Blow-up solutions of a time-fractional diffusion equation with variable exponents

J. Manimaran and L. Shangerganesh

Department of Applied Sciences, National Institute of Technology Goa, Goa - 403 401, India.
E-mail: manimaranj@nitgoa.ac.in, shangerganesh@nitgoa.ac.in

Abstract

In this paper, we investigate the blow-up of solutions of a time-fractional nonlocal reaction-diffusion equation with variable exponents under the Dirichlet boundary condition. We prove the existence of a solution using the Banach fixed point theorem. Finally, we estimate the blow-up of solutions of the considered model using the Kaplan method.

2010 Mathematics Subject Classification. **35B44**. 35K57, 35R11
Keywords. blow-up, time-fractional reaction-diffusion equation, variable exponents.

1 Introduction

In recent years, time-fractional partial differential equations (TFPDEs) have found many applications in biological models and other areas of science and engineering, see, for example, [20, 21, 28, 40] and references therein. Therefore, many researchers are giving great attention to analyze the existence and asymptotic behaviours of solutions of TFPDEs, see, [4, 5, 35, 38, 39]. Parabolic equations with integer-order derivatives concerned, in the literature, reasonable amount of work already done to study the blow-up phenomena of solutions, for instance, see [9, 10, 12, 16, 18, 32]. However, contributions related to mathematical analysis of TFPDEs are very few and briefly, we discuss some of the important works related to our problem. In particular, Alikhanov [3] discussed the existence of solutions for a time-fractional boundary value problem of the diffusion-wave equation using the method of energy inequalities. Yamamoto studied weak solutions of non-homogeneous boundary-value problems with time-fractional diffusion equations in [35]. Global existence of solutions a reaction-diffusion equation with the Caputo fractional derivative in time and with various boundary conditions proved in [2, 4]. Kubica and Yamamoto proved the existence and regularity of weak solutions for the fractional diffusion equation with the Caputo time-fractional derivative where the coefficients are dependent on spatial and assumed the zero Dirichlet boundary condition in [23]. Using the maximum principle and some a priory estimates existence and uniqueness of a solution established for time-fractional wave equations in [24, 25, 26, 27]. Ahmad et al. [5] discussed the blow-up solution of subdiffusion equation with time-fractional derivative. Zhang and Li [36] studied the blow-up property of solutions for the time-fractional subdiffusion equation in an unbounded domain. Further, they also obtained some blow-up results for a subdiffusion equation with nonlinear memory under the Dirichlet boundary condition in [37]. Cao et al. [14] studied the blow-up solution of the time-fractional reaction-diffusion equation with superlinearity. In contrast to the above works, the main novelty of this work is to study the existence and blow-up phenomenon of the nonlinear time-fractional diffusion equation with nonlocal nonlinearities and variable exponents. Moreover, in the literature, this is the first paper to study the blow-up phenomenon of the time-fractional diffusion equation with variable exponents nonlinearities.

Tbilisi Mathematical Journal 12(4) (2019), pp. 149–157.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 31 March 2019.
Accepted for publication: 05 November 2019.

In this paper, we interest to study the following nonlinear time-fractional diffusion equation with nonlocal variable exponent reaction terms,

$$\left.\begin{aligned} &\partial_t^\alpha u = \Delta u + a(x)\int_\Omega u^{p(x)}dx \text{ in } Q_T,\\ &u(x,t) = 0 \text{ in } \Sigma_T,\\ &u(x,0) = u_0(x) \text{ on } \Omega, \end{aligned}\right\} \tag{1.1}$$

where ∂_t^α denotes the left Caputo fractional derivative of order $0 < \alpha \leq 1$ with respect to time t, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and $Q_T = \Omega \times (0,T)$, $\Sigma_T = \partial\Omega \times (0,T)$, $T > 0$ is a final time. Here $u(x,t)$ is the density of population and $a(x)$ and $p(x)$ arise in the mathematical models which describe the process of absorption, see [6, 7, 8] and the references therein. Further, $a(x)\int_\Omega u^{p(x)}dx$ denotes the human-controlled distribution. Nonlocal reaction-diffusion equation is the generalisation of the local reaction-diffusion equation. It represents that the evolution of the species in a point of space does not depend only on nearby density but also on the total amount of species. It is due to the effects of spatial inhomogeneity (see [13, 19, 29, 31, 33] and the references therein). Moreover, the variable exponents $L^{p(x)}$ spaces are of interest for their applications to modelling in a wide variety of physical problems. For instance, modeling of physical phenomena such as flows of electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, filtration processes through a porous media and image processing see, [1, 9, 10, 11] and the references therein. Further, we assume the following hypothesis,

(H1) $1 < p^- \leq p(x) \leq p^+ < +\infty$, and $0 < c_a \leq a(x) \leq C_a < +\infty$.

The outline of this work proceed as follows. In Section 2, we briefly present some basic definitions and theorems related to TFPDEs. Further, we prove the solvability of the proposed equation. In Section 3, we analyze the blow-up property of equation (1.1). Finally, conclusions are drawn in Section 4.

2 Local existence of solutions

In this section, we provide the existence and uniqueness of a mild solution of equation (1.1) using fixed point argument. Here, we used the same notations and definitions as in [22, 38, 39]. Now, we present some definitions and some lemmas, which are used throughout this work.

Definition 2.1. Let $\mathcal{A}$ be the $L^2(\Omega)$ realization of $-\Delta$ under the homogeneous Dirichlet boundary conditions. Let $0 < \kappa_0 \leq \kappa_1 \leq \cdots$ be sequence of nondecreasing eigenvalues of $\mathcal{A}$ and the corresponding orthonormal eigenfunction $\{\psi_n\}_{n\geq 0}$; $\mathcal{A}\psi_n = \kappa_n\psi_n$ and

$$D(\mathcal{A}) = \left\{ u \in L^2(\Omega); u = 0 \text{ on } \partial\Omega \text{ and } \|\mathcal{A}u\|^2_{L^2(\Omega)} = \sum_{l=0}^{+\infty} |\kappa_l(u,\psi_l)|^2 < +\infty \right\}.$$

Now, we introduce the mild solution to the problem (1.1) in terms of result in [5, 36, 37].

Definition 2.2 ([5])**.** For every $T > 0$ and $u_0 \in C(\bar{\Omega})$, any function $u \in C([0,T] : C(\bar{\Omega}))$ is a mild solution of equation (1.1) if it satisfies

$$u(t) = E_\alpha(-t^\alpha\mathcal{A})u_0 + \alpha\int_0^t s^{\alpha-1}E'_\alpha(-s^\alpha\mathcal{A})\left(f(u(t-s))\right)ds, \tag{2.1}$$

for all $t \in [0,T]$, where $E_\alpha(-t^\alpha \mathcal{A})$ and $E'_\alpha(-s^\alpha \mathcal{A})$ are linear operators defined as in [5] and $f(u) = a(x)\int_\Omega u^{p(x)}dx$.

Lemma 2.1. [5] If $u \in L^\infty(\Omega)$, we have

$$\|E_\alpha(-t^\alpha \mathcal{A})u\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Omega)}, \qquad \text{for } t > 0, \tag{2.2}$$

$$\|E'_\alpha(-t^\alpha \mathcal{A})u\|_{L^\infty(\Omega)} \le \frac{1}{\alpha\Gamma(\alpha)}\|u\|_{L^\infty(\Omega)}, \text{ for } t > 0. \tag{2.3}$$

Lemma 2.2. Let $\zeta(t)$ be non-negative, which satisfies the following differential inequality

$$\zeta'(t) \ge c\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\zeta^m(t), \quad \zeta(0) > 0,$$

where $m > 1$, and $c > 0$. Then $\zeta(t)$ blows up at finite time and

$$\zeta(t) \ge \left(\zeta(0)^{1-m} - \frac{m-1}{c(2-\alpha)\Gamma(2-\alpha)}t^{2-\alpha}\right)^{\frac{-1}{m-1}}.$$

This lemma can be easily achieved by the direct integration. Further, we remark that Lemma 2.2 is the generalization of Lemma 2.2 in [30].

Theorem 2.1. If $u_0 \in C(\bar{\Omega})$ and assume $(H1)$ hold true. Then problem (1.1) possesses a local mild solution $u \in C([0, T_{max}]; C(\bar{\Omega}))$ with the alternative

- either $T_{max} = +\infty$;
- or $T_{max} < +\infty$ and in this case $\lim_{t\to T_{\max}} \|u(t)\|_{L^\infty(\Omega)} = +\infty$.

Proof. We apply the Banach fixed point theorem to prove the existence of a solution of (1.1). Now, we introduce the Banach space $\mathcal{B}$ as follows:

$$\mathcal{B} = \left\{ u \in C([0,T]; C(\bar{\Omega})); \ \sup_{t\in[0,T]} \|u(t) - u_0\|_{L^\infty(\Omega)} \le M \right\},$$

where T will be chosen later.

Now, we define

$$\mathcal{T}u := E_\alpha(-t^\alpha \mathcal{A})u_0 + \int_0^t s^{\alpha-1}E'_\alpha(-s^\alpha \mathcal{A})\left(f(u(t-s))\right)ds,$$

for any $u \in \mathcal{B}$. Our main goal of the proof is to ensure the unique fixed point of the operator $\mathcal{T}$. To proceed further, we need the following estimate. From [15, 17, 30] , for any fixed $x \in \Omega$,

$$u_1^{p(x)} - u_2^{p(x)} = p(x)\omega^{p(x)-1}(u_1 - u_2),$$

where $\omega = \lambda(x)u_1 + (1-\lambda(x))u_2,\ \lambda(x) \in (0,1)$.

$$\int_\Omega u_1^{p(x)} - u_2^{p(x)} dx = \int_\Omega p(x)\omega^{p(x)-1}(u_1-u_2)dx \leq (2M)^{p^+-1}\int_\Omega p(x)(u_1-u_2)dx$$

Hence,

$$\left\| \int_\Omega u_1^{p(x)} - u_2^{p(x)} dx \right\|_{L^\infty(\Omega)} \leq (2M)^{p^+-1}\text{meas}(\Omega)\ \|p\|_{L^\infty(\Omega)}\ \|u-v\|_{L^\infty(\Omega)}. \tag{2.4}$$

From (2.4), the nonlinearity is the locally Lipschitz function. Now, our aim is $\mathcal{T}$ maps $\mathcal{B}$ onto $\mathcal{B}$ with the help of (2.2), (2.3) and (2.4).

$$\|E_\alpha(-t^\alpha)u_0 - u_0\|_{L^\infty(\Omega)} \leq \frac{1}{2}M.$$

Similarly,

$$\left\| E'_\alpha(-t^\alpha\mathcal{A})a(x)\int_\Omega u^{p(x)}dx \right\|_{L^\infty(\Omega)} \leq \frac{1}{\alpha\Gamma(\alpha)}\left\| a(x)\int_\Omega u^{p(x)}dx \right\|_{L^\infty(\Omega)} \leq K\frac{1}{\alpha\Gamma(\alpha)},$$

where, $\left\| a(x)\int_\Omega u^{p(y)}dy \right\|_{L^\infty(\Omega)} \leq K$. Now, we fix T such that $T^\alpha \leq \alpha\Gamma(\alpha)M/2K$. Hence $\mathcal{T}$ maps $\mathcal{B}$ onto $\mathcal{B}$.

Now, we claim that $\mathcal{T}$ is a contraction mapping. From (2.4)

$$\left\| \int_\Omega u^{p(x)} - v^{p(x)}dx \right\|_{L^\infty(\Omega)} \leq L\|u-v\|_{L^\infty(\Omega)}.$$

This concludes that

$$\|\mathcal{T}u - \mathcal{T}v\|_{L^\infty(\Omega)} \leq \frac{L}{\alpha\Gamma(\alpha)}T^\alpha \sup_{0\leq s\leq t}\|u-v\|_{L^\infty(\Omega)}.$$

Now, we choose T such that $T^\alpha < \dfrac{\alpha\Gamma(\alpha)}{L}$, this proves that $\mathcal{T}$ is a contraction mapping. Therefore, from the Banach fixed point theorem there exists a unique $u \in \mathcal{B}$. It is the desired mild solution of (1.1). Q.E.D.

3 Blow up of solutions

In this section, we analyze the blow-up property of the solution of the given time-fractional diffusion problem (1.1). We estimate the blow-up solutions of (1.1) by adopting the Kaplan method with some inequalities and assuming that $(H1)$ hold true.

Theorem 3.1. Suppose that there exists a positive solution u for the given TFPDE (1.1) and $(H1)$ hold true. Then for sufficiently large data u_0, there exists a finite time $t_* > 0$ such that

$$\sup_{0\leq t\leq t_*}(\|u\|_{L^\infty(\Omega)}) = +\infty.$$

Proof. Let $\kappa > 0$ be the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ corresponding to the eigenfunction $\psi(x)$, i.e.,

$$\Delta\psi + \kappa\psi = 0 \text{ in } \Omega,\ \psi = 0 \ \text{ on } \ \partial\Omega.$$

Further assume that $\int_\Omega \psi dx = 1$. Now, we define the function

$$E(t) := \int_\Omega u(x,t)\ \psi(x)dx.$$

Then

$$\begin{aligned}\frac{\partial^\alpha E}{\partial t^\alpha} &= \int_\Omega \frac{\partial^\alpha u}{\partial t^\alpha}\psi(x)dx \\ &= \int_\Omega \Delta u\psi(x)dx + \int_\Omega \psi(x)a(x)\left(\int_\Omega u^{p(z)}dz\right)dx \\ &= -\kappa E(t) + \int_\Omega \psi(x)a(x)\left(\int_\Omega u^{p(z)}dz\right)dx \\ &= -\kappa E(t) + I_1, \end{aligned} \tag{3.1}$$

where, $I_1 := \int_\Omega \psi(x)a(x)\left(\int_\Omega u^{p(z)}dz\right)dx$.

Now,

$$\begin{aligned} I_1 &= \int_\Omega \psi(x)a(x)\left(\int_\Omega u^{p(z)}dz\right)dx \\ &\geq \frac{c_a}{\|\psi\|_\infty}\int_\Omega u^{p(z)}\psi(z)dz. \end{aligned} \tag{3.2}$$

Substitute (3.2) in (3.1)

$$\frac{\partial^\alpha E}{\partial t^\alpha} \geq -\kappa E + \frac{c_a}{\|\psi\|_\infty}\int_\Omega u^{p(z)}\psi(z)dz. \tag{3.3}$$

Now, we divided Ω into disjoint two sets such that $\Omega = U_1 \cup U_2$, where

$$U_1 = \{x \in \Omega : u(x,t) < 1\},$$

and

$$U_2 = \{x \in \Omega : u(x,t) \geq 1\},$$

for any $t > 0$.

$$\frac{\partial^\alpha E}{\partial t^\alpha} \geq -\kappa E + I_2,$$

where $I_2 = \frac{c_a}{\|\psi\|_\infty}\int_\Omega u^{p(z)}\psi(z)dz$. Using the representation

$$\begin{aligned} I_2 &= \frac{c_a}{\|\psi\|_\infty}\int_\Omega u^{p(z)}\psi(z)dz \\ &= \frac{c_a}{\|\psi\|_\infty}\int_{U_2} u^{p(z)}\psi(z)dz + \frac{c_a}{\|\psi\|_\infty}\int_{U_1} u^{p(z)}\psi(z)dz. \end{aligned}$$

Now, we evaluate I_2 in the following way

$$\begin{aligned} I_2 &\geq \frac{c_a}{\|\psi\|_\infty} \int_{U_2} u^{p^-} \psi(z) dz \\ &= \frac{c_a}{\|\psi\|_\infty} \int_\Omega u^{p^-} \psi(z) dz - \frac{c_a}{\|\psi\|_\infty} \int_{U_1} u^{p^-} \psi(z) dz \\ &\geq \frac{c_a}{\|\psi\|_\infty} \int_\Omega u^{p^-} \psi(z) dz - \frac{c_a}{\|\psi\|_\infty} \int_\Omega \psi(z) dz \\ &= \frac{c_a}{\|\psi\|_\infty} \int_\Omega u^{p^-} \psi(z) dz - \frac{c_a}{\|\psi\|_\infty}. \end{aligned}$$

Now, we apply the inverse Hölder inequality, we get

$$\int_\Omega u^{p^-} \psi(z) dy \geq \left(\int_\Omega u \psi dz \right)^{p^-}.$$

Substitute all the estimates in (3.3), we get

$$\frac{\partial^\alpha E}{\partial t^\alpha} \geq -\kappa E + \frac{c_a}{\|\psi\|_\infty} E^{p^-} - \frac{c_a}{\|\psi\|_\infty}.$$

Using the monotonicity and maximum principle for subdiffusion equation as in [14, 34], we have

$$\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} E' \geq \frac{\partial^\alpha E}{\partial t^\alpha} \geq -\kappa E + \frac{c_a}{\|\psi\|_\infty} E^{p^-} - \frac{c_a}{\|\psi\|_\infty}.$$

Finally, we get

$$E' \geq \frac{\Gamma(2-\alpha)}{t^{1-\alpha}} \left(-\kappa E + \frac{c_a}{\|\psi\|_\infty} E^{p^-} - \frac{c_a}{\|\psi\|_\infty} \right).$$

and the results follows by Lemma 2.2. Q.E.D.

4 Conclusion

In this paper, we estimated a finite-time blow-up of solutions of a time-fractional nonlocal reaction-diffusion equation. The proposed model involves with nonlocal reaction terms to describe the intra-specific competition, reproduction of the population and random mutations. The local existence and uniqueness of a solution achieved using the Banach fixed point theorem. We investigated the blow-up solutions of a time-fractional nonlocal reaction-diffusion with variable exponents using the Kaplan method.

Acknowledgments

The authors would like to express their sincere gratitude the referee for his/her careful reading and kind suggestions.

References

[1] R. Aboulaich, D. Meskine and A. Souissi, New diffusion models in image processing, Comput. Math. Appl., 56(2008) 874-882.

[2] B. Ahmad, M.S. Alhothuali, H.H. Alsulami, M. Kirane and S.Timoshin, On a time fractional reaction diffusion equation, Appl. Math. Comput., 257(2015) 199-204.

[3] A.A. Alikhanov, A priori estimates for solutions of boundary value problems for fractional-order equations, Differ. Equ., 46(2010) 660-666.

[4] A. Alsaedi, M. Kirane and R. Lassoued, Global existence and asymptotic behavior for a time fractional reaction-diffusion system, Comput. Math. Appl., 73 (2017) 957-958.

[5] B. Ahmad, M. S. Alhothuali, H. H. Alsulami, M. Kirane and S. Timoshin, On a time fractional reaction diffusion equation, Appl. Math. Comput., 257 (2015) 199-204.

[6] S.N. Antontsev and S.I. Shmarev, Parabolic equations with anisotropic nonstandard growth conditions, Internat. Ser. Numer. Math., 154(2007) 33-44.

[7] S.N. Antontsev and S.I. Shmarev, Existence and uniqueness of solutions of degenerate parabolic equations with variable exponents of nonlinearity, J. Math. Sci., 150(2008) 2289-2301.

[8] S.N. Antontsev and S.I. Shmarev, Elliptic equations with anisotropic nonlinearity and non-standard growth conditions, in: Handbook of Differential Equations, in: Stationary Partial Differential Equations, vol. 3, Elsevier, 2006.

[9] S.N. Antontsev and S. Shmarev, Blow-up of solutions to parabolic equations with non- standard growth conditions, J. Comput. Appl. Math., 234 (2010) 2633-2645.

[10] S.N. Antontsev and S. Shmarev, Blow-up solutions to parabolic equations with nonstandard growth conditions, CMAF University of Lisbon, Portugal, 02 (2009) 1-16.

[11] S. Antontsev and V. Zhikov, Higher integrability for parabolic equations of $p(x,t)$-Laplacian type, Adv. Differential Equations, 10(2005) 1053-1080.

[12] V. Bhuvaneswari, L. Shangerganesh and K. Balachandran, Global existence and blow up of solutions of quasilinear chemotaxis system, Math. Meth. Appl. Sci., 38 (2015) 3738-3746.

[13] A. Calsina, C. Perello and J. Saldana, Non-local reaction-diffusion equations modelling predator-prey coevolution, Publ. Mat., 32 (1994) 315-325.

[14] J. Cao, G. Song, J. Wang, Q. Shi and S. Sun, Blow-up and global solutions for a class of time fractional nonlinear reactiondiffusion equation with weakly spatial source, Appl. Math. Lett., 91(2019) 201-206.

[15] D.V. Cruz-Uribe, and A. Fiorenza, Variable Lebesgue spaces: foundations and harmonic analysis. Springer Science & Business Media, 2013.

[16] H. Di, Y. Shang and X. Peng, Blowup phenomena for a pseudoparabolic equation with variable exponents, Appl. Math. Lett., 64(2017), 67-73.

[17] L. Diening, P. Harjulehto, P. Hästö, and M. Ruzicka. Lebesgue and Sobolev spaces with variable exponents. Springer, 2011.

[18] D. Erdem, Blow-up of solutions to quasilinear parabolic equations, Appl. Math. Lett., 12 (1999), 65-69.

[19] J. Furter and M. Grinfield, Local vs. non-local interactions in populations dynamics, J. Math. Biol., 27(1989) 65-80.

[20] J.H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, Comput. Methods Appl. Mech. Engrg. 167(1998) 57-68.

[21] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[22] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and Applications of the Fractional Differential Equations, Elsevier, Amsterda, 2006.

[23] A. Kubica and M. Yamamoto, Initial-boundary value problems for fractional diffusion equations with time-dependent coeffcients, arXiv:1703.07160, 2017.

[24] Z. Li, Y. Luchko and M. Yamamoto, Analyticity of solutions to a distributed order time-fractional diffusion equation and its application to an inverse problem, Comput. Math. Appl., 73(2017) 1041-1052.

[25] Z. Li, Y. Liu and M. Yamamoto, Initial-boundary value problems for multi-term time-fractional diffusion equations with positive constant coefficients, Appl. Math. Comput., 257(2015) 381-397.

[26] Y. Luchko, Initial-boundary-value problems for the generalized multi-term time-fractional diffusion equation, J. Math. Anal. Appl. 374(2011) 538-548.

[27] Y. Luchko and M. Yamamoto, General time-fractional diffusion equation: some uniqueness and existence results for the initial-boundary value problems. Fract. Calc. Appl. Anal. 19(2016), 676-695.

[28] R.L. Magin, Fractional calculus models of complex dynamics in biological tissues, Comput. Math. Appl., 59(2010) 1586-1593.

[29] C.V. Pao, Nonlinear Parabolic and Elliptic Equations, Plenum Press, New York, 1992.

[30] J. P. Pinasco, Blow-up for parabolic and hyperbolic problems with variable exponents, Nonlinear Anal., 71(2009), 1094-1099.

[31] A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov and A.P. Mikhailov, Blow-up in Quasilinear Parabolic Equations, Walter de Gruyter, Berlin, New York, 1995.

[32] L. Shangerganesh, N. Nyamoradi, G. Sathishkumar, and S. Karthikeyan, Finite-time blow-up of solutions to a cancer invasion mathematical model with haptotaxis effects. Comput. Math. Appl., 77 (2019) 2242-2254.

[33] V. Volpert, Elliptic Partial Differential Equations: Volume 2: Reaction-Diffusion Equations, Springer, 2014.

[34] Q. Xu and Y. Xu, Extremely low order time-fractional differential equation and application in combustion process, Commun. Nonlinear Sci. Numer. Simul., 64 (2018), 135-148.

[35] Y. Yamamoto, Weak solutions to non-homogeneous boundary value problems for time-fractional diffusion equations, J. Math. Anal. Appl., 460 (2018), 365-81.

[36] Q.G. Zhang and Y.N. Li, The critical exponent for a time fractional diffusion equation with nonlinear memory, Math. Meth. Appl. Sci., 41 (2018), 6443-6456.

[37] Q.G. Zhang and Y.N. Li, The critical exponents for a time fractional diffusion equation with nonlinear memory in a bounded domain, Appl. Math. Lett., 92(2019) 1-7.

[38] Y. Zhou, J. Manimaran, L. Shangerganesh and A. Debbouche, Weakness and Mittag-Leffler Stability of Solutions for Time-Fractional Keller-Segel Models, Int. J. Nonlinear Sci. Numer. Simul., 19 (2018), 753-761.

[39] Y. Zhou, L. Shangerganesh, J. Manimaran and A. Debbouche, A class of time fractional reaction-diffusion equation with nonlocal boundary condition, Math. Meth. Appl. Sci., 41 (2018), 2987-2999.

[40] Y.Zhou and L. Peng, Weak solutions of the time-fractional Navier-Stokes equations and optimal control, Comput. Math. Appl., 73(2017) 1016-1027.