An efficient method for solving nonlinear time-fractional wave-like equations with variable coefficients

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Abstract

In this paper, fractional residual power series method (FRPSM) is effectively applied for finding the approximate analytical solutions of general nonlinear time-fractional wave-like equations with variable coefficients. This method based on constructing series solutions in a form of rapidly convergent series with easily computable components and without need of linearization, discretization, perturbation or unrealistic assumptions. Numerical results are given and then they are compared with the exact solutions both numerically and graphically. By numerical examples, it is shown that the FRPSM is very simple, efficient and convenient for solving different forms of nonlinear fractional partial differential equations.

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1 Introduction

In recent years, the nonlinear fractional partial differential equations (NFPDEs) have achieved great attention among researchers due to its wide range of applications in many fields of physics, mechanics, electrochemistry, viscoelasticity, nonlinear control theory, image processing, nonlinear biological systems, astrophysics, and other fields of science and engineering [3, 9, 12, 13, 14, 20, 23]. So, there has been significant interest in developing methods for the solutions of NFPDEs. Exact solutions of NFPDEs are sometimes too complicated to be attained by conventional techniques due to the computational complexities of nonlinear parts involving them. Therefore, for the study of solution of NFPDEs there are variety of analytical and approximate methods found in literature. Among them most common methods are: Adomian decomposition method (ADM) [25], variational iteration method (VIM) [27], fractional difference method (FDM) [22], differential transform method (DTM) [6], homotopy perturbation method (HPM) [11].

The residual power series method (RPSM) was initially developed to compute the numerical solutions of the first-order and the second-order fuzzy differential equations [1]. This method provides a power series solution with rapid convergence. It has been advantageously implemented for the nonlinear fractional Korteweg–de Vries–Burgers equation [7], for the fractional foam drainage equation [4], for the time-fractional two-component evolutionary system of order two [5] and for other equations [2]. It has been proven that the RPSM is a convenient and effective method in its application.

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The main objective of the present paper is to use the fractional residual power series method to find the approximate analytical solutions for the genaral time-fractional wave-like equations with variable coefficients of the form

$$\begin{aligned} D_t^{\alpha} u &= \sum_{i,j=1}^{N} F_{1ij}(X,t,u)\frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(u_{x_i}, u_{x_j}) \\ &\quad + \sum_{i=1}^{N} G_{1i}(X,t,u)\frac{\partial^{p}}{\partial x_i^p} G_{2i}(u_{x_i}) + H(X,t,u) + S(X,t), \end{aligned} \tag{1.1}$$

with the initial conditions

$$u(X,0) = a_0(X), u_t(X,0) = a_1(X), \tag{1.2}$$

where D_t^{α} is the Caputo fractional derivative operator of order α, $1 < \alpha \leq 2$, $u = u(X,t)$ is an unknown function where $X = (x_1, x_2, ..., x_N) \in \mathbb{R}^N$ and $t > 0$, F_{1ij}, G_{1i} $i,j \in \{1,2,...,N\}$, are nonlinear functions of X, t and u, F_{2ij}, G_{2i} $i,j \in \{1,2,...,N\}$, are nonlinear functions of derivatives of u with respect to x_i and x_j $i,j \in \{1,2,...,N\}$, respectively. Also H, S are nonlinear functions and k, m, p are integers.

These types of equations are of considerable significance in various fields of applied sciences, mathematical physics, nonlinear hydrodynamics, engineering physics, biophysics, human movement sciences, astrophysics and plasma physics. These equations describe the evolution of erratic motions of small particles that are immersed in fluids, fluctuations of the intensity of laser light, velocity distributions of fluid particles in turbulent flows.

This article is organized as follows. In Section 2, we present some fundamental definitions of fractional calculus. In Section 3, we present the basic idea of the fractional power series (FPS). In Section 4, we introduce our results of fractional residual power series method (FRPSM) for general nonlinear time-fractional partial differential equation, in particular for the general nonlinear time-fractional wave-like equations (1.1) with the initial conditions (1.2) . In Section 5, we have proposed some examples in order to show the validity and effectiveness of this approach. Finally, we present our obtained results (Graphs and Tables), comparing them with their exact associated forms. These results were verified with Matlab (version 7.9.0.529 (R2009b)).

2 Basic definitions of fractional calculus theory

For the reader's convenience, we recall some concepts, definitions and basic results from fractional calculus theory, which are useful in the following discussion. These definitions and concepts can be found in [8, 18, 19, 21, 22, 24, 26], and references therein. Throughout this paper, the set of real numbers and the set of integer numbers are denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively.

Definition 2.1. A real function $u(X,t), X \in \mathbb{R}^N, N \in \mathbb{N}^*, t \in \mathbb{R}^+$, is considered to be in the space $C_{\mu}(\mathbb{R}^N \times \mathbb{R}^+)$, $\mu \in \mathbb{R}$, if there exists a real number $p > \mu$, so that $u(X,t) = t^p v(X,t)$, where $v \in C\left(\mathbb{R}^N \times \mathbb{R}^+\right)$, and it is said to be in the space C_{μ}^n if $u^{(n)} \in C_{\mu}(\mathbb{R}^N \times \mathbb{R}^+)$, $n \in \mathbb{N}$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of $u \in C_{\mu}(\mathbb{R}^N \times \mathbb{R}^+), \mu \geq -1$ is defined as follows

$$I_t^{\alpha} u(X,t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \int\limits_{t_0}^{t} (t-\xi)^{\alpha-1} u(X,\xi) d\xi, \alpha > 0, t > \xi > t_0 \geq 0, \\ u(X,t), \qquad\qquad \alpha = 0, \end{cases} \tag{2.1}$$

where

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \alpha > 0,$$

is the well-known gamma function.

Some properties of the operator I_t^α , which we will need here, are as follows.
For $u \in C_\mu(\mathbb{R}^N \times \mathbb{R}^+), \mu \geq -1, \alpha, \beta \geq 0, \gamma \geq -1$ and $t > t_0 \geq 0$, we have

$$I_t^\alpha I_t^\beta u(X,t) = I_t^{\alpha+\beta} u(X,t) = I_t^\beta I_t^\alpha u(X,t),$$

$$I_t^\alpha (t-t_0) = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} (t-t_0)^{\alpha+\gamma}.$$

Theorem 2.3. [10] Let $t_0 < T, \alpha > 0$ and $u \in C\left(\mathbb{R}^N \times [t_0, T]\right)$. Moreover, assume that v is Lebesgue integrable on $\mathbb{R}^N \times [t_0, T]$ and that v does not change its sign in $\mathbb{R}^N \times [t_0, T]$. Then, for almost every $t \in (t_0, T]$ there exists some $\tau \in (t_0, t) \subset (t_0, T)$ such that

$$I_t^\alpha (uv)(X,t) = u(\tau) I_t^\alpha v(X,t). \tag{2.2}$$

If additionally $\alpha \geq 1$ or $v \in C\left(\mathbb{R}^N \times [0, T]\right)$, then this result holds for every $t \in (0, T]$.

Definition 2.4. The Caputo time-fractional derivative operator of order $\alpha > 0$ of $u \in C_{-1}^n(\mathbb{R}^N \times \mathbb{R}^+), n \in \mathbb{N}$, is defined as follows

$$D_t^\alpha u(X,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^t (t-\xi)^{n-\alpha-1} u^{(n)}(X,\xi) d\xi, n-1 < \alpha < n, \\ u^{(n)}(X,t), \qquad\qquad \alpha = n. \end{cases} \tag{2.3}$$

For the Riemann-Liouville fractional integral and Caputo fractional derivative, we have the following relation

$$I_t^\alpha D_t^\alpha u(X,t) = u(X,t) - \sum_{k=0}^{n-1} u^{(k)}(X, t_0^+) \frac{(t-t_0)^k}{k!}, X \in \mathbb{R}^N, t > t_0 \geq 0. \tag{2.4}$$

Definition 2.5. The Mittag-Leffler function is defined as follows

$$E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha+1)}, \alpha \in \mathbb{C}, Re(\alpha) > 0. \tag{2.5}$$

A further generalization of (2.5) is given in the form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha+\beta)}, \alpha, \beta \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0. \tag{2.6}$$

For $\alpha = 1$, $E_\alpha(z)$ reduces to e^z.

3 Fractional power series (FPS)

In this section, we will introduce a general form of generalized Taylor's formula that contains the Caputo derivatives. For more details, see [2].

Definition 3.1. A power series of the form

$$\sum_{i=0}^{\infty}\sum_{j=0}^{m-1} C_{ij}(X)(t-t_0)^{i\alpha+j}, \tag{3.1}$$

where $0 \leq m-1 < \alpha \leq m$ and $t \geq t_0$, is called a multiple fractional power series (MFPS) about $t = t_0$, t is a variable and $C_{ij}\ (X)$ are functions of $X \in \mathbb{R}^N$, called the coefficients of the series.

Lemma 3.2. Suppose that $u \in C\left(\mathbb{R}^N \times [t_0, t_0+R)\right)$ and $D_t^{i\alpha}u \in C\left(\mathbb{R}^N \times (t_0, t_0+R)\right)$ for $i \in \{0,1,2,...,n+1\}$, where $0 \leq m-1 < \alpha \leq m$. Then we have

$$\begin{aligned} I_t^{(n+1)\alpha} D_t^{(n+1)\alpha} u(X,t) &= \frac{D_t^{(n+1)\alpha}u(X,\tau)}{\Gamma((n+1)\,\alpha+1)}(t-t_0)^{(n+1)\alpha}, \\ X &\in \mathbb{R}^N, t_0 \leq \tau \leq t < t_0+R. \end{aligned} \tag{3.2}$$

Proof. Using the definition of the Riemann-Liouville fractional integral operator (2.1) and the mean value theorem for fractional integrals (2.2), we have

$$\begin{aligned} I_t^{(n+1)\alpha} D_t^{(n+1)\alpha} u(X,t) &= \frac{1}{\Gamma((n+1)\alpha)} \int_{t_0}^{t} (t-\xi)^{(n+1)\alpha-1}\, D_t^{(n+1)\alpha} u(X,\xi) d\xi \\ &= \frac{D_t^{(n+1)\alpha}u(X,\tau)}{\Gamma((n+1)\alpha)} \int_{t_0}^{t} (t-\xi)^{(n+1)\alpha-1}\, d\xi \\ &= \frac{D_t^{(n+1)\alpha}u(X,\tau)}{\Gamma((n+1)\,\alpha+1)}(t-t_0)^{(n+1)\alpha}. \end{aligned}$$

Q.E.D.

Theorem 3.3. Suppose that $u \in C\left(\mathbb{R}^N \times [t_0, t_0+R)\right)$, $D_t^{i\alpha}u \in C\left(\mathbb{R}^N \times (t_0, t_0+R)\right)$, and $D_t^{i\alpha}u$ can be differentiated $(m-1)$-times with respect to t on (t_0, t_0+R) for $i \in \{0,1,2,...,n+1\}$, where $0 \leq m-1 < \alpha \leq m$. Then we have

$$\begin{aligned} u(X,t) &= \sum_{i=0}^{n}\sum_{j=0}^{m-1} \frac{D_t^{i\alpha+j}u(X,t_0)}{\Gamma(i\alpha+j+1)}(t-t_0)^{i\alpha+j} + \frac{D_t^{(n+1)\alpha}u(X,\tau)}{\Gamma((n+1)\,\alpha+1)}(t-t_0)^{(n+1)\alpha}, \\ X &\in \mathbb{R}^N, t_0 \leq \tau \leq t < t_0+R. \end{aligned} \tag{3.3}$$

Proof. From certain properties of the operator I_t^{α}, the concept of sequential fractional derivatives

(See [22]), and the propertie (2.4), we have

$$
\begin{aligned}
I_t^{(n+1)\alpha} D_t^{(n+1)\alpha} u(X,t) &= I_t^{n\alpha}\left(\left(I_t^{\alpha} D_t^{\alpha}\right) D_t^{n\alpha} u(X,t)\right) = I_t^{n\alpha}\left(\left(I_t^{m} D_t^{m}\right) D_t^{n\alpha} u(X,t)\right) \\
&= I_t^{n\alpha}\left(D_t^{n\alpha} u(X,t) - \sum_{j=0}^{m-1} \frac{\partial^j}{\partial t^j} D_t^{n\alpha} u(X,t_0) \frac{(t-t_0)^j}{j!}\right) \\
&= I_t^{n\alpha} D_t^{n\alpha} u(X,t) - I_t^{n\alpha}\left(\sum_{j=0}^{m-1} D^j D_t^{n\alpha} u(X,t_0) \frac{(t-t_0)^j}{j!}\right) \\
&= I_t^{(n-1)\alpha}\left(\left(I_t^{m} D_t^{m}\right) D_t^{(n-1)\alpha} u(X,t)\right) - \sum_{j=0}^{m-1} \frac{D_t^{n\alpha+j} u(X,t_0)}{\Gamma(n\alpha+j+1)} (t-t_0)^{n\alpha+j} \\
&= I_t^{(n-1)\alpha}\left(D_t^{(n-1)\alpha} u(X,t) - \sum_{j=0}^{m-1} \frac{\partial^j}{\partial t^j} D_t^{(n-1)\alpha} u(X,t_0) \frac{(t-t_0)^j}{j!}\right) \\
&\quad - \sum_{j=0}^{m-1} \frac{D_t^{n\alpha+j} u(X,t_0)}{\Gamma(n\alpha+j+1)} (t-t_0)^{n\alpha+j} \\
&= I_t^{(n-1)\alpha} D_t^{(n-1)\alpha} u(X,t) - I_t^{(n-1)\alpha}\left(\sum_{j=0}^{m-1} D^j D_t^{(n-1)\alpha} u(X,t_0) \frac{(t-t_0)^j}{j!}\right) \\
&\quad - \sum_{j=0}^{m-1} \frac{D_t^{n\alpha+j} u(X,t_0)}{\Gamma(n\alpha+j+1)} (t-t_0)^{n\alpha+j} \\
&= I_t^{(n-2)\alpha}\left(\left(I_t^{m} D_t^{m}\right) D_t^{(n-2)\alpha} u(X,t)\right) \\
&\quad - \sum_{j=0}^{m-1} \frac{D_t^{(n-1)\alpha+j} u(X,t_0)}{\Gamma((n-1)\alpha+j+1)} (t-t_0)^{(n-1)\alpha+j} \\
&\quad - \sum_{j=0}^{m-1} \frac{D_t^{n\alpha+j} u(X,t_0)}{\Gamma(n\alpha+j+1)} (t-t_0)^{n\alpha+j}.
\end{aligned}
$$

If we keep repeating this process, then after n-times of computations, we can find that

$$
I_t^{(n+1)\alpha} D_t^{(n+1)\alpha} u(X,t) = u(X,t) - \sum_{i=0}^{n} \sum_{j=0}^{m-1} \frac{D_t^{i\alpha+j} u(X,t_0)}{\Gamma(i\alpha+j+1)} (t-t_0)^{i\alpha+j}.
$$

Using (3.2), we get

$$
\begin{aligned}
u(X,t) &= \sum_{i=0}^{n} \sum_{j=0}^{m-1} \frac{D_t^{i\alpha+j} u(X,t_0)}{\Gamma(i\alpha+j+1)} (t-t_0)^{i\alpha+j} + \frac{D_t^{(n+1)\alpha} u(X,\tau)}{\Gamma((n+1)\alpha+1)} (t-t_0)^{(n+1)\alpha}, \\
t_0 &\leq \tau \leq t < t_0 + R.
\end{aligned}
$$

Q.E.D.

In the generalized Taylor's series the partial sums are

$$T_n(X,t)=\sum_{i=0}^{n}\sum_{j=0}^{m-1}\frac{D_t^{i\alpha+j}u(X,t_0)}{\Gamma(i\alpha+j+1)}(t-t_0)^{i\alpha+j},$$

and in general, $u(X,t)$ is the sum of its generalized Taylor's series if $u(X,t)=\lim_{n\longrightarrow\infty}T_n(X,t)$. On the other aspect as well, if we let $R_n(X,t)=u(X,t)-T_n(X,t)$, then $R_n(X,t)$ is the remainder of the generalized Taylor's series.

Corollary 3.4. If $\left|D_t^{(n+1)\alpha}u(X,t)\right|\leq M(X)$ on $t_0\leq t\leq d$ where $0\leq m-1<\alpha\leq m$, then $R_n(X,t)$ the remainder of the generalized Taylor's series will satisfy the inequality

$$|R_n(X,t)|\leq\frac{M(X)}{\Gamma((n+1)\,\alpha+1)}(t-t_0)^{(n+1)\alpha},t_0\leq t\leq d.$$

4 FRPSM for general nonlinear time-fractional partial differential equation

Theorem 4.1. Consider a general nonlinear time-fractional partial differential equation

$$D_t^{\alpha}u(X,t)+Lu(X,t)+Nu(X,t)=g(X,t),0\leq m-1<\alpha\leq m,X\in\mathbb{R}^N,t>0, \tag{4.1}$$

with the initial conditions

$$\frac{\partial^j u(X,0)}{\partial t^j}=a_j(X),X\in\mathbb{R}^N,j=0,1,2,...,m-1, \tag{4.2}$$

where D_t^{α} is the Caputo fractional derivative of order α, L is a linear differential operator, N represents a general nonlinear differential operator, and g is the source term.

Then, by FRPSM the solution of Eqs. (4.1)-(4.2) is given in the form of infinite series which converges rapidly to the exact solution as follows

$$\begin{aligned}u(X,t) &= \sum_{j=0}^{m-1}\frac{a_j(X)}{j!}t^j+\sum_{i=1}^{\infty}\sum_{j=0}^{m-1}f_{ij}(X)\frac{t^{i\alpha+j}}{\Gamma(i\alpha+j+1)}, \\ X &\in \mathbb{R}^N,0\leq t<R.\end{aligned} \tag{4.3}$$

where $f_{ij}\ i\in\{1,2,3,...,\}$, $j\in\{0,1,2,...,m-1\}$, are the coefficients of the series, and R is the radius of convergence of the MFPS.

Proof. In order to achieve our goal, we consider the following generalized nonlinear time-fractional partial differential equation (4.1) with the initial conditions (4.2).

The FRPS method assumes the solution for Eqs. (4.1)-(4.2) as a fractional power series about the initial point $t=0$, as follows

$$\begin{aligned}u(X,t) &= \sum_{i=0}^{\infty}\sum_{j=0}^{m-1}f_{ij}(X)\frac{t^{i\alpha+j}}{\Gamma(i\alpha+j+1)},\ m-1<\alpha\leq m, \\ X &\in \mathbb{R}^N,0\leq t<R,\end{aligned} \tag{4.4}$$

From Eq. (4.4), we have

$$\frac{\partial^j u(X,0)}{\partial t^j} = j! f_{0j}(X) = a_j(X), j \in \{0,1,2,...,m-1\},$$

then

$$f_{0j}(X) = \frac{a_j(X)}{j!}, j \in \{0,1,2,...,m-1\}.$$

Hence, we can obtain the initial guess approximation of u as

$$u_{(0,m-1)}(X,t) = \sum_{j=0}^{m-1} \frac{a_j(X)}{j!} t^j, X \in \mathbb{R}^N, 0 \le t < R. \tag{4.5}$$

So, Eq. (4.4) could be reformulated as

$$\begin{aligned} u(X,t) &= \sum_{j=0}^{m-1} \frac{a_j(X)}{j!} t^j + \sum_{i=1}^{\infty} \sum_{j=0}^{m-1} f_{ij}(X) \frac{t^{i\alpha+j}}{\Gamma(i\alpha+j+1)}, \\ X &\in \mathbb{R}^N, 0 \le t < R. \end{aligned} \tag{4.6}$$

In the following step, we will let $u_{(k,l)}$ denotes the (k,l)-truncated series, that is

$$\begin{aligned} u_{(k,l)}(X,t) &= \sum_{j=0}^{m-1} \frac{a_j(X)}{j!} t^j + \sum_{i=1}^{k} \sum_{j=0}^{l} f_{ij}(X) \frac{t^{i\alpha+j}}{\Gamma(i\alpha+j+1)}, \\ X &\in \mathbb{R}^N, 0 \le t < R, \end{aligned} \tag{4.7}$$

where $k \in \{1,2,3,...\}$ and $l \in \{0,1,2,...,m-1\}$. Now applying the FRPS for determining the coefficients f_{ij} of Eq. (4.7), we define the residual function as

$$\begin{aligned} \mathrm{Res}(X,t) &= D_t^\alpha u(X,t) + Lu(X,t) + Nu(X,t) - g(X,t), \\ X &\in \mathbb{R}^N, 0 \le t < R. \end{aligned} \tag{4.8}$$

Therefore, the (k,l)-truncated residual function is

$$\begin{aligned} \mathrm{Res}_{(k,l)}(X,t) &= D_t^\alpha u_{(k,l)}(X,t) + Lu_{(k,l)}(X,t) + Nu_{(k,l)}(X,t) - g(X,t), \\ X &\in \mathbb{R}^N, 0 \le t < R. \end{aligned} \tag{4.9}$$

As described in [1],[7], it is clear that $\mathrm{Res}(X,t) = 0$ for each $X \in \mathbb{R}^N$ and $t \in [0,R)$. Therefore $D_t^{(i-1)\alpha} D_t^j \mathrm{Res}(X,t) = 0$ for each $i \in \{1,2,3,...,k\}$ and $j \in \{0,1,2,...,l\}$, since the fractional derivative of a constant function in the Caputo sense is zero, and in the mean time, the fractional derivatives $D_t^{(i-1)\alpha} D_t^j$ for each $i \in \{1,2,3,...,k\}$ and $j \in \{0,1,2,...,l\}$ of $\mathrm{Res}(X,t) = 0$ and $\mathrm{Res}_{(i,j)}(X,t)$ are matching at $(X,t) = (X,0)$.

$$\begin{aligned} D_t^{(i-1)\alpha} D_t^j \mathrm{Res}(X,0) &= D_t^{(i-1)\alpha} D_t^j \mathrm{Res}_{(i,j)}(X,0) = 0, \\ X &\in \mathbb{R}^n, i \in \{1,2,3,...,k\}, j \in \{0,1,2,...,l\}. \end{aligned} \tag{4.10}$$

To get the form of the required coefficients f_{ij} in Eq. (4.7) for $i \in \{1,2,3,...,k\}$ and $j \in \{0,1,2,...,l\}$, we apply the following FRPS technique:

First, we substitute (k,l)-th truncated series of u into Eq. (4.9). Second we find then the fractional derivative formula $D_t^{(i-1)\alpha} D_t^j \operatorname{Res}_{(i,j)}(.,.)$ $i \in \{1,2,3,...,k\}$, $j \in \{0,1,2,...,l\}$, we substitute $t=0$ in the following formula, equate it to zero, and then solve the obtained algebraic Eq. (4.10). Q.E.D.

Let us consider a general nonlinear time-fractional wave-like equation (1.1) with the initial conditions (1.2).

First we define

$$\begin{aligned} Nu &= \sum_{i,j=1}^{N} F_{1ij}(X,t,u) \frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(u_{x_i}, u_{x_j}), \\ Mu &= +\sum_{i=1}^{N} G_{1i}(X,t,u) \frac{\partial^p}{\partial x_i^p} G_{2i}(u_{x_i}), \\ Ku &= H(X,t,u). \end{aligned}$$

Eq.(1.1) is written in the form

$$\begin{aligned} D_t^\alpha u(X,t) &= Nu(X,t) + Mu(X,t) + Ku(X,t) + S(X,t), \\ X &\in \mathbb{R}^N, t > 0, 1 < \alpha \leq 2. \end{aligned}$$

According to Eq. (4.6), the solution of Eqs. (1.1)-(1.2) can be expressed as

$$u(X,t) = a_0(X) + a_1(X)t + \sum_{i=1}^{\infty} \sum_{j=0}^{1} f_{ij}(X) \frac{t^{i\alpha+j}}{\Gamma(i\alpha+j+1)}, \tag{4.11}$$

where

$$u_{(0,1)}(x,y,t) = a_0(X) + a_1(X)t, \tag{4.12}$$

is the initial guess approximation which is obtained by Eq. (4.5).

To find the coefficients f_{ij} for $i \in \{1,2,3,...,\}$ and $j \in \{0,1\}$, we use the same steps as in above.

5 Examples and numerical results

In this section, the applicability of the FRPSM shall be demonstrated by test examples.

We define $E_{(k,l)}$ to be the absolute error between the exact solution u and the FRPS approximate solution $u_{(k,l)}$, where $k \in \{0,1,2,3,...\}$ and $l \in \{0,1,2,...,m-1\}$, as follows

$$\begin{aligned} E_{(k,l)}(X,t) &= \left| u(X,t) - u_{(k,l)}(X,t) \right|, \\ k &\in \{0,1,2,3,...\}, l \in \{0,1,2,...,m-1\}. \end{aligned}$$

Example 5.1. Consider the 2-dimensional nonlinear time-fractional wave-like equation with variable coefficients

$$D_t^\alpha u = \frac{\partial^2}{\partial x \partial y}(u_{xx} u_{yy}) - \frac{\partial^2}{\partial x \partial y}(xy u_x u_y) - u, t > 0, 1 < \alpha \leq 2, \tag{5.1}$$

with initial conditions

$$u(x,y,0)=e^{xy}, u_t(x,y,0)=e^{xy}, (x,y)\in\mathbb{R}^2. \tag{5.2}$$

By applying the steps involved in the FRPSM as presented in Section 4, we have the solution of Eqs (5.1)-(5.2) is in the form

$$u(x,y,t)=e^{xy}+e^{xy}t+\sum_{i=1}^{\infty}\sum_{j=0}^{1}f_{ij}(x,y)\frac{t^{i\alpha+j}}{\Gamma(i\alpha+j+1)}, \tag{5.3}$$

and the coefficients f_{ij} for $i\in\{1,2,3,...,\}$ and $j\in\{0,1\}$, take the following values

$$\begin{cases} f_{i0}(x,y)=f_{i1}(x,y)=-e^{xy} & \text{for } i\in\{1,3,5,...\},\\ f_{i0}(x,y)=f_{i1}(x,y)=e^{xy} & \text{for } i\in\{2,4,8,...\}. \end{cases}$$

Therefore, according to Eq. (5.3), the FRPS solution of Eqs. (5.1)-(5.2) can be written in the form of infinite series as follows

$$\begin{aligned} u(x,y,t) &= \left(1+t-\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}-...\right)e^{xy}\\ &= \left(t\sum_{k=0}^{\infty}\frac{(-1)^k t^{k\alpha}}{\Gamma(k\alpha+2)}+\sum_{k=0}^{\infty}\frac{(-1)^k t^{k\alpha}}{\Gamma(k\alpha+1)}\right)e^{xy}\\ &= \left(tE_{\alpha,2}(-t^{\alpha})+E_{\alpha}(-t^{\alpha})\right)e^{xy}, \end{aligned}$$

where $E_{\alpha}(-t^{\alpha})$ and $E_{\alpha,2}(-t^{\alpha})$ are the Mittag-Leffler functions, defined by Eqs. (2.5) and (2.6).

Particularly, when $\alpha=2$, the exact solution of Eqs. (5.1)-(5.2) is given by

$$\begin{aligned} u(x,y,t) &= \left(1+t-\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!}-...\right)e^{xy}\\ &= \left(\cos t+\sin t\right)e^{xy}. \end{aligned}$$

The above expressions is exactly same as those given by the NIM, NHPM [15], NVIM [16], and FNDM [17].

$t/x,y$	0.1	0.3	0.5	0.7
0.1	1.4226×10^{-9}	1.5411×10^{-9}	1.8085×10^{-9}	2.2991×10^{-9}
0.3	1.0648×10^{-6}	1.1535×10^{-6}	1.3536×10^{-6}	1.7208×10^{-6}
0.5	2.3382×10^{-5}	2.5330×10^{-5}	2.9725×10^{-5}	3.7787×10^{-5}
0.7	1.8000×10^{-4}	1.9499×10^{-4}	2.2882×10^{-4}	2.9089×10^{-4}
0.9	8.2963×10^{-4}	8.9872×10^{-4}	1.0547×10^{-3}	1.3407×10^{-3}

TABLE 1. Comparison of the absolute errors for the obtained results and the exact solution for Example 5.1 when $(k,l)=(2,1)$ and $\alpha=2$.

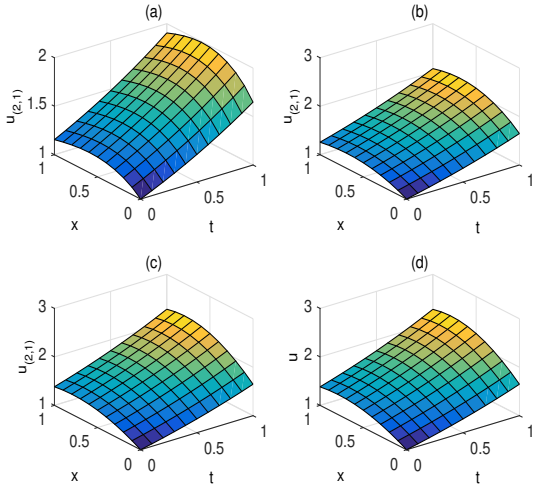


FIGURE 1. The surface graph of the exact solution u and the FRPS approximate solution $u_{(2,1)}$ for Example 5.1 when $y = 0.5$: (a) $u_{(2,1)}$ when $\alpha = 1.5$, (b) $u_{(2,1)}$ when $\alpha = 1.75$, (c) $u_{(2,1)}$ when $\alpha = 2$, and (d) u is exact.

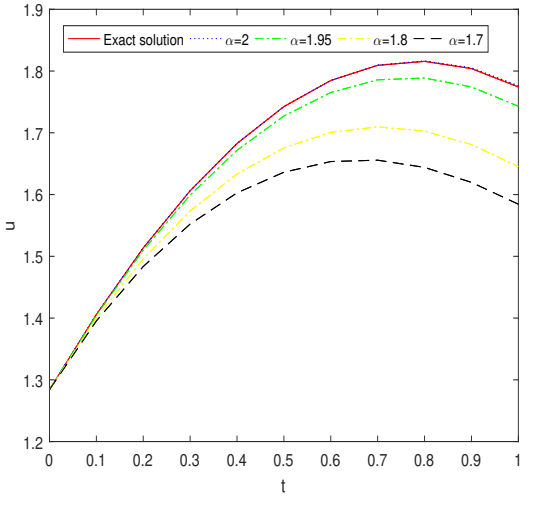


FIGURE 2. The behavior of the exact solution and the FRPS approximate solution $u_{(2,1)}$ for different values of α for Example 5.1 in the case when $x = y = 0.5$.

Example 5.2. Consider the following nonlinear time-fractional wave-like equation with variable coefficients

$$D_t^{\alpha}u = u^2\frac{\partial^2}{\partial x^2}(u_x u_{xx} u_{xxx}) + u_x^2\frac{\partial^2}{\partial x^2}(u_{xx}^3) - 18u^5 + u, t > 0, 1 < \alpha \leq 2, \tag{5.4}$$

with initial conditions

$$u(x,0) = e^x, u_t(x,0) = e^x, x \in]0,1[\,. \tag{5.5}$$

By applying the steps involved in the FRPSM as presented in Section 4, we have the solution of Eqs (5.4)-(5.5) is in the form

$$u(x,t) = e^x + e^x t + \sum_{i=1}^{\infty}\sum_{j=0}^{1} f_{ij}(x)\frac{t^{i\alpha+j}}{\Gamma(i\alpha+j+1)}, \tag{5.6}$$

and the coefficients f_{ij} for $i \in \{1,2,3,...,\}$ and $j \in \{0,1\}$, take the following values

$$f_{i0}(x) = f_{i1}(x) = e^x \quad \text{for} \quad i \in \{1,2,3,...\}\,.$$

Therefore, according to Eq. (5.6), the FRPS solution of Eqs. (5.4)-(5.5) can be written in the form of infinite series as follows

$$\begin{aligned} u(x,t) &= \left(1 + t + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + ...\right)e^x \\ &= \left(\sum_{k=0}^{\infty}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)} + t\sum_{k=0}^{\infty}\frac{t^{k\alpha}}{\Gamma(k\alpha+2)}\right)e^x = \left(E_{\alpha}(t^{\alpha}) + tE_{\alpha,2}(t^{\alpha})\right)e^x, \end{aligned}$$

where $E_{\alpha}(t^{\alpha})$ and $E_{\alpha,2}(t^{\alpha})$ are the Mittag-Leffler functions, defined by Eqs (2.5) and (2.6).

Particularly, when $\alpha = 2$, the exact solution of Eqs. (5.4)-(5.5) is given by

$$u(x,t) = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + ...\right)e^x = e^{x+t}.$$

The above expressions is exactly same as those given by the NIM, NHPM [15], NVIM [16], and FNDM [17].

t/x	0.1	0.3	0.5	0.7
0.1	1.5572×10^{-9}	1.9019×10^{-9}	2.3230×10^{-9}	2.8373×10^{-9}
0.3	1.1688×10^{-6}	1.4276×10^{-6}	1.7436×10^{-6}	2.1297×10^{-6}
0.5	2.5810×10^{-5}	3.1525×10^{-5}	3.8504×10^{-5}	4.7029×10^{-5}
0.7	2.0036×10^{-4}	2.4472×10^{-4}	2.9890×10^{-4}	3.6507×10^{-4}
0.9	9.3372×10^{-4}	1.1404×10^{-3}	1.3929×10^{-3}	1.7013×10^{-3}

TABLE 2. Comparison of the absolute errors for the obtained results and the exact solution for Example 5.2 when $(k,l) = (2,1)$ and $\alpha = 2$.

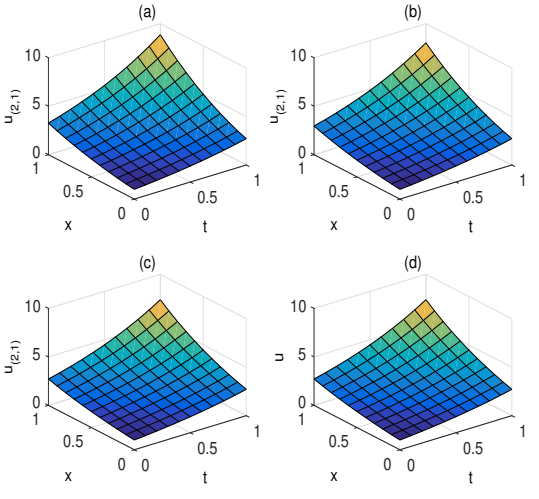


FIGURE 3. The surface graph of the exact solution u and the FRPS approximate solution $u_{(2,1)}$ for Example 5.2: (a) $u_{(2,1)}$ when $\alpha = 1.5$, (b) $u_{(2,1)}$ when $\alpha = 1.75$, (c) $u_{(2,1)}$ when $\alpha = 2$, and (d) u is exact.

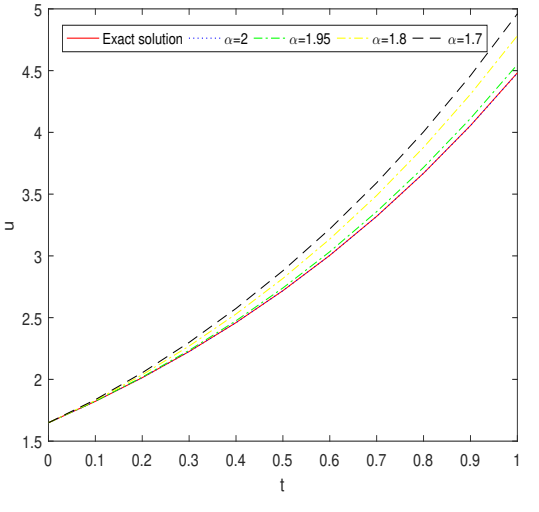


FIGURE 4. The behavior of the exact solution and the FRPS approximate solution $u_{(2,1)}$ for different values of α for Example 5.2 in the case when $x = 0.5$.

Example 5.3. Consider the following one-dimensional nonlinear time-fractional wave-like equation with variable coefficients

$$D_t^{\alpha} u = x^2 \frac{\partial}{\partial x}(u_x u_{xx}) - x^2(u_{xx}^2) - u, t > 0, 1 < \alpha \leq 2, \tag{5.7}$$

with initial conditions

$$u(x,0) = 0, u_t(x,0) = x^2, x \in]0,1[\,. \tag{5.8}$$

By applying the steps involved in the FRPSM as presented in Section 4, we have the solution of Eqs (5.7)-(5.8) is in the form

$$u(x,t) = x^2 t + \sum_{i=1}^{\infty} \sum_{j=0}^{1} f_{ij}(x) \frac{t^{i\alpha+j}}{\Gamma(i\alpha + j + 1)}, \tag{5.9}$$

and the coefficients f_{ij} for $i \in \{1,2,3,...,\}$ and $j \in \{0,1\}$, take the following values

$$\begin{cases} f_{i0}(x) = 0 & \text{for } \; i \in \{1,2,3,...\}\,, \\ f_{i1}(x) = (-1)^i x^2 & \text{for } \; i \in \{1,2,3,...\}\,. \end{cases}$$

Therefore, according to Eq. (5.9), the FRPS solution of Eqs. (5.7)-(5.8) can be written in the form of infinite series as follows

$$\begin{aligned} u(x,t) &= x^2 \left(t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} + ... \right) \\ &= x^2 \left(t \sum_{k=0}^{\infty} \frac{(-1)^k t^{k\alpha}}{\Gamma(k\alpha+2)} \right) = x^2 \left(t E_{\alpha,2}(-t^{\alpha}) \right), \end{aligned}$$

where $E_{\alpha,2}(-t^{\alpha})$ is the Mittag-Leffler function, defined by Eq. (2.6).

Particularly, when $\alpha = 2$, the exact solution of Eqs. (5.7)-(5.8) is given by

$$u(x,t) = x^2 \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + ... \right) = x^2 \sin t.$$

The above expressions is exactly same as those given by the NIM, NHPM [15], NVIM [16], and FNDM [17].

t/x	0.1	0.3	0.5	0.7
0.1	1.9839×10^{-13}	1.7855×10^{-12}	4.9596×10^{-12}	9.7209×10^{-12}
0.3	4.3339×10^{-10}	3.9005×10^{-9}	1.0835×10^{-8}	2.1236×10^{-8}
0.5	1.5447×10^{-8}	1.3903×10^{-7}	3.8618×10^{-7}	7.5692×10^{-7}
0.7	1.6229×10^{-7}	1.4606×10^{-6}	4.0574×10^{-6}	7.9524×10^{-6}
0.9	9.3840×10^{-7}	8.4456×10^{-6}	2.3460×10^{-5}	4.5982×10^{-5}

TABLE 3. Comparison of the absolute errors for the obtained results and the exact solution for Example 5.3 when $(k,l) = (2,1)$ and $\alpha = 2$.

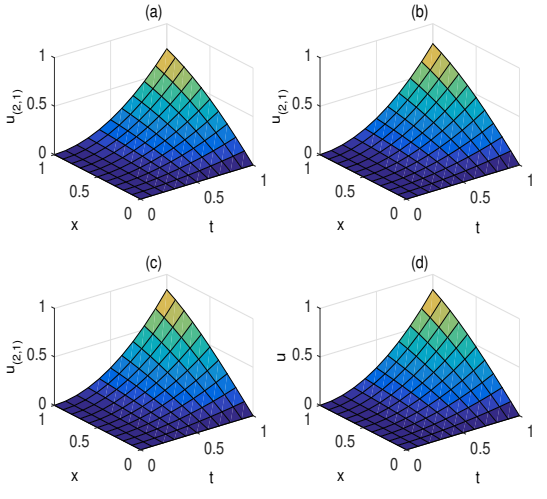


FIGURE 5. The surface graph of the exact solution u and the FRPS approximate solution $u_{(2,1)}$ for Example 5.3: (a) $u_{(2,1)}$ when $\alpha = 1.5$, (b) $u_{(2,1)}$ when $\alpha = 1.75$, (c) $u_{(2,1)}$ when $\alpha = 2$, and (d) u is exact.

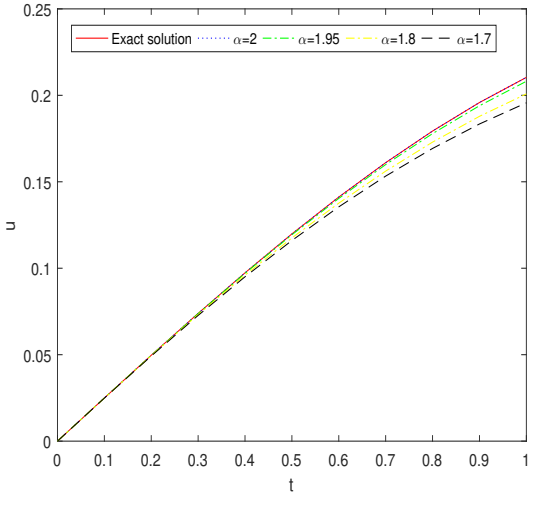


FIGURE 6. The behavior of the exact solution and the FRPS approximate solution $u_{(2,1)}$ for different values of α for Example 5.3 in the case when $x = 0.5$.

Remark 5.4. From the numerical results of $u(x, y, t)$ for different values of the fractional derivative α and for different values of x, y and t, we remark that when the fractional derivative $1 < \alpha \leq 2$ tends to 2, the approximate solutions obtained by FRPSM tends continuously to the exact solutions.

6 Conclusion

It this paper, the fractional residual power series method (FRPSM) is successfully applied to find approximate analytical solutions of general nonlinear time-fractional wave-like equations with variables coefficients. Throughout this article, it has been observed that there exists a very good agreement between the approximate solutions obtained by this method and exact solutions. When we set fractional derivative $\alpha = 2$ in the FRPS approximate solutions, we get exact solutions. Therefore we can conclude that the FRPSM is quite effective, convenient and practically well suited for finding approximate analytical solutions of such types of fractional partial differential equations.

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