7}$$

Since $\mathcal{R}(C^*) \subseteq \mathcal{R}(B^*)$, then $CB^\dagger B = C$. Due to this fact and by multiplying $AA^\dagger A$ on the left and $DD^\dagger D$ on the right to (4.6), we obtain

$$AA^\dagger H_2 D^\dagger D = AA^\dagger C B^\dagger H_3 D^\dagger D = C B^\dagger H_3 D^\dagger D. \tag{4.8}$$

By setting (4.4) and (4.7) equal to each other we derive that

$$A^\dagger H_1 F^\dagger = B^\dagger H_3 D^\dagger + V_4 - B^\dagger B V_4 D D^\dagger - V_1 + A^\dagger A V_1 F F^\dagger. \tag{4.9}$$

By putting $W = V_1$, $U = V_4$ in the above relations and using (4.7) and (4.9) we observe that

$$X = \frac{1}{2} A^\dagger H_1 F^\dagger + \frac{1}{2} B^\dagger H_3 D^\dagger + \frac{1}{2} U - \frac{1}{2} B^\dagger B U D D^\dagger - \frac{1}{2} W + \frac{1}{2} A^\dagger A W F F^\dagger,$$

where U and W are arbitrary operators over $\mathcal{L}(\mathcal{Y}, \mathcal{X})$. Now, by our hypotheses and taking into account (4.8) it can be easily shown that X is a solution of the system (1.4). Q.E.D.

References

[1] Z. Bakhtiari and S.M. Vaezpour, *Positive solutions to the system of operator equations $T_i X = U_i$ and $T_i X V_i = U_i$*, J. Math. Anal. **7**5 (2016) 102–117.

[2] A. Dajić and J.J. Koliha, *Positive solutions to the equations $AX = C$ and $XB = D$ for Hilbert space operators*, J. Math. Anal. Appl. **333** (2007), 567–576.

[3] C.Y. Deng and H.K. Du, *Representations of the Moor-Penrose inverse for a class of 2-by-2 block operator valued partial matrices*, Linear and Multilinear Alg. **58** (2010) 15–26.

[4] X. Fang, J. Yu and H. Yao, *Solutions to operator equations on Hilbert C^*-modules*, Linear Alg. Appl. **431** (2009) 2142–2153.

[5] F.O. Farid, M.S. Moslehian, Q.W. Wang and Z.C. Wu, *On the Hermitian solutions to a system of adjointable operator equations*, Linear Alg. Appl. **437** (2012) 1854–1891.

[6] J.K. Han, H.Y. Lee and W.Y. Lee, *Invertible completions of 2×2 upper triangular operator matrices*, AMS. **128** (1999) 119–123.

[7] C.G. Khatri and S.K. Mitra, *Hermitian and nonnegative definite solutions of linear matrix equations,* SIAM J. Appl. Math. **31** (1976) 578–585.

[8] E.C. Lance, Hilbert C^*-Modules - A Toolkit for Operator Algebraists, LMS Lecture Note Series 210, *Cambridge Univ. Press,* Cambridge, England, 1995.

[9] M. Mohammadzadeh Karizaki, M. Hassani and D. Djordjevic, *The solutions to the operator equation $TXS - SX^*T^* = A$ in Hilbert C^*-modules*, IJNAA, **7** (2) (2016) 127–132.

[10] M. Mohammadzadeh Karizaki, M. Hassani and S.S. Dragomir, *Explicit solution to modular operator equation $TXS^* - SX^*T^* = A$*, Kragujevac J. Math. **40** (2) (2016) 280–289.

[11] G.J. Song, *Common solutions to some operator equations over Hilbert C^* modules and applications*, Linear and Multilinear Alg. **627** (2014) 895–912.

[12] G.J. Song, *G-positive and G-repositive solutions to some adjointable operator equations over Hilbert C^*-modules*, Bull. Iran. Math. Soc. **39** (5) (2013) 971–992.

[13] G.J. Song and Q.W. Wang, *Positive and re-positive solutions to some systems of adjointable operator equations over Hilbert C*- modules*, Electron. J. Linear Alg. **22** (2011) 1070–1084.

[14] M. Vosough and M.S. Moslehian, *Solutions of the system of operator equations $BXA = B = AXB$ via the *-order*, Electron. J. Linear Alg. **32** (2017) 172–183.

[15] Q.W. Wang and C.Z. Dong, *Positive solutions to a system of adjointable operator equations over Hilbert C^*-modules*, Linear Alg. Appl. **433** (2010) 1481–1489.

[16] Q.W. Wang and Z.C. Wu, *Common Hermitian solutions to some operator equations on Hilbert C^*-modules,* Linear Alg. Appl. **432** (2010) 3159–3171.

[17] Q. Xu, *Common Hermitian and positive solutions to the adjointable operator equations $AX = C$, $XB = D$*, Linear Alg. Appl. **429** (2008) 1–11.

[18] Q. Xu and L. Sheng, *Positive semi-definite matrices of adjointable operators on Hilbert C*-modules*, Linear Alg. Appl. **428** (2008) 992–1000.