On moduli of smoothness of functions in Orlicz spaces

Sadulla Z. Jafarov

Department of Mathematics and Science Education, Faculty of Education, Muş Alparslan University, 49250, Muş, Turkey;
Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, 9 B. Vahabzadeh str., AZ 1141, Baku, Azerbaijan

E-mail: s.jafarov@alparslan.edu.tr

Abstract

In this work the estimate about modulus of smoothness of the derivatives of the functions is obtained in Orlicz spaces. The relations between the modulus of smoothness of the functions with nth partial and Vallée-Poussin sums of the Fourier series in Orlicz spaces are studied.

2010 Mathematics Subject Classification. **41A17**. 41A25, 41A27, 42A10, 42A50, 46E30, 46E35, 26A33
Keywords. Orlicz space, modulus of smoothness, Vallée-Poussin sum, polynomial approximation.

1 Introduction, some auxiliary results and main results

Let $M(u)$ be a continuous increasing convex function on $[0,\infty)$ such that $M(u)/u \to 0$ if $u \to 0$, and $M(u)/u \to \infty$ if $u \to \infty$. We denote by N *the complementary of* M *in Young's sense,* i.e. $N(u) = \max\{uv - M(v) : v \geq 0\}$ if $u \geq 0$. We will say that M satisfies the Δ_2-condition if $M(2u) \leq cM(u)$ for any $u \geq u_0$ with some constant c independent of u.

Let $\mathbb{T}$ denote the interval $[-\pi,\pi]$, $\mathbb{C}$ the complex plane, and $L_p(\mathbb{T})$, $1 \leq p \leq \infty$, the Lebesgue space of measurable complex-valued functions on $\mathbb{T}$.

For a given Young function M, let $\widetilde{L}_M(\mathbb{T})$ denote the set of all Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{C}$ for which

$$\int_{\mathbb{T}} M\left(|f(x)|\right) dx < \infty.$$

Let N be the complementary Young function of M. It is well-known [21, p.69], [28, pp.52-68] that the linear span of $\widetilde{L}_M(\mathbb{T})$ equipped with the *Orlicz norm*

$$\|f\|_{L_M(\mathbb{T})} := \sup\left\{\int_{\mathbb{T}} |f(x)g(x)|\, dx : g \in \widetilde{L}_N(\mathbb{T}), \int_{\mathbb{T}} N\left(|g(x)|\right) dx \leq 1\right\},$$

or with the *Luxembourg norm*

$$\|f\|^*_{L_M(\mathbb{T})} := \inf\left\{k > 0 : \int_{\mathbb{T}} M\left(\frac{|f(x)}{k}\right) dx \leq 1\right\}$$

becomes a Banach space. This space is denoted by $L_M(\mathbb{T})$ and it is called as an *Orlicz space* [21, p.26]. The Luxembourg norm is equivalent to the Orlicz norm as

$$\|f\|^*_{L_M(\mathbb{T})} \leq \|f\|_{L_M(\mathbb{T})} \leq 2\,\|f\|_{L_M(\mathbb{T})}, \qquad f \in L_M(\mathbb{T})$$

Tbilisi Mathematical Journal 12(3) (2019), pp. 121–129.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 11 December 2018.
Accepted for publication: 20 July 2019.

holds true [21, p.80].

If we choose $M(u)=u^p/p \ \ (1<p<\infty)$ then the complementary function is $N(u)=u^q/q$ with $1/p+1/q=1$ and we have the relation

$$p^{-1/p}\,|u|_{L_p(\mathbb{T})}=\|u\|^*_{L_M(\mathbb{T})}\le\|u\|_{L_M(\mathbb{T})}\le q^{1/q}\,\|u\|_{L_p(\mathbb{T})}\,,$$

where $\|u\|_{L_p(\mathbb{T})}=\left(\int\limits_{\mathbb{T}}|u(x)|^p\,dx\right)^{1/p}$ denotes the usual norm of the $L_p(\mathbb{T})-$space.

If N is complementary to M in Young's sense and $f\in L_M(\mathbb{T})$, $g\in L_N(T)$ then the so-called strong Hölder inequalities [21, p.80]

$$\int\limits_{\mathbb{T}}|f(x)g(x)|\,dx\le\|f\|_{L_M(\mathbb{T})}\,\|g\|^*_{L_N(\mathbb{T})}\,,$$

$$\int\limits_{\mathbb{T}}|f(x)g(x)|\,dx\le\|f\|^*_{L_M(\mathbb{T})}\,\|g\|_{L_N(\mathbb{T})}\,.$$

are satisfied.

The Orlicz space $L_M(T)$ is *reflexive* if and only if the $N-$function M and its complementary function N both satisfy the Δ_2-condition [28, p.113].

Let $M^{-1}:[0,\infty)\to[0,\infty)$ be the inverse function of the $N-$function M. The *lower* and *upper indices* α_M,β_M

$$\alpha_M:=\lim_{t\to+\infty}-\frac{\log h(t)}{\log t},\ \beta_M:=\lim_{t\to o^+}-\frac{\log h(t)}{\log t}$$

of the function

$$h:(0,\infty)\to(0,\infty],\quad h(t):=\lim_{y\to\infty}\sup\frac{M^{-1}(y)}{M^{-1}(ty)},\quad t>0$$

first considered by Matuszewska and Orlicz [25], are called the *Boyd indices* of the Orlicz spaces $L_M(\mathbb{T})$.

It is known that the indices α_M and β_M satisfy $0\le\alpha_M\le\beta_M\le1,\alpha_N+\beta_M=1$, $\alpha_M+\beta_N=1$ and the space $L_M(\mathbb{T})$ is reflexive if and only if $0<\alpha_M\le\beta_M<1$. The detailed information about the Boyd indices can be found in [7], [24], [25] and [28].

Let $L_M(\mathbb{T})$ be an Orlicz space. Suppose that x,h are real, and let us take into account the following series

$$\Delta_h^\alpha f(x)=\sum_{k=0}^{\infty}(-1)^k\left(\begin{array}{c}\alpha\\k\end{array}\right)f\left(x+(\alpha-k)h\right),\quad\alpha\in Z,\quad f\in L_M(\mathbb{T}).$$

Then, by [3] the last series converges absolutely almost everywhere (a. e). on $\mathbb{T}$. Hence the operator Δ_h^α by [3] is bounded in the space $L_M(\mathbb{T})$. Namely,

$$\Delta_h^\alpha f(x)=\sum_{k=0}^{\infty}(-1)^k\left(\begin{array}{c}\alpha\\k\end{array}\right)f\left(x+(\alpha-k)h\right)=\sum_{k=0}^{\alpha}(-1)^{\alpha-k}\left(\begin{array}{c}\alpha\\k\end{array}\right)f\left(x+kh\right).$$

We can define $\alpha-th\ (\alpha\in\mathbb{N})$ *modulus of smoothness of a function* $f\in L_M(\mathbb{T})$ as

$$\omega_\alpha(f,\delta)_M := \sup_{0<\ h_i\leq\delta} \|\Delta_h^\alpha f\|_{L_M(T)}\,.$$

The modulus of smoothness $\omega_\alpha(f,\delta)_M$, $\alpha\in\mathbb{N}$, is a non-decreasing, non-negative function of $\delta\geq 0$, and

$$\begin{aligned}\omega_\alpha(f+g,\cdot)_M &\leq \omega_\alpha(f,\cdot)_M+\omega_\alpha(g,\cdot)_M,\\ \lim_{\delta\to 0}\omega_\alpha(f,\delta)_M &= 0\end{aligned}$$

for $f,g\in L_M(\mathbb{T})$.

Let

$$\frac{a_0}{2}+\sum_{k=1}^{\infty} A_k(f,x),\ A_k(f,x):=a_k(f)\cos kx+b_k(f)\sin kx \tag{1.1}$$

be the Fourier series of the function $f\in L^1(\mathbb{T})$, where $a_k(f)$ and $b_k(f)$ are Fourier coefficients of the function f. The *$n-th$ partial sums* and *de la Vallé e-Poussin sums* of the series (1.1) are defined, respectively, as:

$$\begin{aligned}S_n(f,x) &= \frac{a_0}{2}+\sum_{k=1}^{n} A_k(f,x),\\ V_n(f,x) &:= \frac{1}{n}\sum_{\nu=n}^{2n-1} S_\nu(f,x).\end{aligned}$$

We denote by $E_n(f)_M$ the best approximation of $f\in L_M(\mathbb{T})$ by trigonometric polynomials of degree not exceeding n, i.e.,

$$E_n(f)_M=\inf\left\{\|f-T_n\|_{L_M(\mathbb{T})}: T_n\in\Pi_n\right\},$$

where Π_n denotes the class of trigonometric polynomials of degree at most n.

We use the constants $c,c_1,c_2,...$ (in general, different in different relations) which depend on only the quantities that are not important for the questions of interest.

In the present paper, in under the condition that $M\left(u^{\frac{1}{s}}\right)$ is a convex function for some $1<s\leq 2$, we obtain the estimation about modulus of smoothness of the derivatives of the functions in the Orlicz spaces. The relations between the modulus of smoothness of the functions with *nth* partial and Vallée-Poussin sums of the Fourier seriesin in Orlicz spaces are also studied. .

Similar problems in different spaces have been investigated by several authors (see, for example, [1]- [6], [8]- [20], [22], [23], [26], [27], [29] -[36]).

Main results in the present work are the following theorems:

Theorem 1.1. Let $L_M(\mathbb{T})$ be a reflexive Orlicz space such that $M\left(u^{\frac{1}{s}}\right)$ is a convex function for some $0<s\leq 2$. If $f\in L_M(\mathbb{T})$ then *for* α,r integers satisfying $r<\alpha$ we obtain

$$\omega_{\alpha-r}\left(f^{(r)},t\right)_M \leq c_1\left\{\int_0^t \frac{\omega_\alpha\left(f,u\right)_M^s}{u^{sr+1}}du\right\}^{\frac{1}{s}},$$

where the constant c_1 depends on M, α and r.

Theorem 1.2. If $f \in L_M\left(\mathbb{T}\right)$ we have

1.

$$\begin{aligned} c_2\omega_\alpha(f,\frac{1}{n})_M &\leq \left(n^{-\alpha}\left\|V_n^{(\alpha)}(f,\cdot)\right\|_{L_M(\mathbb{T})} + \|f-V_n(f,\cdot)\|_{L_M(\mathbb{T})}\right) \\ &\leq c_3\omega_\alpha(f,\frac{1}{n})_M\ , \end{aligned} \tag{1.2}$$

where the constants c_2 and c_3 are dependent on M and α.

2.

$$\begin{aligned} c_4\omega_\alpha(f,\frac{1}{n})_M &\leq \left(n^{-\alpha}\left\|S_n^{(\alpha)}(f,\cdot)\right\|_{L_M(\mathbb{T})} + \|f-S_n(f,\cdot)\|_{L_M(\mathbb{T})}\right) \\ &\leq c_5\omega_\alpha(f,\frac{1}{n})_M \end{aligned} \tag{1.3}$$

where the constants c_4 and c_5 are dependent on M and α.

2 Proofs of the main results

Proof of Theorem 1.1. The function $\omega_m(F,t)_{L_M(\mathbb{T})}$ non-decreasing and according to reference [3] the following inequality holds:

$$\omega_\alpha(F,2t)_M \leq c_6\omega_\alpha(F,\ t)_M \tag{2.1}$$

It is sufficient to prove theorem for $t=2^{-n}$. If we consider (2.1), we obtain

$$\left\{\int_0^{2^{-n}} \frac{\omega_\alpha\left(f,\ u\right)_M^s}{u^{sr+1}}du\right\}^{1/s} \asymp \left\{\sum_{\nu=n}^{\infty} 2^{\nu sr}\omega_\alpha(f,\ 2^{-\nu})_M^s\right\}^{1/s}.$$

Therefore for all n it is sufficient to prove the following inequality:

$$\omega_{\alpha-r}(f^{(r)}\ ,\ 2^{-n})_M \leq \left\{\sum_{\nu=n}^{\infty} 2^{\nu sr}\omega_\alpha(f,\ 2^{-\nu})_M^s\right\}^{1/s}. \tag{2.2}$$

By [3] for any trigonometric polynomial Q_n of degree cn and $F \in L_M(\mathbb{T})$ we obtain

$$\omega_\alpha(F\ ,\ 1/n)_M \leq c_7\left(\|F-Q_n\|_{L_M(\mathbb{T})} + n^{-\alpha}\left\|Q_n^{(\alpha)}\right\|_{L_M(\mathbb{T}}\right). \tag{2.3}$$

Therefore we aim to find Q_{2^n} of degree $c2^n$ such that both

$$\left\|f^{(r)} - Q_{2^n}\right\|_{L_M(\mathbb{T})} \text{ and } 2^{-n(\alpha-r)} \left\|Q_{2^n}^{(\alpha-r)}\right\|_{L_M(\mathbb{T})}$$

are bounded by the right-hand side of inequality (2.2). Let $T_n \in \Pi_n$ $(n = 0, 1, 2, ...)$ be the polynomial of best approximation to f. It is known that [1] the set of trigonometric polynomials is dense in $L_M(\mathbb{T})$. Then we obtain $\|f - T_{2^\nu}\|_{L_M(\mathbb{T})} \to 0$, $\nu \to \infty$.

Let $f \in L_M(\mathbb{T})$ has the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) = \sum_{k=0}^{\infty} A_k(f).$$

We define trigonometric polynomial $\nu_N f$ as

$$\nu_N f \quad = \sum_{k=0}^{\infty} \nu(\frac{k}{N}) A_k(f),$$

where $\nu \in C^\infty\ [0, \infty)$, $\nu(x) = 1$ for $x \leq 1$ and $\nu(x) = 0$ for $x \geq 1$. Note that trigonometric polynomial $\nu_N f$ has the following properties:

I) $\nu_N f$ is a trigonometric polynomial of degree smaller than N;

II) If g is a trigonometric polynomial of degree $[N/2]$, then $\nu_N g = g$;

III) $\|\nu_N f\|_{L_M(\mathbb{T})} \leq c \|f\|_{L_M(\mathbb{T})}$.

Henceto referemnce[3] we have

$$\|\nu_N f - f\|_{L_M(\mathbb{T})} \leq c_8 E_{N/2}(f)_{L_M(\mathbb{T})},$$

where $E_k(f)_{L_M(\mathbb{T})}$ is the best approximation of $f \in L_M(\mathbb{T})$ trigonometric polynomials of degree not exceeding k. We now choose the Q_n of (2.3) for $F = f^{(r)}$ to be $(\nu_n f)^{(r)}$. It is cleary that $\|f - \nu_n f\|_{L_M(\mathbb{T})} = o(1)$ as $n \to \infty$.

The identity

$$\nu_{2^k} f - \nu_{2^n} f = \sum_{m=n}^{k-1} (\nu_{2^{m+1}} f - \nu_{2^m} f) \equiv \sum_{m=n}^{k-1} \gamma_m f.$$

holds. Then

$$(\nu_{2^k} f)^{(r)} - (\nu_{2^n} f)^{(r)} = \sum_{m=1}^{k-1} (\gamma_m f)^{(r)}.$$

Using the Littlewood- Paley inequality for the Orlicz spaces $L_M(\mathbb{T})$ in [3] we have

$$\begin{aligned} & c_9 \left\|(\nu_{2^k} f)^{(r)} - (\nu_{2^n} f)^{(r)}\right\|_{L_M(\mathbb{T})} \\ \leq & \left\| \left(\sum_{m=n}^{k-1} \left\{ (\gamma_m f)^{(r)} \right\}^2 \right)^{1/2} \right\|_{L_M(\mathbb{T})} \\ \leq & c_{10} \left\|(\nu_{2^k} f)^{(r)} - (\nu_{2^n} f)^{(r)}\right\|_{L_M(\mathbb{T})}. \end{aligned} \tag{2.4}$$

We get by [3]

$$\left\| \left(\sum_{m=n}^{k-1} \left\{ (\gamma_m f)^{(r)} \right\}^2 \right)^{1/2} \right\|_{L_M(\mathbb{T})} \leq \left(\sum_{m=n}^{k-1} \left\| (\gamma_m f)^{(r)} \right\|_{L_M(\mathbb{T})}^s \right)^{1/s}, \tag{2.5}$$

Note that $\nu_n f$ is the near best approximation to f in $L_M(\mathbb{T})$. Then according to [3], we reach the following equivalence

$$\omega_\alpha(f, 1/n)_M \asymp \|f - \nu_n f\|_{L_M(\mathbb{T})} + n^{-\alpha} \left\| (\nu_n f)^{(\alpha)} \right\|_{L_M(\mathbb{T})}. \tag{2.6}$$

From (2.4)- (2.6) and the Bernstein inequality for the Orlicz spaces [3, Lemma 2.3], we conclude that

$$\begin{aligned} & \left\| (\nu_{2^k} f)^{(r)} - (\nu_{2^n} f)^{(r)} \right\|_{L_M(\mathbb{T})} \\ \leq & \quad c_{11} \left(\sum_{m=n}^{k-1} 2^{mrs} \|(\gamma_m f)\|_{L_M(\mathbb{T})}^s \right)^{1/s} \\ \leq & \quad c_{12} \left(\sum_{m=n}^{k-1} 2^{mrs} \omega_\alpha(f, 2^{-m})_M^s \right)^{1/s}, \end{aligned}$$

where c_1 independent of m, k and f.

Consideration of $Q_{2^n} = \nu_{2^n} f$ and (2.6) gives us

$$\begin{aligned} & 2^{-n(\alpha-r)} \left\| ((\nu_{2^n} f)^{(r)})^{(\alpha-r)} \right\|_{L_M(\mathbb{T})} \\ = & \quad 2^{-n(\alpha-r)} \left\| (\nu_{2^n} f)^{(\alpha)} \right\|_{L_M(\mathbb{T})} \\ \leq & \quad 2^{nr} \omega_\alpha(f, 2^{-n})_M \leq c_{13} \left(\sum_{m=n}^{\infty} 2^{mrs} \omega_\alpha(f, 2^{-m})_M^s \right)^{1/s}. \end{aligned}$$

The proof of Theorem 1.1 is completed. ∎

Proof of Theorem 1.2. By reference [3] the following inequality holds:

$$\omega_\alpha(T_n, \frac{1}{n})_M \leq c_{14}(\alpha, M) n^{-\alpha} \left\| T_n^{(\alpha)} \right\|_{L_M(\mathbb{T})}, \tag{2.7}$$

where $T_n \in \Pi_n$. By using the properties of modulus of smoothness $\omega_\alpha(f, \cdot)_M$ and (2.7), we obtain

$$\begin{aligned} \omega_\alpha(f, \frac{1}{n})_M & \leq \left(\omega_\alpha(f - T_n, \frac{1}{n})_M + \omega_\alpha(T_n, \frac{1}{n})_M \right) \\ & \leq c_{15}(M, \alpha) \left(\|f - T_n\|_{L_M(\mathbb{T})} + n^{-\alpha} \left\| T_n^{(\alpha)} \right\|_{L_M(\mathbb{T})} \right). \end{aligned}$$

Now we estimate the modulus of smoothness $\omega_\alpha(f,\cdot)_M$ from below. By [3] the inequalities

$$E_n(f)_M \leq c_{16}\,(M,\alpha)\,\omega_\alpha(f,\frac{\pi}{n})_M\ , \tag{2.8}$$

$$n^{-\alpha}\left\|T_n^{(\alpha)}\right\|_{L_M(\mathbb{T})} \leq c_{17}\,(M,\alpha)\,\omega_\alpha(f,\frac{1}{n})_M. \tag{2.9}$$

hold. Let $V_v(f,x)$ be the de la Vallée-Poussin sums of the series (1.1) and let $T_n^* \in \Pi_n$ be the polynomial of best approximation to f in $L_M(\mathbb{T})$, that is $\|f-T_n^*\|_{L_M(\mathbb{T})} = E_n(f)_M$. Then we obtain

$$\begin{aligned}\|f-V_n(f,\cdot)\|_{L_M(\mathbb{T})} &\leq \|f-T_n^*\|_{L_M(\mathbb{T})} + \|T_n^*-V_n(f,\cdot)\|_{L_M(\mathbb{T})}\\ &\leq c_{18}(M)E_n(f)_M + \|V_n(T_n^*-f,\cdot)\|_{L_M(\mathbb{T})}\\ &\leq c_{19}(M)E_n(f)_M.\end{aligned} \tag{2.10}$$

Now combining (2.8), (2.9) and (2.10) we get

$$\begin{aligned}& n^{-2\alpha}\left\|V_n^{(2\alpha)}(f,\cdot)\right\|_{L_M(\mathbb{T})} + \|f-V_n(f,\cdot)\|_{L_M(\mathbb{T})}\\ \leq\ & c_{20}\,(M,\alpha)\left(\omega_\alpha(V_n,\frac{1}{n})_M + E_n(f)_M\right)\\ \leq\ & c_{21}\,(M,\alpha)\left(\omega_\alpha(f,\frac{1}{n})_M + \omega_\alpha(f-V_n,\frac{1}{n})_M + E_n(f)_M\right)\\ \leq\ & c_{22}\,(M,\alpha)\,\omega_\alpha(f,\frac{1}{n})_M,\end{aligned}$$

which completes the estimation (1.2) of Theorem 1.2.

According to [3], [4] there exists a constant $c_{23}(M)$ such that

$$\|f-S_n(f,\cdot)\|_{L_M(\mathbb{T})} \leq c_{23}(M)E_n(f)_M. \tag{2.11}$$

Using the inequality (2.11) and the scheme of proof of the estimation (1.2) we obtain the estimation (1.3).

Theorem 1.2 is proved. ∎

Acknowledgements.

This work was supported by Scientific Research Projects Cordination Unit of Muş Alparslan University. Project No: BAP-18-EMF-4901-07.

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