Exact solutions of nonlinear evolution equations using the extended modified Exp(-$\Omega(\xi)$) function method

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Abstract

Obtaining exact solutions of the evolution equation is one of the very important subjects in mathematics, science and technology. For this purpose, many different methods have been constructed and developed. In this article, a new technique which is called extended modified Exp(-$\Omega(\xi)$) function method is going to be studied for seeking new exact solutions of Burger-Fisher equation and Phi Four equation. The method is capable of deriving many number of solutions. With the aid of the method, various exact solutions including trigonometric, hyperbolic and rational solutions have been obtained and using a software the graphical representation of the solutions have been presented. In conclusion, we can say that the present method can also be used for the solutions of a wide range of problems.

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1 Introduction

The nonlinear evolution equations arise in modelling fundamental laws of nature. When scientist want to understand structure and behaviour of phenomena in nature, analysing and investigating the mathematics models based on nonlinear partial differential equations which describe complex structures plays a vital role. For this intention, many scientists focused on finding exact and numerical solutions of partial differential equations and many numerical methods have been developed to determine solutions. Sub- equation method and fractional sub-equation method [15, 27], the (G'/G)-expansion method and its modifications [25,28], the Exp-function method and its modification [1, 8, 11], the extended modified direct algebraic method [16, 21], the modified simple equation method [2, 5] and many different methods get involved in various papers [9, 12–14, 17, 18].

In this article, we use the extended modified Exp(-$\Omega(\xi)$) function method (MEFM) which is further development of by Khater et. al. The method is based on exp-(-$\Omega(\xi)$) function method. But difference between two method is; the extended modified Exp(-$\Omega(\xi)$) function method is more general and can be produce different new solutions. Under specific conditions, the method can be convert into some different known methods. After presenting the main steps of the method, we will explain this transformations in short. In present study, In order to expand the application area of the method, we are going to consider Burger-Fisher equation and Phi-four equation given as follows;

i) The Burger-Fisher equation illustrated in [22] is given as

$$u_t = u_{xx} + \mu u\ u_x + u(1-u) \tag{1.1}$$

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where μ is a arbitrary constant. The Burger-Fisher equation have a wide application area such as biology, gas dynamics, heat and mass transfer,financial mathematics and genetics. Also, especially, The interaction of reaction mechanism, and effect of convection and diffusion are modelled by the equation [26].

ii) The Phi-Four equation is given as

$$u_{tt} = u_{xx} - m^2 u - \lambda u^3 \tag{1.2}$$

the parameters m and λ are real-valued parameters. The equation is a particular version of Klein Gordon equation that arise in various phenomenon in particle physics and quantum mechanics [3].

The remainder of the article is prepared as; in following section, we will explain main steps of the extended modified exp(-$\Omega(\xi)$) function method. Applications of the method to the handled equations, the newly obtained results and graphical simulations are going to present in Section 3. At the last of the article some conclusions are going to given.

2 Basic idea of the extended modified exp(-$\Omega(\xi)$) function method

Assume, the nonlinear partial differential equation can be symbolize as follow;

$$R(u, u_{x_1}, u_{x_2}, u_{x_3}, u_{x_1x_1}, u_{x_1x_2}, u_{x_1x_3}, ...) = 0 \tag{2.3}$$

where $x_1, x_2, x_3, ..., x_n$ are independent variables, R is a polynomial which consist of $u(x_1, x_2, x_3, ..., x_n)$ unknown function and its various order mixed partial derivatives. In the following way, let us give main steps of the extended modified exp(-$\Omega(\xi)$) function method (MEFM);

- step1: Using the change of the $x_1, x_2, x_3, ..., x_n$ independent variables using a suitable ξ travelling wave transformation as it will be $u(x_1, x_2, x_3, ..., x_n) = u(\xi)$, one can convert the partial differential equation given in (2.3) into an ordinary differential equation given as following

$$R'(u, u', u'', u''', ...) = 0 \tag{2.4}$$

  where the prime symbolize derivative respect to ξ independent variable and the general solution of last equation is $u = u(\xi)$. The derivatives of the transformed equation can be obtained by means of invariance of the differential [23] such as $du = \sum_{i=1}^{n} u_{x_i} dx_i = \sum_{j=1}^{n} u_{\xi_j} d\xi_j$.

- step2: this step is based on construction of particular solutions. Now, solution can be construct as a linear combination of an arbitrary functions with many constants such as

$$u(\xi) = \sum_{i=1}^{N} a_i a^{if(\xi)} \tag{2.5}$$

  where a,a_i are constants and they will be determined using symbolic computation and $a_N \neq 0$. $f(\xi)$ is solution of the following second order linear sub-equation;

$$f'(\xi) = \frac{1}{\ln a} \left(\alpha a^{-f(\xi)} + \beta + \sigma a^{f(\xi)} \right). \tag{2.6}$$

  Using the advantages of all kinds of special solutions which is belong to sub equation is benefits of such sub equation methods.

- step3: N parameter is degree of solution in the polynomial form and it is determined by homogeneous balance between linear term of highest order derivative with highest order nonlinear term.

- step4: substituting $u(\xi)$ into transformed equation (2.4) yields to a new equation in terms of a, a_i, α, β , σ and also some constants seen in travelling wave transformation. When we arrange the newly obtained equation according to power of $a^{if(\xi)}$ where $(i = 1, 2, 3, ...N)$, we can encounter a system of algebraic equation. In order to determine unknown variables such as a, a_i, α, β and σ, the system is solved by symbolic computation.

  Before presenting the solutions of the Eq. (2.6) , we should remark that the method proposed by Khater et. al in [19]. However unfortunately, in [29], Zayed et al. proved that the method is wrong by a simple calculation. During the process on studying mentioned method, we emerge the solution presented in the paper of Khater do not satisfy the considered equations. Thus, in this study we present an application of the method with the new solutions of the Eq.(2.6) which are obtained with the aid of simple calculation. The newly obtained solutions which satisfies the equation given as follows;

1)when $\beta^2 - 4\alpha\sigma < 0 \& \sigma \neq 0 \Rightarrow a^{f(\xi)} = \frac{-\beta}{2\sigma} + \frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2\sigma} \tan\left(\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2}\xi\right)$;

$$\text{or} \Rightarrow \ a^{f(\xi)} = \frac{-\beta}{2\sigma} + \frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2\sigma} \cot\left(\frac{\sqrt{-(\beta^2-4\alpha\sigma)}}{2}\xi\right)$$

2)when $\beta^2 - 4\alpha\sigma > 0 \& \sigma \neq 0 \Rightarrow a^{f(\xi)} = \frac{-\beta}{2\sigma} - \frac{\sqrt{\beta^2-4\alpha\sigma}}{2\sigma} \tanh\left(\frac{\sqrt{\beta^2-4\alpha\sigma}}{2}\xi\right)$

$$\text{or} \Rightarrow a^{f(\xi)} = \frac{-\beta}{2\sigma} - \frac{\sqrt{\beta^2-4\alpha\sigma}}{2\sigma} \coth\left(\frac{\sqrt{\beta^2-4\alpha\sigma}}{2}\xi\right)$$

3)when $\beta^2 + 4\alpha^2 > 0 \& \sigma \neq 0 \ \& \ \sigma = -\alpha \Rightarrow a^{f(\xi)} = \frac{\beta}{2\alpha} - \frac{\sqrt{\beta^2+4\alpha^2}}{2\alpha} \tanh\left(\frac{\sqrt{\beta^2+4\alpha^2}}{2}\xi\right)$

$$\text{or} \Rightarrow \ a^{f(\xi)} = \frac{\beta}{2\alpha} - \frac{\sqrt{\beta^2+4\alpha^2}}{2\alpha} \coth\left(\frac{\sqrt{\beta^2+4\alpha^2}}{2}\xi\right)$$

4)when $\beta^2 + 4\alpha^2 < 0 \ \& \ \sigma \neq 0 \ \& \ \sigma = -\alpha \Rightarrow a^{f(\xi)} = \frac{\beta}{2\alpha} + \frac{\sqrt{-(\beta^2+4\alpha^2)}}{2\alpha} \tan\left(\frac{\sqrt{-(\beta^2+4\alpha^2)}}{2}\xi\right)$

$$\text{or} \Rightarrow a^{f(\xi)} = \frac{\beta}{2\alpha} + \frac{\sqrt{-(\beta^2+4\alpha^2)}}{2\alpha} \cot\left(\frac{\sqrt{-(\beta^2+4\alpha^2)}}{2}\xi\right)$$

5)when $\beta^2 - 4\alpha^2 < 0 \ \& \ \ \sigma = \alpha \Rightarrow a^{f(\xi)} = \frac{-\beta}{2\alpha} + \frac{\sqrt{-(\beta^2-4\alpha^2)}}{2\alpha} \tan\left(\frac{\sqrt{-(\beta^2-4\alpha^2)}}{2}\xi\right)$

$$\text{or} \Rightarrow a^{f(\xi)} = \frac{-\beta}{2\alpha} + \frac{\sqrt{-(\beta^2-4\alpha^2)}}{2\alpha} \cot\left(\frac{\sqrt{-(\beta^2-4\alpha^2)}}{2}\xi\right)$$

$$6)\text{when } \beta^2-4\alpha^2>0 \ \&\ \ \sigma=\alpha \Rightarrow a^{f(\xi)}=\frac{-\beta}{2\alpha}-\frac{\sqrt{\beta^2-4\alpha^2}}{2\alpha}\tanh\left(\frac{\sqrt{\beta^2-4\alpha^2}}{2}\xi\right)$$
$$\text{or} \Rightarrow a^{f(\xi)}=\frac{-\beta}{2\alpha}-\frac{\sqrt{\beta^2-4\alpha^2}}{2\alpha}\coth\left(\frac{\sqrt{\beta^2-4\alpha^2}}{2}\xi\right)$$
$$7)\text{when } \alpha\ \sigma<0 \ \&\ \sigma\neq 0 \ \&\ \beta=0 \Rightarrow a^{f(\xi)}=\sqrt{\frac{\alpha}{\sigma}}\tan\left(\sqrt{\alpha\sigma}\xi\right)$$
$$\text{or} \Rightarrow a^{f(\xi)}=\sqrt{\frac{\alpha}{\sigma}}\cot\left(\sqrt{\alpha\sigma}\xi\right)$$
$$8)\text{when } \beta=0 \ \&\ \alpha=-\sigma \Rightarrow a^{f(\xi)}=\frac{-e^{2\sigma\xi}+1}{e^{2\sigma\xi}+1}$$
$$9)\text{when } \ \alpha=\sigma=0 \Rightarrow a^{f(\xi)}=\ln(\beta\xi)$$
$$10)\text{when } \ \beta^2=\alpha\sigma \Rightarrow a^{f(\xi)}=\frac{-\sqrt{\alpha\sigma}}{2\sigma}+\frac{\sqrt{3\alpha\sigma}}{2\sigma}\tan\left(\frac{\sqrt{3\alpha\sigma}}{2}\xi\right)$$

$$11)\text{when } \ \beta=p,\alpha=2p,\sigma=0 \Rightarrow a^{f(\xi)}=e^{p\xi}-2$$
$$12)\text{when } \ \beta=p,\sigma=2p,\alpha=0 \Rightarrow a^{f(\xi)}=\left(\frac{1}{e^{p\xi}}-2\right)^{-1}$$
$$13)\text{when } \ 2\beta=\sigma+\alpha \Rightarrow a^{f(\xi)}=\frac{-\alpha-\sigma}{4\sigma}+\frac{\sqrt{-\alpha^2+14\alpha\sigma-\sigma^2}}{4\sigma}\tan\left(\frac{\sqrt{-\alpha^2+14\alpha\sigma-\sigma^2}}{4}\xi\right)$$
$$or \Rightarrow a^{f(\xi)}=\frac{-\alpha-\sigma}{4\sigma}-\frac{\sqrt{\alpha^2-14\alpha\sigma+\sigma^2}}{4\sigma}\tanh\left(\frac{\sqrt{\alpha^2-14\alpha\sigma+\sigma^2}}{4}\xi\right)$$
$$14)\text{when } \ -2\beta=\sigma+\alpha \Rightarrow a^{f(\xi)}=\frac{\alpha+\sigma}{4\sigma}-\frac{\sqrt{-\alpha^2+14\alpha\sigma-\sigma^2}}{4\sigma}\tan\left(\frac{\sqrt{-\alpha^2+14\alpha\sigma-\sigma^2}}{4}\xi\right)$$
$$15)\text{when } \ \alpha=0 \Rightarrow a^{f(\xi)}=\left(\frac{e^{-\beta\xi}-\sigma}{\beta}\right)^{-1}$$

$$16)\text{when } \ \beta=\alpha=\sigma\neq 0 \Rightarrow a^{f(\xi)}=\frac{1}{2}\left(-1+\sqrt{3}\tan\left(\frac{\sqrt{3}}{2}\alpha\xi\right)\right)$$
$$17)\text{when } \ \beta=\sigma=0 \Rightarrow a^{f(\xi)}=\alpha\xi$$
$$18)\text{when } \ \beta=\alpha=0 \Rightarrow a^{f(\xi)}=(-\sigma\xi)^{-1}$$
$$19)\text{when } \ \beta=0,\alpha=\sigma \Rightarrow a^{f(\xi)}=\tan(\sigma\xi)$$
$$20)\text{when } \ \sigma=0 \Rightarrow a^{f(\xi)}=\frac{e^{\beta\xi}-\alpha}{\beta}$$

where C is an arbitrary constant. For the considered method;

i) when $a^{f(\xi)}=\left(\frac{G'}{G}\right),\ \alpha=-\mu,\beta=-\lambda,\sigma=1$, the method same to $\left(\frac{G'}{G}\right)$ −expansion method.

ii) when $a^{f(\xi)}=e^{(-i\varphi(\xi))},\alpha=-\mu,\beta=-\lambda,\sigma=-1$, the method is similar to $e^{(-i\varphi(\xi))}$ expansion method.

iii) when $a^{f(\xi)}=\varphi,\alpha=b,\beta=0,\sigma=1$, the method is quite closed tanh-function method.

iv)when $a^{f(\xi)}=\varphi,\alpha=0,\beta=-\sigma=-\ln(a)$, the method is smilar to The Kudryashov and modified Kudryashov methods. The modified exp-(-$\Omega(\xi)$) function method produces more solutions than above methods [19].

3 Applications

3.1 Application of the extended modified exp(-$\Omega(\xi)$) function method to the Burger-Fisher equation

In this subsection, we are going to apply the mentioned method to obtain various exact solutions of the Burger-Fisher equation, first of all, we implement the travelling wave transformation

$$\begin{aligned} &u(x,t)=u(\xi),\\ &\xi=kx+ct, \end{aligned} \tag{3.7}$$

where k is wave number and c is wave speed. Then we can transform the Eq.(1.1) into ordinary differential equation in the form

$$-k^2u'' + cu\prime - \mu ku\ u' - u(1-u) = 0 \tag{3.8}$$

when we use homogeneous balance between u'' with $u\ u'$, we get $2+N = 2N+1 \Rightarrow N = 1$.Therefore, the solution acquired from step 2 is

$$u(\xi) = a_0 + a_1 a^{f(\xi)}. \tag{3.9}$$

If the solution (3.9) is taken place in transformed equation (3.8), we acquire a new algebraic equation in terms of $a_0, a_1, a^{f(\xi)}, \alpha, \beta$ and σ. Next, arranging all terms of the algebraic equation with the same order of $a^{f(\xi)}$ and equating each coefficients to zero, following algebraic equation system is acquired;

$$\begin{aligned}
\left(a^{f(\xi)}\right)^0 &= -a_0 + a_0^2 + c\alpha a_1 - k^2\alpha\beta a_1 - k\alpha\mu a_0 a_1 \\
\left(a^{f(\xi)}\right)^1 &= -a_1 + c\beta a_1 - k^2\beta^2 a_1 - 2k^2\alpha\sigma a_1 + 2a_0 a_1 - k\beta\mu a_0 a_1 - k\alpha\mu a_1^2 \\
\left(a^{f(\xi)}\right)^2 &= c\sigma a_1 - 3k^2\beta\sigma a_1 - k\mu\sigma a_0 a_1 + a_1^2 - k\beta\mu a_1^2 \\
\left(a^{f(\xi)}\right)^3 &= -2k^2\sigma^2 a_1 - k\mu\sigma a_1^2.
\end{aligned}$$

solving the system with any computer program such as Mathematica, Mapple, Matlab, etc., we get

$$a_0 = \frac{-2k\beta+\mu}{2\mu}, a_1 = \frac{-2k\sigma}{\mu}, \alpha = \frac{4k^2\beta^2-\mu^2}{16k^2\sigma}, c = \frac{k(4+\mu^2)}{2\mu}.$$

In this study, we chose Mathematica software. Thus, exact travelling wave solution of Burger-Fisher equation given in Eq.(1.1) be formed as

$$\begin{aligned}
u(\xi) &= a_0 + a_1 a^{f(\xi)} \\
u(\xi) &= \frac{-2k\beta+\mu}{2\mu} - \frac{2k\sigma}{\mu} a^{f(\xi)}
\end{aligned}$$

Thus, here are different selected solutions;

$$\begin{array}{ll}
\text{when } \beta^2 - 4\alpha\sigma < 0 \& \sigma \neq 0 & u_1(x,t) = \frac{1}{2}\left(\tanh\left(\frac{1}{8}\left(\left(\mu^2+4\right)t + 2\mu x\right)\right) + 1\right) \\
\text{when } \beta^2 - 4\alpha\sigma > 0 \& \sigma \neq 0 & u_2(x,t) = \frac{1}{2}\left(\coth\left(\frac{1}{8}\left(\left(\mu^2+4\right)t + 2\mu x\right)\right) + 1\right) \\
\text{when } \beta = 0 \ \& \ \alpha = -\sigma & u_3(x,t) = 1 - \frac{1}{e^{\frac{\mu^2 t}{4}+t+\frac{\mu x}{2}}+1} \\
\text{when } \ \beta^2 = \alpha\sigma & u_4(x,t) = \frac{1}{e^{-t}+1} \\
\text{when } \beta = p, \sigma = 2p, \alpha = 0 & u_5(x,t) = \frac{2}{e^{\frac{\mu^2 t}{4}+t+\frac{\mu x}{2}}-2} + 1 \\
\text{when } \ \alpha = 0 & u_6(x,t) = \frac{\sigma}{e^{\frac{\mu^2 t}{4}+t+\frac{\mu x}{2}}-\sigma} + 1 \\
\text{when } \ \sigma = 0 & u_7(x,t) = \frac{a_1 e^{\frac{t}{\mu^2}+t+\frac{x}{\mu}}}{\beta}
\end{array} \tag{3.10}$$

when one looked at the literature, solutions u_1 and u_2 can be encountered in Refs [20, 24, 30]. However, the rest of the solutions are newly obtained. Graphical simulation of some of the selected solutions are presented in Figure 1 and 2 for selected values of $\alpha, \beta, \sigma, \mu$ and k parameters.

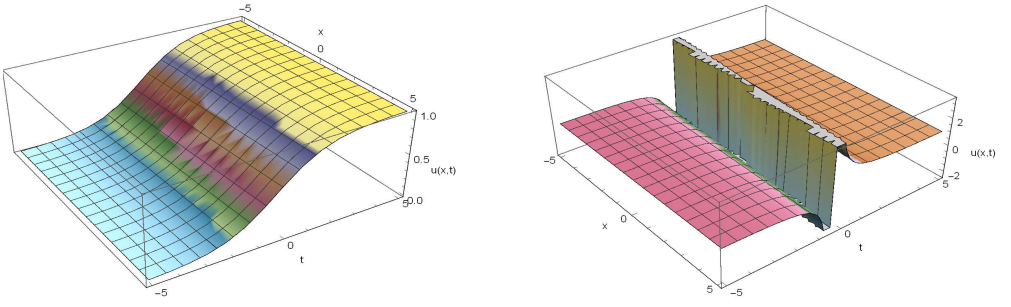


FIGURE 1. The exact solutions u_1 and u_2 of Burger fisher equation for selected $\beta = 2, \sigma = -0.2, k = 0.5$ and $\mu = 0.1$ values

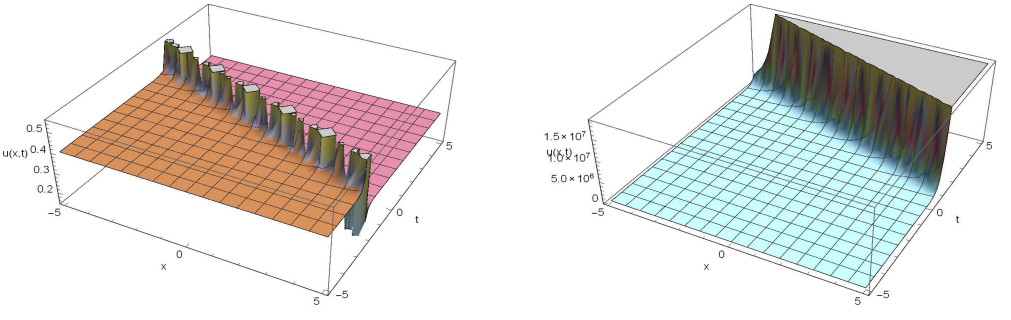


FIGURE 2. The exact solutions u_5 and u_7 of Burger fisher equation for selected $a_1 = 0.2, \mu = 0.5, p = 0.2$ and $\beta = 0.3$ values

3.2 Application of the modified exp-(-$\Omega(\xi)$) function method to the Phi-Four equation

Consider The Phi-Four equation given in Eq.(1.2). Using the transform $u(x,t) = u(\xi)$, $\xi = kx + ct$, we get the converted equation in the following form as

$$\left(c^2 - k^2\right) u'' + m^2 u + \lambda^2 u^3 = 0. \tag{3.11}$$

The homogeneous balance between u'' with u^3, yields $N = 1$. When solution $u(\xi) = a_0 + a_1 a^{f(\xi)}$ substitute into Eq.(3.11) and the procedure of considered method is applied, we get

$$\begin{aligned}
\left(a^{f(\xi)}\right)^0 &= m^2 a_0 + \lambda a_0^3 + c^2 \alpha\beta a_1 - k^2 \alpha\beta a_1 \\
\left(a^{f(\xi)}\right)^1 &= m^2 a_1 + c^2\beta^2 a_1 - k^2\beta^2 a_1 + 2c^2\alpha\sigma a_1 - 2k^2\alpha\sigma a_1 + 3\lambda a_0^2 a_1 \\
\left(a^{f(\xi)}\right)^2 &= 3c^2\beta\sigma a_1 - 3k^2\sigma\beta a_1 + 3\lambda a_0 a_1^2 \\
\left(a^{f(\xi)}\right)^3 &= 2c^2\sigma^2 a_1 - 2k^2\sigma^2 a_1 + \lambda a_1^3
\end{aligned}$$

solving of the above system yields

$$a_0 = \mp\frac{\sqrt{-c^2 + k^2}\beta}{\sqrt{2\lambda}}, a_1 = \mp\frac{\sqrt{2}\sqrt{-c^2 + k^2}\sigma}{\sqrt{\lambda}}, \alpha = \frac{2m^2 - c^2\beta^2 + k^2\beta^2}{4\left(-c^2 + k^2\right)\sigma}$$

Thus, we can derive many solutions with same procedure. Selected different solutions and their graphical simulations presented as following;

when $\beta^2-4\alpha\sigma<0\&\sigma\neq 0$ $\quad u_1(x,t)=-\dfrac{m\sqrt{k^2-c^2}\tanh\left(\frac{m(ct+kx)}{\sqrt{2c^2-2k^2}}\right)}{\sqrt{\lambda}\sqrt{(c-k)(c+k)}}$

when $\beta^2-4\alpha\sigma>0\&\sigma\neq 0$ $\quad u_2(x,t)=-\dfrac{m\sqrt{k^2-c^2}\coth\left(\frac{m(ct+kx)}{\sqrt{2c^2-2k^2}}\right)}{\sqrt{\lambda}\sqrt{(c-k)(c+k)}}$

when $\beta^2+4\alpha^2>0\&\sigma\neq 0\&\sigma=-\alpha$ $\quad u_3(x,t)=\dfrac{m\sqrt{4\alpha^2+\beta^2}\tanh\left(\frac{1}{2}\left(t\sqrt{k^2(4\alpha^2+\beta^2)+2m^2}+kx\sqrt{4\alpha^2+\beta^2}\right)\right)}{\sqrt{\lambda(-(4\alpha^2+\beta^2))}}$

or $\quad u_4(x,t)=\dfrac{\lambda m\left(4\alpha^2+\beta^2\right)^{3/2}\coth\left(\frac{1}{2}\left(t\sqrt{k^2(4\alpha^2+\beta^2)+2m^2}+kx\sqrt{4\alpha^2+\beta^2}\right)\right)}{(\lambda(-(4\alpha^2+\beta^2)))^{3/2}}$

when $\beta^2-4\alpha^2<0\ \&\ \ \sigma=\alpha$ $\quad u_5(x,t)=\dfrac{m\sqrt{4\alpha^2-\beta^2}\tan\left(\frac{1}{2}\sqrt{4\alpha^2-\beta^2}\left(kx-\frac{t\sqrt{k^2(\beta^2-4\alpha^2)+2m^2}}{\sqrt{\beta^2-4\alpha^2}}\right)\right)}{\sqrt{\lambda(4\alpha^2-\beta^2)}}$

or $\quad u_6(x,t)=\dfrac{m\sqrt{4\alpha^2-\beta^2}\cot\left(\frac{1}{2}\sqrt{4\alpha^2-\beta^2}\left(kx-\frac{t\sqrt{k^2(\beta^2-4\alpha^2)+2m^2}}{\sqrt{\beta^2-4\alpha^2}}\right)\right)}{\sqrt{\lambda(4\alpha^2-\beta^2)}}$

when $\alpha\ \sigma<0\ \&\ \sigma\neq 0\ \&\ \beta=0$ $\quad u_7(x,t)=\dfrac{m\sqrt{\sigma^2(k^2-c^2)}\tanh\left(\frac{m\sqrt{\sigma}(ct+kx)}{\sqrt{2c^2\sigma-2k^2\sigma}}\right)}{\sqrt{\lambda}\sqrt{\sigma}\sqrt{\sigma(c-k)(c+k)}}$

or $\quad u_8(x,t)=-\dfrac{m\sqrt{\sigma^2(k^2-c^2)}\coth\left(\frac{m\sqrt{\sigma}(ct+kx)}{\sqrt{2c^2\sigma-2k^2\sigma}}\right)}{\sqrt{\lambda}\sqrt{\sigma}\sqrt{\sigma(c-k)(c+k)}}$

when $\beta=0\ \&\ \alpha=-\sigma$ $\quad u_9(x,t)=-\dfrac{im\tanh\left(t\sqrt{k^2\sigma^2+\frac{m^2}{2}}+k\sigma x\right)}{\sqrt{\lambda}}$

when $\alpha=\sigma=0$ $\quad u_{10}(x,t)=-\dfrac{im}{\sqrt{\lambda}}$

when $\beta=p,\sigma=2p,\alpha=0$ $\quad u_{11}(x,t)=\dfrac{im\left(\frac{4}{e^{t\left(-\sqrt{k^2p^2+2m^2}\right)-kpx}-2}+1\right)}{\sqrt{\lambda}}$

when $\alpha=0$ $\quad u_{12}(x,t)=-\dfrac{im\left(\frac{2}{\sigma e^{t\sqrt{\beta^2k^2+2m^2}+\beta kx}-1}+1\right)}{\sqrt{\lambda}}$

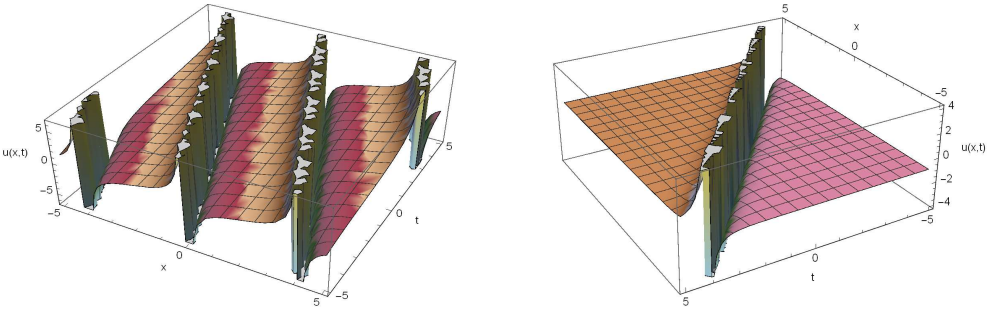


FIGURE 3. The exact solutions u_1 and u_4 of phi-four equation for selected $\beta=0.2,\sigma=\mp 2,k=0.5,\lambda=\pm 1$ and $\mu=1$ values,respectively

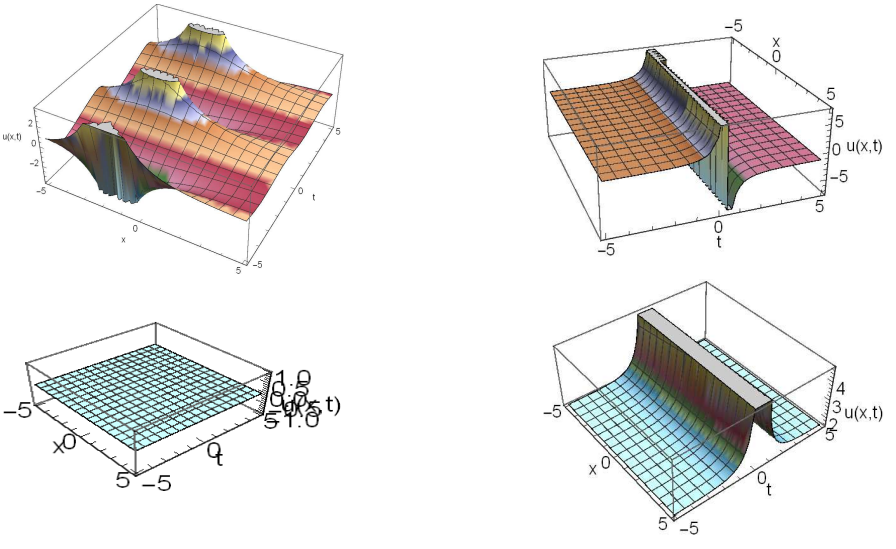


FIGURE 4. The solutions u_{11} and u_{12} of the Phi-four equation for selected values $k = 0.2, p = 2$, $\lambda = 0.4, \sigma = 0.5$ and $\mu = i$ and $\mu = 1$, respectively

The exact solutions of the Phi-four equation have been considered in many papers and tan, tanh, sec and sec h types solutions obtained [3, 4, 6, 7]. Especially, Demiray and Bulut [10] applied modified exp(-$\Omega(\xi)$) function method to the mentioned equation successfully and present many new exact solutions. In additionally, in this study, the generalized exp(-$\Omega(\xi)$) function method have been add newly solutions. As far as the authors know, the solutions of Burger Fisher and Phi Four equations that we found in this study are newly obtained and are not indicated before.

4 Conclusion

In summary, with a view to further extension of exp(-$\Omega(\xi)$) function method, we have applied the extended modified exp(-$\Omega(\xi)$) function method (MEFM) to two important equations in mathematics and physics which are Burger-Fisher equation and the Phi-Four equation. Using the considered method causes to obtain enriched new exact solutions, successfully. We have reached trigonometric,hyperbolic, rational and exponential solutions of the equations. The newly obtained results show that, the method is effective, concise and powerful and it can be used to handle a wide range of evolution equations.

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