Some new refinement of Hermite-Hadamard type inequalities and their applications

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Abstract

In this paper first, we prove some new refinement of Hermite-Hadamard type inequalities for the convex function f. Second, by using five new integral identities, we present some new Riemann-Liouville fractional trapezoid and midpoint type inequalities. Third, using these results, we present applications to f-divergence measures. At the end, some new bounds for special means of different positive real numbers and new error estimates for the trapezoidal and midpoint formula are provided as well. These results give us the generalizations and improvements of the earlier results.

2010 Mathematics Subject Classification. **26A51**. 26A33, 26D07, 26D10, 26D15
Keywords. Hermite-Hadamard inequality, Hölder inequality, power mean inequality, Riemann-Liouville fractional integrals, convex function, special means.

1 Introduction

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° is the interior of I. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The class of convex functions is well known in the literature and is usually defined in the following way: Let I be an interval in $\mathbb{R}$, then a function $f : I \longrightarrow \mathbb{R}$, is said to be convex on I if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$. Also, we say that f is concave, if the inequality in (1.1) holds in the reverse direction. There are several generalizations of the convex function.

Many important inequalities have been obtained for this class of functions but here we will present only one of them.
If $f : I \longrightarrow \mathbb{R}$ is a convex function on the interval I, then for any $a, b \in I$ with $a \neq b$, we have the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

Both the inequalities hold in reverse direction if the function f is concave on the interval I. This remarkable result was given in ([7], 1893) and is well known in the literature as Hermite-Hadamard

Tbilisi Mathematical Journal 12(4) (2019), pp. 159–188.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 08 April 2019.
Accepted for publication: 10 November 2019.

inequality. Since, its discovery this inequality has become the center of interest for many prolific researchers and received a considerable attention. Also, a number of extensions, generalizations and variants of (1.2) have been provided in the theory of mathematical inequalities. For example, see [1,3–5,9–11,15,18,21] and the references cited therein. In [4,16], the authors used the following equality to obtain trapezoid type inequalities and some applications:

Lemma 1.1. Let $a, b \in I$ with $a < b$ and $f : I^\circ \longrightarrow \mathbb{R}$ be a differentiable mapping (I° is the interior of I). If $f' \in L_1[a, b]$, then we have

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt. \tag{1.3}$$

In [12], Kirmaci used the following equality to obtain midpoint type inequalities and some applications:

Lemma 1.2. Let $a, b \in I$ with $a < b$ and $f : I^\circ \longrightarrow \mathbb{R}$ be a differentiable mapping. If $f' \in L_1[a, b]$, then we have

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right)\\ &= (b-a)\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 (t-1)f'(ta+(1-t)b)dt.\end{aligned} \tag{1.4}$$

Definition 1.3. [17] A function f defined on I has a support at $x_0 \in I$ if there exists an affine functions $A(x) = f(x_0) + m(x - x_0)$ such that $A(x) \leq f(x)$ for all $x \in I$.

Theorem 1.4. [17] $f : (a, b) \longrightarrow \mathbb{R}$ is a convex function if and only if there is at least one line of support for f at each $x_0 \in (a, b)$. The graph of the support function A is called a line of support for f at x_0.

Following definitions of the left and right side Riemann-Liouville fractional integrals are well known in the literature.

Definition 1.5. Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a$$

and

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u}u^{\alpha-1}du$. Here $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.
In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes, see [6,14] and the references cited therein.
In [19], Sarikaya et al. proved the following Hermite-Hadamard type fractional integral inequality:

Theorem 1.6. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b) + J^{\alpha}_{b^-}f(a)\right] \leq \frac{f(a)+f(b)}{2},$$

with $\alpha > 0$.

The main purpose of this paper is to give in Section 2 some new refinement of Hermite-Hadamard type inequalities for the convex function f. In Section 3, we present five useful integral identities. In Section 4, by using five lemmas, we obtain some new Riemann-Liouville fractional trapezoid and midpoint type inequalities. In Section 5, using these results, we present applications to f-divergence measures. In Section 6, some bounds for special means of different positive real numbers and new error estimates for the trapezoidal and midpoint formula are provided as well.

2 Main results

New refinement of Hermite-Hadamard's inequalities can be represented in fractional integral forms as follows.

Theorem 2.1. Let $0 < r \leq 1$, $\rho > 0$ and $f : \left[(ra)^{\rho}, \left(\frac{b}{r}\right)^{\rho}\right] \longrightarrow \mathbb{R}$ be a convex function with $0 < a < b$. If $f \in L\left[(ra)^{\rho}, \left(\frac{b}{r}\right)^{\rho}\right]$, then the following inequalities for fractional integrals hold:

$$f\left(\frac{(ra)^{\rho}+b^{\rho}}{r^{\rho}+1}\right) \leq \frac{\Gamma(\alpha+1)}{(r^{\rho}+1)(b^{\rho}-(ra)^{\rho})^{\alpha}}\left[J^{\alpha}_{((ra)^{\rho})^+}f(b^{\rho}) + J^{\alpha}_{(b^{\rho})^-}f((ra)^{\rho})\right]$$

$$\leq \frac{f((ra)^{\rho})+f(b^{\rho})}{r^{\rho}+1}, \tag{2.1}$$

with $\alpha > 0$.

Proof. Since f is a convex function, then for any $x^{\rho}, y^{\rho} \in \left[(ra)^{\rho}, \left(\frac{b}{r}\right)^{\rho}\right]$, we have

$$f\left(\frac{(rx)^{\rho}+y^{\rho}}{r^{\rho}+1}\right) \leq \frac{f((rx)^{\rho})+f(y^{\rho})}{r^{\rho}+1}. \tag{2.2}$$

Let us define, respectively $x^{\rho} = (ta)^{\rho} + (1-t^{\rho})\left(\frac{b}{r}\right)^{\rho}$ and $y^{\rho} = r^{\rho}(1-t^{\rho})a^{\rho} + (tb)^{\rho}$. From (2.2), we get

$$(r^{\rho}+1)f\left(\frac{(ra)^{\rho}+b^{\rho}}{r^{\rho}+1}\right) \leq f((rta)^{\rho} + (1-t^{\rho})b^{\rho}) + f(r^{\rho}(1-t^{\rho})a^{\rho} + (tb)^{\rho}). \tag{2.3}$$

Multiplying both sides of (2.3) by $t^{\alpha\rho-1}$, then integrating the resulting inequality with respect to t over $(0, 1)$, we obtain

$$\left(\frac{r^{\rho}+1}{\alpha\rho}\right)f\left(\frac{(ra)^{\rho}+b^{\rho}}{r^{\rho}+1}\right)$$

$$\leq \int_0^1 t^{\alpha\rho-1}f((rta)^{\rho} + (1-t^{\rho})b^{\rho})dt + \int_0^1 t^{\alpha\rho-1}f(r^{\rho}(1-t^{\rho})a^{\rho} + (tb)^{\rho})dt$$

$$= \int_{(ra)^\rho}^{b^\rho} \left(\frac{b^\rho - u}{b^\rho - (ra)^\rho}\right)^{\alpha-1} f(u)\frac{du}{\rho(b^\rho - (ra)^\rho)}$$
$$+ \int_{(ra)^\rho}^{b^\rho} \left(\frac{u - (ra)^\rho}{b^\rho - (ra)^\rho}\right)^{\alpha-1} f(u)\frac{du}{\rho(b^\rho - (ra)^\rho)}$$
$$= \frac{\Gamma(\alpha)}{\rho(b^\rho - (ra)^\rho)^\alpha}\Big[J^\alpha_{((ra)^\rho)^+} f(b^\rho) + J^\alpha_{(b^\rho)^-} f((ra)^\rho)\Big]. \tag{2.4}$$

Then multiplying both sides of (2.4) by the factor $\frac{\alpha\rho}{r^\rho+1}$, we deduce the left-hand side inequality in (2.1).

For the proof of the right-hand side inequality in (2.1), we first note that if f is a convex function, then

$$f((rta)^\rho + (1-t^\rho)b^\rho) \le t^\rho f((ra)^\rho) + (1-t^\rho)f(b^\rho) \tag{2.5}$$

and

$$f(r^\rho(1-t^\rho)a^\rho + (tb)^\rho) \le (1-t^\rho)f((ra)^\rho) + t^\rho f(b^\rho). \tag{2.6}$$

By adding inequalities (2.5) and (2.6), we have

$$f((rta)^\rho + (1-t^\rho)b^\rho) + f(r^\rho(1-t^\rho)a^\rho + (tb)^\rho)$$
$$\le t^\rho f((ra)^\rho) + (1-t^\rho)f(b^\rho) + (1-t^\rho)f((ra)^\rho) + t^\rho f(b^\rho). \tag{2.7}$$

Then multiplying both sides of (2.7) by $t^{\alpha\rho-1}$ and integrating the resulting inequality with respect to t over $(0,1)$, we obtain

$$\int_0^1 t^{\alpha\rho-1} f((rta)^\rho + (1-t^\rho)b^\rho)dt + \int_0^1 t^{\alpha\rho-1} f(r^\rho(1-t^\rho)a^\rho + (tb)^\rho)dt$$
$$\le \int_0^1 t^{\alpha\rho-1}\big[t^\rho f((ra)^\rho) + (1-t^\rho)f(b^\rho)\big]dt + \int_0^1 t^{\alpha\rho-1}\big[(1-t^\rho)f((ra)^\rho) + t^\rho f(b^\rho)\big]dt \tag{2.8}$$
$$= \frac{f((ra)^\rho) + f(b^\rho)}{\alpha\rho}.$$

Then multiplying both sides of (2.8) by the factor $\frac{\alpha\rho}{r^\rho+1}$, we deduce the right-hand side inequality in (2.1). The proof of Theorem 2.1 is completed. Q.E.D.

Corollary 2.2. Under the assumptions of Theorem 2.1, if we put $r = \rho = 1$, we get (see [19], Theorem 2).

Corollary 2.3. Under the assumptions of Theorem 2.1, if we put $\alpha = 1$, we get the following new refinement of Hermite-Hadamard type via classical integrals:

$$f\left(\frac{(ra)^\rho + b^\rho}{r^\rho+1}\right) \le \frac{2}{(r^\rho+1)(b^\rho - (ra)^\rho)}\int_{(ra)^\rho}^{b^\rho} f(t)dt \le \frac{f((ra)^\rho) + f(b^\rho)}{r^\rho+1}. \tag{2.9}$$

Some other versions of Hermite-Hadamard's inequalities can be represented as follows.

Theorem 2.4. Let $0 < r \leq 1$, $\rho > 0$ and $f : \left[a^\rho, \left(\frac{b}{r}\right)^\rho\right] \longrightarrow \mathbb{R}$ be a convex function with $0 < a < b$. If $f \in L\left[a^\rho, \left(\frac{b}{r}\right)^\rho\right]$, then the following inequalities for the left Riemann-Liouville fractional integral holds:

$$f\left(\frac{\alpha(ra)^\rho + b^\rho}{r^\rho(\alpha+1)}\right) \leq \Gamma(\alpha+1)\left(\frac{r^\rho}{b^\rho-(ra)^\rho}\right)^\alpha J^\alpha_{(a^\rho)^+} f\left(\left(\frac{b}{r}\right)^\rho\right) \leq \frac{\alpha f(a^\rho) + f\left(\left(\frac{b}{r}\right)^\rho\right)}{\alpha+1}, \tag{2.10}$$

with $\alpha > 0$.

Proof. Since f is convex on $\left[a^\rho, \left(\frac{b}{r}\right)^\rho\right]$, using Theorem 1.4, there is at least one line of support

$$A(x) = f\left(\frac{\alpha(ra)^\rho + b^\rho}{r^\rho(\alpha+1)}\right) + m\left(x - \frac{\alpha(ra)^\rho + b^\rho}{r^\rho(\alpha+1)}\right) \leq f(x), \tag{2.11}$$

for any $m \in \left[f^-\left(\frac{\alpha(ra)^\rho+b^\rho}{r^\rho(\alpha+1)}\right), f^+\left(\frac{\alpha(ra)^\rho+b^\rho}{r^\rho(\alpha+1)}\right)\right]$, where $f^-(x)$ and $f^+(x)$ are the left and right derivatives for all $x \in \left[a^\rho, \left(\frac{b}{r}\right)^\rho\right]$. From (2.11), we have

$$\begin{aligned} &A\left((ta)^\rho + (1-t^\rho)\left(\frac{b}{r}\right)^\rho\right) \\ &= f\left(\frac{\alpha(ra)^\rho + b^\rho}{r^\rho(\alpha+1)}\right) + m\left((ta)^\rho + (1-t^\rho)\left(\frac{b}{r}\right)^\rho - \frac{\alpha(ra)^\rho + b^\rho}{r^\rho(\alpha+1)}\right) \\ &\leq f\left((ta)^\rho + (1-t^\rho)\left(\frac{b}{r}\right)^\rho\right), \quad \forall\, t \in [0,1]. \end{aligned} \tag{2.12}$$

Multiplying both sides of (2.12) by $\alpha\rho t^{\alpha\rho-1}$, then integrating the resulting inequality with respect to t over $(0,1)$, we obtain

$$\begin{aligned} &\alpha\rho\int_0^1 t^{\alpha\rho-1} A\left((ta)^\rho + (1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)dt \\ &= \alpha\rho\int_0^1 t^{\alpha\rho-1}\left[f\left(\frac{\alpha(ra)^\rho + b^\rho}{r^\rho(\alpha+1)}\right) + m\left((ta)^\rho + (1-t^\rho)\left(\frac{b}{r}\right)^\rho - \frac{\alpha(ra)^\rho + b^\rho}{r^\rho(\alpha+1)}\right)\right]dt \\ &= \alpha\rho f\left(\frac{\alpha(ra)^\rho + b^\rho}{r^\rho(\alpha+1)}\right)\int_0^1 t^{\alpha\rho-1}dt \\ &+m\left\{\int_0^1 \alpha\rho\left[t^{\alpha\rho-1+\rho}a^\rho + \left(t^{\alpha\rho-1} - t^{\alpha\rho-1+\rho}\right)\left(\frac{b}{r}\right)^\rho\right]dt - \left(\frac{\alpha(ra)^\rho + b^\rho}{r^\rho(\alpha+1)}\right)\alpha\rho\int_0^1 t^{\alpha\rho-1}dt\right\} \\ &= f\left(\frac{\alpha(ra)^\rho + b^\rho}{r^\rho(\alpha+1)}\right) \leq \alpha\rho\int_0^1 t^{\alpha\rho-1} f\left((ta)^\rho + (1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)dt \\ &= \Gamma(\alpha+1)\left(\frac{r^\rho}{b^\rho-(ra)^\rho}\right)^\alpha J^\alpha_{(a^\rho)^+} f\left(\left(\frac{b}{r}\right)^\rho\right). \end{aligned} \tag{2.13}$$

On the other hand, using the convexity of f on $\left[a^\rho, \left(\frac{b}{r}\right)^\rho\right]$, we have

$$f\left((ta)^\rho + (1-t^\rho)\left(\frac{b}{r}\right)^\rho\right) \leq t^\rho f(a^\rho) + (1-t^\rho) f\left(\left(\frac{b}{r}\right)^\rho\right), \quad \forall\, t \in [0,1]. \tag{2.14}$$

Multiplying both sides of (2.14) by $\alpha\rho t^{\alpha\rho-1}$, then integrating the resulting inequality with respect to t over $(0,1)$, we get

$$\begin{aligned} &\Gamma(\alpha+1)\left(\frac{r^\rho}{b^\rho-(ra)^\rho}\right)^\alpha J^\alpha_{(a^\rho)^+} f\left(\left(\frac{b}{r}\right)^\rho\right) \\ &\leq \alpha\rho f(a^\rho)\int_0^1 t^{\alpha\rho-1+\rho}dt + \alpha\rho f\left(\left(\frac{b}{r}\right)^\rho\right)\int_0^1 \left(t^{\alpha\rho-1} - t^{\alpha\rho-1+\rho}\right)dt \\ &= \frac{\alpha f(a^\rho) + f\left(\left(\frac{b}{r}\right)^\rho\right)}{\alpha+1}. \end{aligned} \tag{2.15}$$

By using (2.13) and (2.15), we have (2.10). The proof of Theorem 2.4 is completed. Q.E.D.

Corollary 2.5. Under the assumptions of Theorem 2.4, if we put $r = 1$, we get the following inequality for the left Riemann-Liouville fractional integral

$$f\left(\frac{\alpha a^\rho + b^\rho}{\alpha+1}\right) \leq \frac{\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha} J^\alpha_{(a^\rho)^+} f(b^\rho) \leq \frac{\alpha f(a^\rho) + f(b^\rho)}{\alpha+1}. \tag{2.16}$$

Remark 2.6. In Corollary 2.5, if we put $\alpha = \rho = 1$, we get (1.2).

Theorem 2.7. Let $0 < r \leq 1$, $\rho > 0$ and $f : [(ra)^\rho, b^\rho] \longrightarrow \mathbb{R}$ be a convex function with $0 < a < b$. If $f \in L[(ra)^\rho, b^\rho]$, then the following inequalities for the right Riemann-Liouville fractional integral holds:

$$f\left(\frac{(ra)^\rho + \alpha b^\rho}{\alpha+1}\right) \leq \frac{\Gamma(\alpha+1)}{(b^\rho-(ra)^\rho)^\alpha} J^\alpha_{(b^\rho)^-} f((ra)^\rho) \leq \frac{f((ra)^\rho) + \alpha f(b^\rho)}{\alpha+1}, \tag{2.17}$$

with $\alpha > 0$.

Proof. Since f is convex on $[(ra)^\rho, b^\rho]$, using Theorem 1.4, there is at least one line of support

$$A(x) = f\left(\frac{(ra)^\rho + \alpha b^\rho}{\alpha+1}\right) + m\left(x - \frac{(ra)^\rho + \alpha b^\rho}{\alpha+1}\right) \leq f(x), \tag{2.18}$$

for any $m \in \left[f^-\left(\frac{(ra)^\rho+\alpha b^\rho}{\alpha+1}\right), f^+\left(\frac{(ra)^\rho+\alpha b^\rho}{\alpha+1}\right)\right]$, where $f^-(x)$ and $f^+(x)$ are the left and right derivatives for all $x \in [(ra)^\rho, b^\rho]$. From (2.18), we have

$$\begin{aligned} A(r^\rho(1-t^\rho)a^\rho + (tb)^\rho) &= f\left(\frac{(ra)^\rho + \alpha b^\rho}{\alpha+1}\right) + m\left(r^\rho(1-t^\rho)a^\rho + (tb)^\rho - \frac{(ra)^\rho + \alpha b^\rho}{\alpha+1}\right) \\ &\leq f(r^\rho(1-t^\rho)a^\rho + (tb)^\rho), \quad \forall\, t \in [0,1]. \end{aligned} \tag{2.19}$$

Multiplying both sides of (2.19) by $\alpha\rho t^{\alpha\rho-1}$, then integrating the resulting inequality with respect to t over $(0,1)$, we obtain

$$\alpha\rho\int_0^1 t^{\alpha\rho-1} A\left(r^\rho(1-t^\rho)a^\rho + (tb)^\rho\right)dt$$

$$= \alpha\rho \int_0^1 t^{\alpha\rho-1}\left[f\left(\frac{(ra)^\rho+\alpha b^\rho}{\alpha+1}\right)+m\left(r^\rho(1-t^\rho)a^\rho+(tb)^\rho-\frac{(ra)^\rho+\alpha b^\rho}{\alpha+1}\right)\right]dt$$

$$= \alpha\rho f\left(\frac{(ra)^\rho+\alpha b^\rho}{\alpha+1}\right)\int_0^1 t^{\alpha\rho-1}dt$$

$$+m\left\{\int_0^1 \alpha\rho\left[t^{\alpha\rho-1+\rho}b^\rho+r^\rho\left(t^{\alpha\rho-1}-t^{\alpha\rho-1+\rho}\right)a^\rho\right]dt-\left(\frac{(ra)^\rho+\alpha b^\rho}{\alpha+1}\right)\alpha\rho\int_0^1 t^{\alpha\rho-1}dt\right\}$$

$$= f\left(\frac{(ra)^\rho+\alpha b^\rho}{\alpha+1}\right)\leq \alpha\rho\int_0^1 t^{\alpha\rho-1}f\left(r^\rho(1-t^\rho)a^\rho+(tb)^\rho\right)dt \tag{2.20}$$

$$= \frac{\Gamma(\alpha+1)}{(b^\rho-(ra)^\rho)^\alpha}J^\alpha_{(b^\rho)^-}f((ra)^\rho).$$

On the other hand, using the convexity of f on $[(ra)^\rho, b^\rho]$, we have

$$f\left(r^\rho(1-t^\rho)a^\rho+(tb)^\rho\right)\leq t^\rho f(b^\rho)+(1-t^\rho)f\left((ra)^\rho\right),\ \ \forall\, t\in[0,1]. \tag{2.21}$$

Multiplying both sides of (2.21) by $\alpha\rho t^{\alpha\rho-1}$, then integrating the resulting inequality with respect to t over $(0,1)$, we get

$$\frac{\Gamma(\alpha+1)}{(b^\rho-(ra)^\rho)^\alpha}J^\alpha_{(b^\rho)^-}f((ra)^\rho)$$

$$\leq \alpha\rho f(b^\rho)\int_0^1 t^{\alpha\rho-1+\rho}dt+\alpha\rho f\left((ra)^\rho\right)\int_0^1\left(t^{\alpha\rho-1}-t^{\alpha\rho-1+\rho}\right)dt \tag{2.22}$$

$$= \frac{f((ra)^\rho)+\alpha f(b^\rho)}{\alpha+1}.$$

By using (2.20) and (2.22), we have (2.17). The proof of Theorem 2.7 is completed. Q.E.D.

Corollary 2.8. Under the assumptions of Theorem 2.7, if we put $r=\rho=1$, we get (see [13], Theorem 2.1).

3 Lemmas

In this section we will prove five new useful integral identities as follows.

Lemma 3.1. Let $0<r\leq 1$, $\rho>0$ and $f:\left[a^\rho,\left(\frac{b}{r}\right)^\rho\right]\longrightarrow\mathbb{R}$ be a differentiable function on $\left(a^\rho,\left(\frac{b}{r}\right)^\rho\right)$ with $0<a<b$. If $f'\in L\left[a^\rho,\left(\frac{b}{r}\right)^\rho\right]$, then the following equality holds:

$$\frac{\alpha f(a^\rho)+f\left(\left(\frac{b}{r}\right)^\rho\right)}{\alpha+1}-\Gamma(\alpha+1)\left(\frac{r^\rho}{b^\rho-(ra)^\rho}\right)^\alpha J^\alpha_{(a^\rho)^+}f\left(\left(\frac{b}{r}\right)^\rho\right)$$

$$= \frac{b^\rho-(ra)^\rho}{r^\rho(\alpha+1)}\int_0^1\left[1-(\alpha+1)t^{\alpha\rho}\right]f'\left((ta)^\rho+(1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)dt, \tag{3.1}$$

with $\alpha>0$.
We denote

$$I_1:=\frac{b^\rho-(ra)^\rho}{r^\rho(\alpha+1)}\int_0^1\left[1-(\alpha+1)t^{\alpha\rho}\right]f'\left((ta)^\rho+(1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)dt. \tag{3.2}$$

Proof. Integrating by parts I_1, we have

$$I_1 = \frac{b^\rho - (ra)^\rho}{r^\rho(\alpha+1)}\Bigg[\int_0^1 f'\left((ta)^\rho + (1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)dt$$

$$-(\alpha+1)\int_0^1 t^{\alpha\rho} f'\left((ta)^\rho + (1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)dt\Bigg]$$

$$= \frac{b^\rho - (ra)^\rho}{r^\rho(\alpha+1)}\Bigg\{\frac{r^\rho}{b^\rho-(ra)^\rho}\left[f\left(\left(\frac{b}{r}\right)^\rho\right) - f(a^\rho)\right]$$

$$-(\alpha+1)\Bigg[\frac{t^{\alpha\rho} f\left((ta)^\rho + (1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)}{a^\rho - \left(\frac{b}{r}\right)^\rho}\Bigg|_0^1$$

$$-\frac{\alpha\rho}{a^\rho - \left(\frac{b}{r}\right)^\rho}\int_0^1 t^{\alpha\rho-1} f\left((ta)^\rho + (1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)dt\Bigg]\Bigg\}$$

$$= \frac{\alpha f(a^\rho) + f\left(\left(\frac{b}{r}\right)^\rho\right)}{\alpha+1} - \Gamma(\alpha+1)\left(\frac{r^\rho}{b^\rho-(ra)^\rho}\right)^\alpha J^\alpha_{(a^\rho)^+} f\left(\left(\frac{b}{r}\right)^\rho\right).$$

The proof of Lemma 3.1 is completed. Q.E.D.

Remark 3.2. In Lemma 3.1, if one takes $r = \rho = \alpha = 1$, one has the Lemma 1.1.

Lemma 3.3. Let $0 < r \le 1$, $\rho > 0$ and $f : [(ra)^\rho, b^\rho] \longrightarrow \mathbb{R}$ be a differentiable function on $((ra)^\rho, b^\rho)$ with $0 < a < b$. If $f' \in L[(ra)^\rho, b^\rho]$, then the following equality holds:

$$\frac{f((ra)^\rho) + \alpha f(b^\rho)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(b^\rho-(ra)^\rho)^\alpha} J^\alpha_{(b^\rho)^-} f((ra)^\rho)$$

$$= \frac{b^\rho - (ra)^\rho}{\alpha+1}\int_0^1 \left[(\alpha+1)t^{\alpha\rho} - 1\right] f'(r^\rho(1-t^\rho)a^\rho + (tb)^\rho)dt, \tag{3.3}$$

with $\alpha > 0$.
We denote

$$I_2 := \frac{b^\rho - (ra)^\rho}{\alpha+1}\int_0^1 \left[(\alpha+1)t^{\alpha\rho} - 1\right] f'(r^\rho(1-t^\rho)a^\rho + (tb)^\rho)dt. \tag{3.4}$$

Proof. Integrating by parts I_2, we have

$$I_2 = \frac{b^\rho - (ra)^\rho}{\alpha+1}\left[(\alpha+1)\int_0^1 t^{\alpha\rho} f'(r^\rho(1-t^\rho)a^\rho + (tb)^\rho)dt - \int_0^1 f'(r^\rho(1-t^\rho)a^\rho + (tb)^\rho)dt\right]$$

$$= \frac{b^\rho - (ra)^\rho}{\alpha+1}\Bigg\{(\alpha+1)\left[\frac{t^{\alpha\rho} f(r^\rho(1-t^\rho)a^\rho + (tb)^\rho)}{b^\rho-(ra)^\rho}\Bigg|_0^1\right.$$

$$-\frac{\alpha\rho}{b^\rho-(ra)^\rho}\int_0^1 t^{\alpha\rho-1}f(r^\rho(1-t^\rho)a^\rho+(tb)^\rho)dt\Bigg]-\frac{f(b^\rho)-f((ra)^\rho)}{b^\rho-(ra)^\rho}\Bigg\}$$
$$=\frac{f((ra)^\rho)+\alpha f(b^\rho)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b^\rho-(ra)^\rho)^\alpha}J^\alpha_{(b^\rho)^-}f((ra)^\rho).$$

The proof of Lemma 3.3 is completed. Q.E.D.

Remark 3.4. In Lemma 3.3, if one takes $r=\rho=1$, we get (see [13], Lemma 3.1).

Lemma 3.5. Let $0<r\leq 1$, $\rho>0$ and $f:[(ra)^\rho,b^\rho]\longrightarrow\mathbb{R}$ be a differentiable function on $((ra)^\rho,b^\rho)$ with $0<a<b$. If $f'\in L[(ra)^\rho,b^\rho]$, then the following equality holds:

$$\frac{f((ra)^\rho)+f(b^\rho)}{r^\rho+1}-\frac{\Gamma(\alpha+1)}{(r^\rho+1)(b^\rho-(ra)^\rho)^\alpha}\Big[J^\alpha_{((ra)^\rho)^+}f(b^\rho)+J^\alpha_{(b^\rho)^-}f((ra)^\rho)\Big]$$
$$=\frac{b^\rho-(ra)^\rho}{r^\rho+1}\int_0^1\big[t^{\alpha\rho}-(1-t)^{\alpha\rho}\big]f'(r^\rho(1-t^\rho)a^\rho+(tb)^\rho)dt, \tag{3.5}$$

with $\alpha>0$.
We denote

$$I_3:=\frac{b^\rho-(ra)^\rho}{r^\rho+1}\int_0^1\big[t^{\alpha\rho}-(1-t)^{\alpha\rho}\big]f'(r^\rho(1-t^\rho)a^\rho+(tb)^\rho)dt. \tag{3.6}$$

Proof. Integrating by parts I_3, we have

$$I_3=\frac{b^\rho-(ra)^\rho}{r^\rho+1}$$
$$\times\left[\int_0^1 t^{\alpha\rho}f'(r^\rho(1-t^\rho)a^\rho+(tb)^\rho)dt-\int_0^1(1-t)^{\alpha\rho}f'(r^\rho(1-t^\rho)a^\rho+(tb)^\rho)dt\right]$$
$$=\frac{b^\rho-(ra)^\rho}{r^\rho+1}\Bigg\{\Bigg[\frac{t^{\alpha\rho}f(r^\rho(1-t^\rho)a^\rho+(tb)^\rho)}{b^\rho-(ra)^\rho}\Bigg|_0^1$$
$$-\frac{\alpha\rho}{b^\rho-(ra)^\rho}\int_0^1 t^{\alpha\rho-1}f(r^\rho(1-t^\rho)a^\rho+(tb)^\rho)dt\Bigg]$$
$$-\Bigg[\frac{(1-t)^{\alpha\rho}f(r^\rho(1-t^\rho)a^\rho+(tb)^\rho)}{b^\rho-(ra)^\rho}\Bigg|_0^1$$
$$+\frac{\alpha\rho}{b^\rho-(ra)^\rho}\int_0^1(1-t)^{\alpha\rho-1}f(r^\rho(1-t^\rho)a^\rho+(tb)^\rho)dt\Bigg]\Bigg\}$$
$$=\frac{f((ra)^\rho)+f(b^\rho)}{r^\rho+1}-\frac{\Gamma(\alpha+1)}{(r^\rho+1)(b^\rho-(ra)^\rho)^\alpha}\Big[J^\alpha_{((ra)^\rho)^+}f(b^\rho)+J^\alpha_{(b^\rho)^-}f((ra)^\rho)\Big].$$

The proof of Lemma 3.5 is completed. Q.E.D.

Lemma 3.6. Let $0 < r \leq 1$, $\rho > 0$ and $f : \left[a^\rho, \left(\frac{b}{r}\right)^\rho\right] \longrightarrow \mathbb{R}$ be a differentiable function on $\left(a^\rho, \left(\frac{b}{r}\right)^\rho\right)$ with $0 < a < b$. If $f' \in L\left[a^\rho, \left(\frac{b}{r}\right)^\rho\right]$, then the following equality holds:

$$\Gamma(\alpha+1)\left(\frac{r^\rho}{b^\rho-(ra)^\rho}\right)^\alpha J^\alpha_{(a^\rho)^+} f\left(\left(\frac{b}{r}\right)^\rho\right) - f\left(\frac{\alpha(ra)^\rho+b^\rho}{r^\rho(\alpha+1)}\right) \tag{3.7}$$

$$= \frac{b^\rho-(ra)^\rho}{r^\rho}$$

$$\times\left[\int_0^1 t^{\alpha\rho} f'\left((ta)^\rho+(1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)dt - \int_{\frac{\alpha}{\alpha+1}}^1 f'\left((ta)^\rho+(1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)dt\right],$$

with $\alpha > 0$.
We denote

$$I_4 := \frac{b^\rho-(ra)^\rho}{r^\rho}$$

$$\times\left[\int_0^1 t^{\alpha\rho} f'\left((ta)^\rho+(1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)dt - \int_{\frac{\alpha}{\alpha+1}}^1 f'\left((ta)^\rho+(1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)dt\right]. \tag{3.8}$$

Proof. Integrating by parts I_4, we have

$$I_4 = \frac{b^\rho-(ra)^\rho}{r^\rho}\left\{\left[\frac{t^{\alpha\rho} f\left((ta)^\rho+(1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)}{a^\rho-\left(\frac{b}{r}\right)^\rho}\right|_0^1 - \frac{\alpha\rho}{a^\rho-\left(\frac{b}{r}\right)^\rho}\int_0^1 t^{\alpha\rho-1} f\left((ta)^\rho+(1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)dt\right]$$

$$-\frac{r^\rho}{b^\rho-(ra)^\rho}\left[f\left(\frac{\alpha(ra)^\rho+b^\rho}{r^\rho(\alpha+1)}\right)-f(a^\rho)\right]\Bigg\}$$

$$= \Gamma(\alpha+1)\left(\frac{r^\rho}{b^\rho-(ra)^\rho}\right)^\alpha J^\alpha_{(a^\rho)^+} f\left(\left(\frac{b}{r}\right)^\rho\right) - f\left(\frac{\alpha(ra)^\rho+b^\rho}{r^\rho(\alpha+1)}\right).$$

The proof of Lemma 3.6 is completed. Q.E.D.

Lemma 3.7. Let $0 < r \leq 1$, $\rho > 0$ and $f : [(ra)^\rho, b^\rho] \longrightarrow \mathbb{R}$ be a differentiable function on $((ra)^\rho, b^\rho)$ with $0 < a < b$. If $f' \in L[(ra)^\rho, b^\rho]$, then the following equality holds:

$$\frac{\Gamma(\alpha+1)}{(b^\rho-(ra)^\rho)^\alpha} J^\alpha_{(b^\rho)^-} f((ra)^\rho) - f\left(\frac{(ra)^\rho+\alpha b^\rho}{\alpha+1}\right)$$

$$= (b^\rho-(ra)^\rho)\left[\int_0^1 -t^{\alpha\rho} f'(r^\rho(1-t^\rho)a^\rho+(tb)^\rho)dt + \int_{\frac{\alpha}{\alpha+1}}^1 f'(r^\rho(1-t^\rho)a^\rho+(tb)^\rho)dt\right], \tag{3.9}$$

with $\alpha > 0$.
We denote

$$I_5 := (b^\rho-(ra)^\rho)\left[\int_0^1 -t^{\alpha\rho} f'(r^\rho(1-t^\rho)a^\rho+(tb)^\rho)dt + \int_{\frac{\alpha}{\alpha+1}}^1 f'(r^\rho(1-t^\rho)a^\rho+(tb)^\rho)dt\right]. \tag{3.10}$$

Proof. Integrating by parts I_5, we have

$$I_5 = (b^\rho - (ra)^\rho)\Bigg\{ -\left[\frac{t^{\alpha\rho} f(r^\rho(1-t^\rho)a^\rho + (tb)^\rho)}{b^\rho - (ra)^\rho}\right|_0^1$$

$$-\frac{\alpha\rho}{b^\rho - (ra)^\rho}\int_0^1 t^{\alpha\rho-1} f(r^\rho(1-t^\rho)a^\rho + (tb)^\rho)dt\Bigg]$$

$$+\frac{1}{b^\rho - (ra)^\rho}\left[f(b^\rho) - f\left(\frac{(ra)^\rho + \alpha b^\rho}{\alpha+1}\right)\right]\Bigg\}$$

$$= \frac{\Gamma(\alpha+1)}{(b^\rho - (ra)^\rho)^\alpha} J^\alpha_{(b^\rho)^-} f((ra)^\rho) - f\left(\frac{(ra)^\rho + \alpha b^\rho}{\alpha+1}\right).$$

The proof of Lemma 3.7 is completed. Q.E.D.

Remark 3.8. In Lemma 3.7, if one takes $r = \rho = 1$, we get (see [13], Lemma 3.2).

4 Fractional trapezoid and midpoint type inequalities

In this section we will obtain some new Riemann-Liouville fractional trapezoid and midpoint type inequalities by using Lemmas 3.1, 3.3, 3.5, 3.6 and Lemma 3.7.

Theorem 4.1. Let $0 < r \le 1$, $\rho > 0$ and $f : \left[a^\rho, \left(\frac{b}{r}\right)^\rho\right] \longrightarrow \mathbb{R}$ be a differentiable function on $\left(a^\rho, \left(\frac{b}{r}\right)^\rho\right)$ with $0 < a < b$. If $|f'|^q$ is convex on $\left[a^\rho, \left(\frac{b}{r}\right)^\rho\right]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality for the left Riemann-Liouville fractional integral holds:

$$\left|I_1\right| = \left|\frac{\alpha f(a^\rho) + f\left(\left(\frac{b}{r}\right)^\rho\right)}{\alpha+1} - \Gamma(\alpha+1)\left(\frac{r^\rho}{b^\rho - (ra)^\rho}\right)^\alpha J^\alpha_{(a^\rho)^+} f\left(\left(\frac{b}{r}\right)^\rho\right)\right|$$

$$\le \left(\frac{b^\rho - (ra)^\rho}{r^\rho(\alpha+1)}\right)\left[A_1(\alpha,\rho,p) + A_2(\alpha,\rho,p)\right]^{\frac{1}{p}}\left[\frac{|f'(a^\rho)|^q + \left|f'\left(\left(\frac{b}{r}\right)^\rho\right)\right|^q}{\rho+1}\right]^{\frac{1}{q}}, \tag{4.1}$$

where

$$A_1(\alpha,\rho,p) := \int_0^{\frac{1}{\sqrt[\alpha\rho]{\alpha+1}}} (1-(\alpha+1)t^{\alpha\rho})^p\, dt,$$

$$A_2(\alpha,\rho,p) := \int_{\frac{1}{\sqrt[\alpha\rho]{\alpha+1}}}^1 ((\alpha+1)t^{\alpha\rho} - 1)^p\, dt$$

with $\alpha > 0$.

Proof. Using Lemma 3.1, Hölder's inequality and the convexity of $|f'|^q$, we have

$$\left|I_1\right| \le \left(\frac{b^\rho - (ra)^\rho}{r^\rho(\alpha+1)}\right)\int_0^1 \left|1-(\alpha+1)t^{\alpha\rho}\right|\left|f'\left((ta)^\rho + (1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)\right| dt$$

$$\leq \left(\frac{b^\rho-(ra)^\rho}{r^\rho(\alpha+1)}\right)\left(\int_0^1 \left|1-(\alpha+1)t^{\alpha\rho}\right|^p dt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1 \left|f'\left((ta)^\rho+(1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq \left(\frac{b^\rho-(ra)^\rho}{r^\rho(\alpha+1)}\right)\left[A_1(\alpha,\rho,p)+A_2(\alpha,\rho,p)\right]^{\frac{1}{p}}$$
$$\times\left(\int_0^1 \left[t^\rho|f'(a^\rho)|^q+(1-t^\rho)\left|f'\left(\left(\frac{b}{r}\right)^\rho\right)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$=\left(\frac{b^\rho-(ra)^\rho}{r^\rho(\alpha+1)}\right)\left[A_1(\alpha,\rho,p)+A_2(\alpha,\rho,p)\right]^{\frac{1}{p}}\left[\frac{|f'(a^\rho)|^q+\left|f'\left(\left(\frac{b}{r}\right)^\rho\right)\right|^q}{\rho+1}\right]^{\frac{1}{q}}.$$

The proof of Theorem 4.1 is completed. Q.E.D.

Theorem 4.2. Let $0<r\leq 1$, $\rho>0$ and $f:\left[a^\rho,\left(\frac{b}{r}\right)^\rho\right]\longrightarrow\mathbb{R}$ be a differentiable function on $\left(a^\rho,\left(\frac{b}{r}\right)^\rho\right)$ with $0<a<b$. If $|f'|^q$ is convex on $\left[a^\rho,\left(\frac{b}{r}\right)^\rho\right]$ for $q\geq 1$, then the following inequality holds:

$$\left|I_1\right|\leq\left(\frac{b^\rho-(ra)^\rho}{r^\rho(\alpha+1)}\right)\left[\frac{2\alpha\rho}{(\alpha\rho+1)(\alpha+1)^{\frac{1}{\alpha\rho}}}\right]^{1-\frac{1}{q}}$$
$$\times\left[|f'(a^\rho)|^q(C_1(\alpha,\rho)+C_3(\alpha,\rho))+\left|f'\left(\left(\frac{b}{r}\right)^\rho\right)\right|^q(C_2(\alpha,\rho)+C_4(\alpha,\rho))\right]^{\frac{1}{q}}, \tag{4.2}$$

where

$$C_1(\alpha,\rho):=\int_0^{\frac{1}{\sqrt[\alpha\rho]{\alpha+1}}}\left(1-(\alpha+1)t^{\alpha\rho}\right)t^\rho dt,$$
$$C_2(\alpha,\rho):=\int_0^{\frac{1}{\sqrt[\alpha\rho]{\alpha+1}}}\left(1-(\alpha+1)t^{\alpha\rho}\right)(1-t^\rho)dt,$$
$$C_3(\alpha,\rho):=\int_{\frac{1}{\sqrt[\alpha\rho]{\alpha+1}}}^1\left((\alpha+1)t^{\alpha\rho}-1\right)t^\rho dt,$$
$$C_4(\alpha,\rho):=\int_{\frac{1}{\sqrt[\alpha\rho]{\alpha+1}}}^1\left((\alpha+1)t^{\alpha\rho}-1\right)(1-t^\rho)dt,$$

with $\alpha>0$.

Proof. Using Lemma 3.1, power mean inequality and the convexity of $|f'|^q$, we have

$$\begin{aligned}
\left|I_1\right| &\leq \frac{b^\rho-(ra)^\rho}{r^\rho(\alpha+1)}\int_0^1 \left|1-(\alpha+1)t^{\alpha\rho}\right|\left|f'\left((ta)^\rho+(1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)\right|dt \\
&\leq \frac{b^\rho-(ra)^\rho}{r^\rho(\alpha+1)}\left(\int_0^1 \left|1-(\alpha+1)t^{\alpha\rho}\right|dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 \left|1-(\alpha+1)t^{\alpha\rho}\right|\left|f'\left((ta)^\rho+(1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \left(\frac{b^\rho-(ra)^\rho}{r^\rho(\alpha+1)}\right)\left[\frac{2\alpha\rho}{(\alpha\rho+1)(\alpha+1)^{\frac{1}{\alpha\rho}}}\right]^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 \left|1-(\alpha+1)t^{\alpha\rho}\right|\left[t^\rho|f'(a^\rho)|^q+(1-t^\rho)\left|f'\left(\left(\frac{b}{r}\right)^\rho\right)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&= \left(\frac{b^\rho-(ra)^\rho}{r^\rho(\alpha+1)}\right)\left[\frac{2\alpha\rho}{(\alpha\rho+1)(\alpha+1)^{\frac{1}{\alpha\rho}}}\right]^{1-\frac{1}{q}} \\
&\times\left[|f'(a^\rho)|^q(C_1(\alpha,\rho)+C_3(\alpha,\rho))+\left|f'\left(\left(\frac{b}{r}\right)^\rho\right)\right|^q(C_2(\alpha,\rho)+C_4(\alpha,\rho))\right]^{\frac{1}{q}}.
\end{aligned}$$

The proof of Theorem 4.2 is completed. Q.E.D.

Theorem 4.3. Let $0<r\leq 1$, $\rho>0$ and $f:[(ra)^\rho,b^\rho]\longrightarrow\mathbb{R}$ be a differentiable function on $((ra)^\rho,b^\rho)$ with $0<a<b$. If $|f'|^q$ is convex on $[(ra)^\rho,b^\rho]$ for $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:

$$\begin{aligned}
\left|I_2\right| &= \left|\frac{f((ra)^\rho)+\alpha f(b^\rho)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b^\rho-(ra)^\rho)^\alpha}J^\alpha_{(b^\rho)^-}f((ra)^\rho)\right| \\
&\leq \left(\frac{b^\rho-(ra)^\rho}{\alpha+1}\right)\left[A_1(\alpha,\rho,p)+A_2(\alpha,\rho,p)\right]^{\frac{1}{p}}\left[\frac{|f'((ra)^\rho)|^q+|f'(b^\rho)|^q}{\rho+1}\right]^{\frac{1}{q}}, \qquad (4.3)
\end{aligned}$$

where $A_1(\alpha,\rho,p)$ and $A_2(\alpha,\rho,p)$ are defined as in Theorem 4.1, with $\alpha>0$.

Proof. Using Lemma 3.3, Hölder's inequality and the convexity of $|f'|^q$, we have

$$\begin{aligned}
\left|I_2\right| &\leq \frac{b^\rho-(ra)^\rho}{\alpha+1}\int_0^1 \left|(\alpha+1)t^{\alpha\rho}-1\right||f'(r^\rho(1-t^\rho)a^\rho+(tb)^\rho)|dt \\
&\leq \frac{b^\rho-(ra)^\rho}{\alpha+1}\left(\int_0^1 \left|(\alpha+1)t^{\alpha\rho}-1\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(r^\rho(1-t^\rho)a^\rho+(tb)^\rho)|^q dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$\leq \frac{b^\rho-(ra)^\rho}{\alpha+1}\big[A_1(\alpha,\rho,p)+A_2(\alpha,\rho,p)\big]^{\frac{1}{p}}\left(\int_0^1\big[(1-t^\rho)|f'((ra)^\rho)|^q+t^\rho|f'(b^\rho)|^q\big]dt\right)^{\frac{1}{q}}$$

$$=\left(\frac{b^\rho-(ra)^\rho}{\alpha+1}\right)\big[A_1(\alpha,\rho,p)+A_2(\alpha,\rho,p)\big]^{\frac{1}{p}}\left[\frac{|f'((ra)^\rho)|^q+|f'(b^\rho)|^q}{\rho+1}\right]^{\frac{1}{q}}.$$

The proof of Theorem 4.3 is completed. Q.E.D.

Theorem 4.4. Let $0<r\leq 1$, $\rho>0$ and $f:[(ra)^\rho,b^\rho]\longrightarrow\mathbb{R}$ be a differentiable function on $((ra)^\rho,b^\rho)$ with $0<a<b$. If $|f'|^q$ is convex on $[(ra)^\rho,b^\rho]$ for $q\geq 1$, then the following inequality holds:

$$\big|I_2\big|\leq\left(\frac{b^\rho-(ra)^\rho}{\alpha+1}\right)\left[\frac{2\alpha\rho}{(\alpha\rho+1)(\alpha+1)^{\frac{1}{\alpha\rho}}}\right]^{1-\frac{1}{q}}$$

$$\times\Big[|f'((ra)^\rho)|^q(C_2(\alpha,\rho)+C_4(\alpha,\rho))+|f'(b^\rho)|^q(C_1(\alpha,\rho)+C_3(\alpha,\rho))\Big]^{\frac{1}{q}},\tag{4.4}$$

where $C_1(\alpha,\rho),C_2(\alpha,\rho),C_3(\alpha,\rho)$ and $C_4(\alpha,\rho)$ are defined as in Theorem 4.2, with $\alpha>0$.

Proof. Using Lemma 3.3, power mean inequality and the convexity of $|f'|^q$, we have

$$\big|I_2\big|\leq\frac{b^\rho-(ra)^\rho}{\alpha+1}\int_0^1\big|(\alpha+1)t^{\alpha\rho}-1\big||f'(r^\rho(1-t^\rho)a^\rho+(tb)^\rho)|dt$$

$$\leq\frac{b^\rho-(ra)^\rho}{\alpha+1}\left(\int_0^1\big|(\alpha+1)t^{\alpha\rho}-1\big|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\big|(\alpha+1)t^{\alpha\rho}-1\big||f'(r^\rho(1-t^\rho)a^\rho+(tb)^\rho)|^q dt\right)^{\frac{1}{q}}$$

$$\leq\left(\frac{b^\rho-(ra)^\rho}{\alpha+1}\right)\left[\frac{2\alpha\rho}{(\alpha\rho+1)(\alpha+1)^{\frac{1}{\alpha\rho}}}\right]^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\big|(\alpha+1)t^{\alpha\rho}-1\big|\big[(1-t^\rho)|f'((ra)^\rho)|^q+t^\rho|f'(b^\rho)|^q\big]dt\right)^{\frac{1}{q}}$$

$$=\left(\frac{b^\rho-(ra)^\rho}{\alpha+1}\right)\left[\frac{2\alpha\rho}{(\alpha\rho+1)(\alpha+1)^{\frac{1}{\alpha\rho}}}\right]^{1-\frac{1}{q}}$$

$$\times\Big[|f'((ra)^\rho)|^q(C_2(\alpha,\rho)+C_4(\alpha,\rho))+|f'(b^\rho)|^q(C_1(\alpha,\rho)+C_3(\alpha,\rho))\Big]^{\frac{1}{q}}.$$

The proof of Theorem 4.4 is completed. Q.E.D.

Theorem 4.5. Let $0 < r \leq 1$, $\rho > 0$ and $f : [(ra)^\rho, b^\rho] \longrightarrow \mathbb{R}$ be a differentiable function on $((ra)^\rho, b^\rho)$ with $0 < a < b$. If $|f'|^q$ is convex on $[(ra)^\rho, b^\rho]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:

$$\begin{aligned}\left|I_3\right| &= \left| \frac{f((ra)^\rho) + f(b^\rho)}{r^\rho + 1} - \frac{\Gamma(\alpha+1)}{(r^\rho+1)(b^\rho-(ra)^\rho)^\alpha} \left[J^\alpha_{((ra)^\rho)^+} f(b^\rho) + J^\alpha_{(b^\rho)^-} f((ra)^\rho) \right] \right| \\ &\leq \frac{2(b^\rho-(ra)^\rho)}{r^\rho+1} \left(\frac{1}{p\alpha\rho+1} \right)^{\frac{1}{p}} \left[\frac{|f'((ra)^\rho)|^q + |f'(b^\rho)|^q}{\rho+1} \right]^{\frac{1}{q}}, \end{aligned} \tag{4.5}$$

with $\alpha > 0$.

Proof. Using Lemma 3.5, Hölder's inequality and the convexity of $|f'|^q$, we have

$$\begin{aligned}\left|I_3\right| &\leq \frac{b^\rho-(ra)^\rho}{r^\rho+1} \\ &\times \left[\int_0^1 t^{\alpha\rho} |f'(r^\rho(1-t^\rho)a^\rho + (tb)^\rho)| dt + \int_0^1 (1-t)^{\alpha\rho} \left| |f'(r^\rho(1-t^\rho)a^\rho + (tb)^\rho)| dt \right] \right. \\ &\leq \frac{2(b^\rho-(ra)^\rho)}{r^\rho+1} \left(\int_0^1 t^{p\alpha\rho} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left[(1-t^\rho)|f'((ra)^\rho)|^q + t^\rho |f'(b^\rho)|^q \right] dt \right)^{\frac{1}{q}} \\ &= \frac{2(b^\rho-(ra)^\rho)}{r^\rho+1} \left(\frac{1}{p\alpha\rho+1} \right)^{\frac{1}{p}} \left[\frac{|f'((ra)^\rho)|^q + |f'(b^\rho)|^q}{\rho+1} \right]^{\frac{1}{q}}. \end{aligned}$$

The proof of Theorem 4.5 is completed. Q.E.D.

Theorem 4.6. Let $0 < r \leq 1$, $\rho > 0$ and $f : [(ra)^\rho, b^\rho] \longrightarrow \mathbb{R}$ be a differentiable function on $((ra)^\rho, b^\rho)$ with $0 < a < b$. If $|f'|^q$ is convex on $[(ra)^\rho, b^\rho]$ for $q \geq 1$, then the following inequality holds:

$$\begin{aligned}\left|I_3\right| &\leq \frac{b^\rho-(ra)^\rho}{r^\rho+1} \left(\frac{1}{\alpha\rho+1} \right)^{1-\frac{1}{q}} \\ &\times \left\{ \left(\frac{1}{\rho(\alpha+1)+1} \right)^{\frac{1}{q}} \left[\frac{\rho}{\alpha\rho+1} |f'((ra)^\rho)|^q + |f'(b^\rho)|^q \right]^{\frac{1}{q}} \right. \\ &+ \left. \left[\left(\frac{\rho}{\alpha\rho+1} - \beta(\alpha\rho+1, \rho+1) \right) |f'((ra)^\rho)|^q + \beta(\alpha\rho+1, \rho+1) |f'(b^\rho)|^q \right]^{\frac{1}{q}} \right\}, \end{aligned} \tag{4.6}$$

with $\alpha > 0$.

Proof. Using Lemma 3.5, power mean inequality and the convexity of $|f'|^q$, we have

$$\left|I_3\right| \leq \frac{b^\rho-(ra)^\rho}{r^\rho+1}$$

$$\times\left[\int_0^1 t^{\alpha\rho}|f'(r^\rho(1-t^\rho)a^\rho+(tb)^\rho)|dt+\int_0^1(1-t)^{\alpha\rho}|f'(r^\rho(1-t^\rho)a^\rho+(tb)^\rho)|dt\right]$$

$$\leq \frac{b^\rho-(ra)^\rho}{r^\rho+1}\left(\int_0^1 t^{\alpha\rho}dt\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_0^1 t^{\alpha\rho}|f'(r^\rho(1-t^\rho)a^\rho+(tb)^\rho)|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1(1-t)^{\alpha\rho}|f'(r^\rho(1-t^\rho)a^\rho+(tb)^\rho)|^q dt\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{b^\rho-(ra)^\rho}{r^\rho+1}\left(\frac{1}{\alpha\rho+1}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_0^1 t^{\alpha\rho}\left[(1-t^\rho)|f'((ra)^\rho)|^q+t^\rho|f'(b^\rho)|^q\right]dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1(1-t)^{\alpha\rho}\left[(1-t^\rho)|f'((ra)^\rho)|^q+t^\rho|f'(b^\rho)|^q\right]dt\right)^{\frac{1}{q}}\right]$$

$$=\frac{b^\rho-(ra)^\rho}{r^\rho+1}\left(\frac{1}{\alpha\rho+1}\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left(\frac{1}{\rho(\alpha+1)+1}\right)^{\frac{1}{q}}\left[\frac{\rho}{\alpha\rho+1}|f'((ra)^\rho)|^q+|f'(b^\rho)|^q\right]^{\frac{1}{q}}\right.$$

$$\left.+\left[\left(\frac{\rho}{\alpha\rho+1}-\beta\left(\alpha\rho+1,\rho+1\right)\right)|f'((ra)^\rho)|^q+\beta\left(\alpha\rho+1,\rho+1\right)|f'(b^\rho)|^q\right]^{\frac{1}{q}}\right\}.$$

The proof of Theorem 4.6 is completed. Q.E.D.

Theorem 4.7. Let $0<r\leq 1$, $\rho>0$ and $f:\left[a^\rho,\left(\frac{b}{r}\right)^\rho\right]\longrightarrow\mathbb{R}$ be a differentiable function on $\left(a^\rho,\left(\frac{b}{r}\right)^\rho\right)$ with $0<a<b$. If $|f'|^q$ is convex on $\left[a^\rho,\left(\frac{b}{r}\right)^\rho\right]$ for $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:

$$|I_4|=\left|\Gamma(\alpha+1)\left(\frac{r^\rho}{b^\rho-(ra)^\rho}\right)^\alpha J^\alpha_{(a^\rho)^+}f\left(\left(\frac{b}{r}\right)^\rho\right)-f\left(\frac{\alpha(ra)^\rho+b^\rho}{r^\rho(\alpha+1)}\right)\right|$$

$$\leq\left(\frac{b^\rho-(ra)^\rho}{r^\rho}\right)\left\{\left(\frac{1}{p\alpha\rho+1}\right)^{\frac{1}{p}}\left[\frac{|f'(a^\rho)|^q+\left|f'\left(\left(\frac{b}{r}\right)^\rho\right)\right|^q}{\rho+1}\right]^{\frac{1}{q}}\right.$$

$$+\left[\frac{1}{\rho+1}\left(1-\left(\frac{\alpha}{\alpha+1}\right)^{\rho+1}\right)|f'(a^\rho)|^q \tag{4.7}$$
$$+\left(\frac{1}{\alpha+1}-\frac{1}{\rho+1}\left(1-\left(\frac{\alpha}{\alpha+1}\right)^{\rho+1}\right)\right)\left|f'\left(\left(\frac{b}{r}\right)^\rho\right)\right|^q\Bigg]\Bigg\},$$

with $\alpha > 0$.

Proof. Using Lemma 3.6, Hölder's inequality and the convexity of $|f'|^q$, we have

$$\begin{aligned}
\left|I_4\right| &\le \frac{b^\rho-(ra)^\rho}{r^\rho}\left[\int_0^1 t^{\alpha\rho}\left|f'\left((ta)^\rho+(1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)\right|dt\right.\\
&\quad\left.+\int_{\frac{\alpha}{\alpha+1}}^1\left|f'\left((ta)^\rho+(1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)\right|dt\right]\\
&\le \frac{b^\rho-(ra)^\rho}{r^\rho}\left[\left(\int_0^1 t^{p\alpha\rho}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left((ta)^\rho+(1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\int_{\frac{\alpha}{\alpha+1}}^1\left|f'\left((ta)^\rho+(1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)\right|dt\right]\\
&\le \frac{b^\rho-(ra)^\rho}{r^\rho}\left\{\left(\frac{1}{p\alpha\rho+1}\right)^{\frac{1}{p}}\left(\int_0^1\left[t^\rho|f'(a^\rho)|^q+(1-t^\rho)\left|f'\left(\left(\frac{b}{r}\right)^\rho\right)\right|^q\right]dt\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\int_{\frac{\alpha}{\alpha+1}}^1\left[t^\rho|f'(a^\rho)|^q+(1-t^\rho)\left|f'\left(\left(\frac{b}{r}\right)^\rho\right)\right|^q\right]dt\right\}\\
&= \left(\frac{b^\rho-(ra)^\rho}{r^\rho}\right)\left\{\left(\frac{1}{p\alpha\rho+1}\right)^{\frac{1}{p}}\left[\frac{|f'(a^\rho)|^q+\left|f'\left(\left(\frac{b}{r}\right)^\rho\right)\right|^q}{\rho+1}\right]^{\frac{1}{q}}\right.\\
&\quad+\left[\frac{1}{\rho+1}\left(1-\left(\frac{\alpha}{\alpha+1}\right)^{\rho+1}\right)|f'(a^\rho)|^q\right.\\
&\quad\left.\left.+\left(\frac{1}{\alpha+1}-\frac{1}{\rho+1}\left(1-\left(\frac{\alpha}{\alpha+1}\right)^{\rho+1}\right)\right)\left|f'\left(\left(\frac{b}{r}\right)^\rho\right)\right|^q\right]\right\}.
\end{aligned}$$

The proof of Theorem 4.7 is completed. Q.E.D.

Theorem 4.8. Let $0 < r \le 1$, $\rho > 0$ and $f : \left[a^\rho, \left(\frac{b}{r}\right)^\rho\right] \longrightarrow \mathbb{R}$ be a differentiable function on $\left(a^\rho, \left(\frac{b}{r}\right)^\rho\right)$ with $0 < a < b$. If $|f'|^q$ is convex on $\left[a^\rho, \left(\frac{b}{r}\right)^\rho\right]$ for $q \ge 1$, then the following inequality holds:

$$\left|I_4\right| \le \left(\frac{b^\rho-(ra)^\rho}{r^\rho}\right)\left\{\left(\frac{1}{\alpha\rho+1}\right)^{1-\frac{1}{q}}\left(\frac{1}{\rho(\alpha+1)+1}\right)^{\frac{1}{q}}\right. \tag{4.8}$$

$$\times\left[|f'(a^\rho)|^q+\frac{\rho}{\alpha\rho+1}\left|f'\left(\left(\frac{b}{r}\right)^\rho\right)\right|^q\right]^{\frac{1}{q}}$$
$$+\left[\frac{1}{\rho+1}\left(1-\left(\frac{\alpha}{\alpha+1}\right)^{\rho+1}\right)|f'(a^\rho)|^q\right.$$
$$\left.+\left(\frac{1}{\alpha+1}-\frac{1}{\rho+1}\left(1-\left(\frac{\alpha}{\alpha+1}\right)^{\rho+1}\right)\right)\left|f'\left(\left(\frac{b}{r}\right)^\rho\right)\right|^q\right]\Bigg\},$$

with $\alpha>0$.

Proof. Using Lemma 3.6, power mean inequality and the convexity of $|f'|^q$, we have

$$\left|I_4\right|\le\frac{b^\rho-(ra)^\rho}{r^\rho}\left[\int_0^1 t^{\alpha\rho}\left|f'\left((ta)^\rho+(1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)\right|dt\right.$$
$$\left.+\int_{\frac{\alpha}{\alpha+1}}^1\left|f'\left((ta)^\rho+(1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)\right|dt\right]$$
$$\le\frac{b^\rho-(ra)^\rho}{r^\rho}\left[\left(\int_0^1 t^{\alpha\rho}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha\rho}\left|f'\left((ta)^\rho+(1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\int_{\frac{\alpha}{\alpha+1}}^1\left|f'\left((ta)^\rho+(1-t^\rho)\left(\frac{b}{r}\right)^\rho\right)\right|dt\right]$$
$$\le\frac{b^\rho-(ra)^\rho}{r^\rho}\left\{\left(\frac{1}{\alpha\rho+1}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha\rho}\left[t^\rho|f'(a^\rho)|^q+(1-t^\rho)\left|f'\left(\left(\frac{b}{r}\right)^\rho\right)\right|^q\right]dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\int_{\frac{\alpha}{\alpha+1}}^1\left[t^\rho|f'(a^\rho)|^q+(1-t^\rho)\left|f'\left(\left(\frac{b}{r}\right)^\rho\right)\right|^q\right]dt\right\}$$
$$=\left(\frac{b^\rho-(ra)^\rho}{r^\rho}\right)\left\{\left(\frac{1}{\alpha\rho+1}\right)^{1-\frac{1}{q}}\left(\frac{1}{\rho(\alpha+1)+1}\right)^{\frac{1}{q}}\right.$$
$$\times\left[|f'(a^\rho)|^q+\frac{\rho}{\alpha\rho+1}\left|f'\left(\left(\frac{b}{r}\right)^\rho\right)\right|^q\right]^{\frac{1}{q}}$$
$$+\left[\frac{1}{\rho+1}\left(1-\left(\frac{\alpha}{\alpha+1}\right)^{\rho+1}\right)|f'(a^\rho)|^q\right.$$
$$\left.\left.+\left(\frac{1}{\alpha+1}-\frac{1}{\rho+1}\left(1-\left(\frac{\alpha}{\alpha+1}\right)^{\rho+1}\right)\right)\left|f'\left(\left(\frac{b}{r}\right)^\rho\right)\right|^q\right]\right\}.$$

The proof of Theorem 4.8 is completed. Q.E.D.

Theorem 4.9. Let $0 < r \leq 1$, $\rho > 0$ and $f : [(ra)^\rho, b^\rho] \longrightarrow \mathbb{R}$ be a differentiable function on $((ra)^\rho, b^\rho)$ with $0 < a < b$. If $|f'|^q$ is convex on $[(ra)^\rho, b^\rho]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:

$$\begin{aligned}
\left|I_5\right| &= \left| \frac{\Gamma(\alpha+1)}{(b^\rho-(ra)^\rho)^\alpha} J^\alpha_{(b^\rho)^-} f((ra)^\rho) - f\left(\frac{(ra)^\rho + \alpha b^\rho}{\alpha+1}\right) \right| \\
&\leq (b^\rho-(ra)^\rho)\Bigg\{ \left(\frac{1}{p\alpha\rho+1}\right)^{\frac{1}{p}} \left[\frac{|f'((ra)^\rho)|^q + |f'(b^\rho)|^q}{\rho+1}\right]^{\frac{1}{q}} \\
&\quad + \Bigg[\left(\frac{1}{\alpha+1} - \frac{1}{\rho+1}\left(1-\left(\frac{\alpha}{\alpha+1}\right)^{\rho+1}\right)\right) \left|f'((ra)^\rho)\right|^q \\
&\qquad + \frac{1}{\rho+1}\left(1-\left(\frac{\alpha}{\alpha+1}\right)^{\rho+1}\right) |f'(b^\rho)|^q \Bigg] \Bigg\},
\end{aligned} \tag{4.9}$$

with $\alpha > 0$.

Proof. Using Lemma 3.7, Hölder's inequality and the convexity of $|f'|^q$, we have

$$\begin{aligned}
\left|I_5\right| &\leq (b^\rho-(ra)^\rho)\left[\int_0^1 |-t^{\alpha\rho}||f'(r^\rho(1-t^\rho)a^\rho + (tb)^\rho)|dt + \int_{\frac{\alpha}{\alpha+1}}^1 |f'(r^\rho(1-t^\rho)a^\rho + (tb)^\rho)|dt\right] \\
&\leq (b^\rho-(ra)^\rho)\Bigg[\left(\int_0^1 t^{p\alpha\rho}dt\right)^{\frac{1}{p}} \left(\int_0^1 |f'(r^\rho(1-t^\rho)a^\rho + (tb)^\rho)|^q dt\right)^{\frac{1}{q}} \\
&\qquad + \int_{\frac{\alpha}{\alpha+1}}^1 |f'(r^\rho(1-t^\rho)a^\rho + (tb)^\rho)|dt \Bigg] \\
&\leq (b^\rho-(ra)^\rho)\Bigg\{ \left(\frac{1}{p\alpha\rho+1}\right)^{\frac{1}{p}} \left(\int_0^1 \left[(1-t^\rho)|f'((ra)^\rho)|^q + t^\rho|f'(b^\rho)|^q\right]dt\right)^{\frac{1}{q}} \\
&\qquad + \int_{\frac{\alpha}{\alpha+1}}^1 \left[(1-t^\rho)|f'((ra)^\rho)|^q + t^\rho|f'(b^\rho)|^q\right]dt \Bigg\} \\
&= (b^\rho-(ra)^\rho)\Bigg\{ \left(\frac{1}{p\alpha\rho+1}\right)^{\frac{1}{p}} \left[\frac{|f'((ra)^\rho)|^q + |f'(b^\rho)|^q}{\rho+1}\right]^{\frac{1}{q}} \\
&\quad + \Bigg[\left(\frac{1}{\alpha+1} - \frac{1}{\rho+1}\left(1-\left(\frac{\alpha}{\alpha+1}\right)^{\rho+1}\right)\right) \left|f'((ra)^\rho)\right|^q \\
&\qquad + \frac{1}{\rho+1}\left(1-\left(\frac{\alpha}{\alpha+1}\right)^{\rho+1}\right) |f'(b^\rho)|^q \Bigg] \Bigg\}.
\end{aligned}$$

The proof of Theorem 4.9 is completed. Q.E.D.

Theorem 4.10. Let $0 < r \leq 1$, $\rho > 0$ and $f : [(ra)^\rho, b^\rho] \longrightarrow \mathbb{R}$ be a differentiable function on $((ra)^\rho, b^\rho)$ with $0 < a < b$. If $|f'|^q$ is convex on $[(ra)^\rho, b^\rho]$ for $q \geq 1$, then the following inequality holds:

$$\begin{aligned}\left|I_5\right| \leq (b^\rho - (ra)^\rho)\Bigg\{ &\left(\frac{1}{\alpha\rho+1}\right)^{1-\frac{1}{q}} \left(\frac{1}{\rho(\alpha+1)+1}\right)^{\frac{1}{q}} \left[\frac{\rho}{\alpha\rho+1}|f'((ra)^\rho)|^q + |f'(b^\rho)|^q\right]^{\frac{1}{q}} \\ &+\left[\left(\frac{1}{\alpha+1} - \frac{1}{\rho+1}\left(1-\left(\frac{\alpha}{\alpha+1}\right)^{\rho+1}\right)\right)\left|f'((ra)^\rho)\right|^q \right. \\ &\left. +\frac{1}{\rho+1}\left(1-\left(\frac{\alpha}{\alpha+1}\right)^{\rho+1}\right)|f'(b^\rho)|^q\right]\Bigg\},\end{aligned}$$

with $\alpha > 0$.

Proof. Using Lemma 3.7, power mean inequality and the convexity of $|f'|^q$, we have

$$\begin{aligned}\left|I_5\right| &\leq (b^\rho - (ra)^\rho)\left[\int_0^1 |-t^{\alpha\rho}||f'(r^\rho(1-t^\rho)a^\rho + (tb)^\rho)|dt \right. \\ &\qquad \left. + \int_{\frac{\alpha}{\alpha+1}}^1 |f'(r^\rho(1-t^\rho)a^\rho + (tb)^\rho)|dt\right] \\ &\leq (b^\rho - (ra)^\rho)\left[\left(\int_0^1 t^{\alpha\rho}dt\right)^{1-\frac{1}{q}} \left(\int_0^1 t^{\alpha\rho}|f'(r^\rho(1-t^\rho)a^\rho + (tb)^\rho)|^q dt\right)^{\frac{1}{q}} \right. \\ &\qquad \left. + \int_{\frac{\alpha}{\alpha+1}}^1 |f'(r^\rho(1-t^\rho)a^\rho + (tb)^\rho)|dt\right] \\ &\leq (b^\rho - (ra)^\rho)\Bigg\{\left(\frac{1}{\alpha\rho+1}\right)^{1-\frac{1}{q}} \left(\int_0^1 t^{\alpha\rho}\left[(1-t^\rho)|f'((ra)^\rho)|^q + t^\rho|f'(b^\rho)|^q\right]dt\right)^{\frac{1}{q}} \\ &\qquad + \int_{\frac{\alpha}{\alpha+1}}^1 \left[(1-t^\rho)|f'((ra)^\rho)|^q + t^\rho|f'(b^\rho)|^q\right]dt\Bigg\} \\ &= (b^\rho - (ra)^\rho)\Bigg\{\left(\frac{1}{\alpha\rho+1}\right)^{1-\frac{1}{q}} \left(\frac{1}{\rho(\alpha+1)+1}\right)^{\frac{1}{q}} \left[\frac{\rho}{\alpha\rho+1}|f'((ra)^\rho)|^q + |f'(b^\rho)|^q\right]^{\frac{1}{q}} \qquad (4.10) \\ &\qquad +\left[\left(\frac{1}{\alpha+1} - \frac{1}{\rho+1}\left(1-\left(\frac{\alpha}{\alpha+1}\right)^{\rho+1}\right)\right)\left|f'((ra)^\rho)\right|^q \right. \\ &\qquad \left. +\frac{1}{\rho+1}\left(1-\left(\frac{\alpha}{\alpha+1}\right)^{\rho+1}\right)|f'(b^\rho)|^q\right]\Bigg\}.\end{aligned}$$

The proof of Theorem 4.10 is completed. Q.E.D.

5 Applications to f-divergence measures

One of the basic problems in various applications of Probability Theory is finding an appropriate measure of distance between any two probability distributions. For this purpose, a lot of divergence measures have been proposed and extensively studied, for example, see [8] and the references therein. Let the set Σ and the σ-finite measure μ be given and consider the set of all probability densities on μ to be defined on $\Omega := \left\{p|p : \Sigma \to \mathbb{R}, p(x) > 0, \int_\Sigma p(x)d\mu(x) = 1\right\}$. Let $f : (0,\infty) \longrightarrow \mathbb{R}$ be given function and consider $D_f(p,q)$ be defined by

$$D_f(p,q) := \int_\Sigma p(x) f\left[\frac{q(x)}{p(x)}\right] d\mu(x), \quad p,q \in \Omega. \tag{5.1}$$

If f is convex function, then (5.1) is known as the Csisźar f-divergence [2].
In [20], Shioya and Da-te introduced the Hermite-Hadamard (HH) divergence

$$D^f_{HH}(p,q) := \int_\Sigma p(x) \frac{\int_1^{\frac{q(x)}{p(x)}} f(t)dt}{\frac{q(x)}{p(x)} - 1} d\mu(x), \quad p,q \in \Omega. \tag{5.2}$$

where f is convex function on $(0,\infty)$ with $f(1) = 0$. In [20], the authors gave the property of (HH) divergence that $D^f_{HH}(p,q) \geq 0$ with the equality holds if and only if $p = q$.

Proposition 5.1. Let all the assumptions of Theorem 4.3 hold with $\alpha = r = 1$ and $f(1) = 0$. If $p,q \in \Omega$, then the following inequality holds:

$$\left|\frac{1}{2}D_f(p,q) - D^f_{HH}(p,q)\right| \leq \frac{1}{2}\left(\frac{1}{\rho+1}\right)^{\frac{1}{q}} \sqrt[\rho]{2E(p,\rho)} \tag{5.3}$$

$$\times\left\{\int_{X_1}\left(\frac{q^\rho(x) - p^\rho(x)}{p^{\rho-1}(x)}\right)\left[|f'(1)|^q + \left|f'\left(\left(\frac{q(x)}{p(x)}\right)^\rho\right)\right|^q\right]^{\frac{1}{q}} d\mu(x)\right.$$

$$\left.+\int_{X_2}\left(\frac{p^\rho(x) - q^\rho(x)}{p^{\rho-1}(x)}\right)\left[|f'(1)|^q + \left|f'\left(\left(\frac{q(x)}{p(x)}\right)^\rho\right)\right|^q\right]^{\frac{1}{q}} d\mu(x)\right\},$$

where

$$E(p,\rho) := \frac{1}{\rho}\left(\frac{1}{2}\right)^{\frac{1}{\rho}} \beta\left(p+1, \frac{1}{\rho}\right).$$

Proof. Let $X_1 = \left\{x \in \Sigma : q(x) > p(x)\right\}$, $X_2 = \left\{x \in \Sigma : p(x) > q(x)\right\}$ and $X_3 = \left\{x \in \Sigma : q(x) = p(x)\right\}$.
If $x \in X_3$, then obviously equality holds in (5.3). Now if $x \in X_1$, then by using Theorem 4.3 for $a = 1$, $b = \frac{q(x)}{p(x)}$, multiplying both hand sides of the obtained results by $p(x)$ and then integrating over X_1, we get

$$\left|\frac{1}{2}\int_{X_1} p(x) f\left[\frac{q(x)}{p(x)}\right] d\mu(x) - \int_{X_1} p(x) \frac{\int_1^{\frac{q(x)}{p(x)}} f(t)dt}{\frac{q(x)}{p(x)} - 1} d\mu(x)\right|$$

$$\leq \frac{1}{2}\left(\frac{1}{\rho+1}\right)^{\frac{1}{q}} \sqrt[p]{2E(p,\rho)} \tag{5.4}$$

$$\times \int_{X_1}\left(\frac{q^{\rho}(x)-p^{\rho}(x)}{p^{\rho-1}(x)}\right)\left[|f'(1)|^q+\left|f'\left(\left(\frac{q(x)}{p(x)}\right)^{\rho}\right)\right|^q\right]^{\frac{1}{q}} d\mu(x).$$

Similarly if $x \in X_2$, then by using Theorem 4.3 for $a=\frac{q(x)}{p(x)}, b=1$, multiplying both hand sides of the obtained results by $p(x)$ and then integrating over X_2, we get

$$\left|\frac{1}{2}\int_{X_2} p(x) f\left[\frac{q(x)}{p(x)}\right] d\mu(x) - \int_{X_2} p(x)\frac{\int_1^{\frac{q(x)}{p(x)}} f(t)dt}{\frac{q(x)}{p(x)}-1} d\mu(x)\right|$$

$$\leq \frac{1}{2}\left(\frac{1}{\rho+1}\right)^{\frac{1}{q}} \sqrt[p]{2E(p,\rho)} \tag{5.5}$$

$$\times \int_{X_2}\left(\frac{p^{\rho}(x)-q^{\rho}(x)}{p^{\rho-1}(x)}\right)\left[|f'(1)|^q+\left|f'\left(\left(\frac{q(x)}{p(x)}\right)^{\rho}\right)\right|^q\right]^{\frac{1}{q}} d\mu(x).$$

By adding inequalities (5.4) and (5.5) and then using triangular inequality, we get (5.3). The proof of Proposition 5.1 is completed. Q.E.D.

Proposition 5.2. Let all the assumptions of Theorem 4.4 hold with $\alpha = r = \rho = 1$ and $f(1) = 0$. If $p, q \in \Omega$, then the following inequality holds:

$$\left|\frac{1}{2}D_f(p,q) - D^f_{HH}(p,q)\right| \leq \left(\frac{1}{2}\right)^{2+\frac{1}{q}} \tag{5.6}$$

$$\times\left\{\int_{X_1}(q(x)-p(x))\left[|f'(1)|^q+\left|f'\left(\frac{q(x)}{p(x)}\right)\right|^q\right]^{\frac{1}{q}} d\mu(x)\right.$$

$$\left.+\int_{X_2}(p(x)-q(x))\left[|f'(1)|^q+\left|f'\left(\frac{q(x)}{p(x)}\right)\right|^q\right]^{\frac{1}{q}} d\mu(x)\right\}.$$

Proof. The proof is similar as to that of Proposition 5.1 but use Theorem 4.4. Q.E.D.

Proposition 5.3. Let all the assumptions of Theorem 4.5 hold with $\alpha = r = 1$ and $f(1) = 0$. If $p, q \in \Omega$, then the following inequality holds:

$$\left|\frac{1}{2}D_f(p,q) - D^f_{HH}(p,q)\right| \leq \left(\frac{1}{\rho+1}\right)^{\frac{1}{q}}\left(\frac{1}{p\rho+1}\right)^{\frac{1}{p}} \tag{5.7}$$

$$\times\left\{\int_{X_1}\left(\frac{q^{\rho}(x)-p^{\rho}(x)}{p^{\rho-1}(x)}\right)\left[|f'(1)|^q+\left|f'\left(\left(\frac{q(x)}{p(x)}\right)^{\rho}\right)\right|^q\right]^{\frac{1}{q}} d\mu(x)\right.$$

$$+\int_{X_2}\left(\frac{p^{\rho}(x)-q^{\rho}(x)}{p^{\rho-1}(x)}\right)\left[|f'(1)|^q+\left|f'\left(\left(\frac{q(x)}{p(x)}\right)^{\rho}\right)\right|^q\right]^{\frac{1}{q}}d\mu(x)\Bigg\}.$$

Proof. The proof is similar as to that of Proposition 5.1 but use Theorem 4.5. Q.E.D.

Proposition 5.4. Let all the assumptions of Theorem 4.6 hold with $\alpha = r = \rho = 1$ and $f(1) = 0$. If $p, q \in \Omega$, then the following inequality holds:

$$\left|\frac{1}{2}D_f(p,q)-D^f_{HH}(p,q)\right|\le\frac{1}{4}\left(\frac{1}{3}\right)^{\frac{1}{q}} \tag{5.8}$$

$$\times\Bigg\{\int_{X_1}(q(x)-p(x))$$

$$\times\left(\left[|f'(1)|^q+2\left|f'\left(\frac{q(x)}{p(x)}\right)\right|^q\right]^{\frac{1}{q}}+\left[2|f'(1)|^q+\left|f'\left(\frac{q(x)}{p(x)}\right)\right|^q\right]^{\frac{1}{q}}\right)d\mu(x)$$

$$+\int_{X_2}(p(x)-q(x))$$

$$\times\left(\left[|f'(1)|^q+2\left|f'\left(\frac{q(x)}{p(x)}\right)\right|^q\right]^{\frac{1}{q}}+\left[2|f'(1)|^q+\left|f'\left(\frac{q(x)}{p(x)}\right)\right|^q\right]^{\frac{1}{q}}\right)d\mu(x)\Bigg\}.$$

Proof. The proof is similar as to that of Proposition 5.1 but use Theorem 4.6. Q.E.D.

6 Applications to special means and new error estimates

In this section, we are concerned with the applications of some our results given in Section 4 to some special means of real numbers as well as new error estimates of trapezoidal and midpoint formula. For this purpose, we consider the following means of different positive real numbers a and b.

1. The arithmetic mean:
$$A := A(a,b) = \frac{a+b}{2}.$$
2. The harmonic mean:
$$H := H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}}.$$
3. The logarithmic mean:
$$L := L(a,b) = \frac{b-a}{\ln b - \ln a}.$$
4. The generalized log-mean:
$$L_n := L_n(a,b) = \left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}};\ \ n\in\mathbb{Z}\setminus\{-1,0\}.$$

In the following propositions applications of the above results to certain means of different positive real numbers have been incorporated.

Proposition 6.1. Let $0 < a < b$, $n \in \{2, 3, \dots\}$, $0 < r \leq 1$, $\rho > 0$ and $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have

$$\left| A\left((ra)^{n\rho}, b^{n\rho}\right) - L_n^n((ra)^{\rho}, b^{\rho}) \right|$$

$$\leq \left(\frac{2}{\rho+1}\right)^{\frac{1}{q}} \frac{|n|(b^{\rho} - (ra)^{\rho})}{2} \sqrt[p]{2E(p,\rho)} A^{\frac{1}{q}}\left((ra)^{q\rho(n-1)}, b^{q\rho(n-1)}\right), \tag{6.1}$$

where $E(p, \rho)$ is defined as in Proposition 5.1.

Proof. Consider the convex function $f(x) = x^n$, $x > 0$ and using Theorem 4.3 for $\alpha = 1$, we obtain the required inequality. Q.E.D.

Proposition 6.2. Let $0 < a < b$, $n \in \mathbb{Z}$, $|n| \geq 2$, $0 < r \leq 1$ and $q \geq 1$. Then, we have

$$\left| A\left((ra)^n, b^n\right) - L_n^n(ra, b) \right| \leq \frac{|n|(b - ra)}{4} A^{\frac{1}{q}}\left((ra)^{q(n-1)}, b^{q(n-1)}\right). \tag{6.2}$$

Proof. Consider the function $f(x) = x^n$, $x > 0$ but use Theorem 4.4 for $\alpha = \rho = 1$. Q.E.D.

Proposition 6.3. Let $0 < a < b$, $n \in \mathbb{Z}$, $|n| \geq 2$, $0 < r \leq 1$, $\rho > 0$ and $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have

$$\left| A\left((ra)^{n\rho}, b^{n\rho}\right) - L_n^n((ra)^{\rho}, b^{\rho}) \right|$$

$$\leq \left(\frac{2}{\rho+1}\right)^{\frac{1}{q}} |n|(b^{\rho} - (ra)^{\rho}) \left(\frac{1}{p\rho+1}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left((ra)^{q\rho(n-1)}, b^{q\rho(n-1)}\right). \tag{6.3}$$

Proof. Consider the function $f(x) = x^n$, $x > 0$ but use Theorem 4.5 for $\alpha = 1$. Q.E.D.

Proposition 6.4. Let $0 < a < b$, $n \in \mathbb{Z}$, $|n| \geq 2$, $0 < r \leq 1$ and $q \geq 1$. Then, we have

$$\left| A\left((ra)^n, b^n\right) - L_n^n(ra, b) \right| \leq \frac{|n|(b - ra)}{4} \left(\frac{1}{3}\right)^{\frac{1}{q}} \tag{6.4}$$

$$\times \left\{ A^{\frac{1}{q}}\left(2(ra)^{q(n-1)}, 4b^{q(n-1)}\right) + A^{\frac{1}{q}}\left(4(ra)^{q(n-1)}, 2b^{q(n-1)}\right) \right\}.$$

Proof. Consider the function $f(x) = x^n$, $x > 0$ but use Theorem 4.6 for $\alpha = \rho = 1$. Q.E.D.

Proposition 6.5. Let $0 < a < b$, $n \in \mathbb{Z}$, $|n| \geq 2$, $0 < r \leq 1$ and $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have

$$\left| L_n^n(ra, b) - A^n(ra, b) \right| \leq |n|(b - ra) \tag{6.5}$$

$$\times \left\{ \left(\frac{1}{p+1}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left((ra)^{q(n-1)}, b^{q(n-1)}\right) + \frac{1}{4} A\left((ra)^{q(n-1)}, 3b^{q(n-1)}\right) \right\}.$$

Proof. Consider the function $f(x) = x^n$, $x > 0$ but use Theorem 4.9 for $\alpha = \rho = 1$. Q.E.D.

Proposition 6.6. Let $0 < a < b$, $n \in \mathbb{Z}$, $|n| \geq 2$, $0 < r \leq 1$ and $q \geq 1$. Then, we have

$$\left|L_n^n(ra,b) - A^n(ra,b)\right| \leq |n|(b-ra) \tag{6.6}$$

$$\times\left\{\frac{1}{2}\left(\frac{1}{6}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(2(ra)^{q(n-1)}, 4b^{q(n-1)}\right) + \frac{1}{4}A\left((ra)^{q(n-1)}, 3b^{q(n-1)}\right)\right\}.$$

Proof. Consider the function $f(x) = x^n$, $x > 0$ but use Theorem 4.10 for $\alpha = \rho = 1$. Q.E.D.

Proposition 6.7. Let $0 < a < b$, $0 < r \leq 1$, $\rho > 0$ and $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have

$$\left|\frac{1}{H((ra)^\rho, b^\rho)} - \frac{1}{L((ra)^\rho, b^\rho)}\right|$$

$$\leq \left(\frac{2}{\rho+1}\right)^{\frac{1}{q}} \frac{(b^\rho - (ra)^\rho)}{2} \sqrt[p]{2E(p,\rho)} \left(\frac{1}{H\left((ra)^{2q\rho}, b^{2q\rho}\right)}\right)^{\frac{1}{q}}, \tag{6.7}$$

where $E(p,\rho)$ is defined as in Proposition 5.1.

Proof. Consider the convex function $f(x) = \frac{1}{x}$, $x > 0$ and using Theorem 4.3 for $\alpha = 1$, we obtain the required inequality. Q.E.D.

Proposition 6.8. Let $0 < a < b$, $0 < r \leq 1$ and $q \geq 1$. Then, we have

$$\left|\frac{1}{H(ra,b)} - \frac{1}{L(ra,b)}\right| \leq \frac{(b-ra)}{4}\left(\frac{1}{H\left((ra)^{2q}, b^{2q}\right)}\right)^{\frac{1}{q}}. \tag{6.8}$$

Proof. Consider the function $f(x) = \frac{1}{x}$, $x > 0$ but use Theorem 4.4 for $\alpha = \rho = 1$. Q.E.D.

Proposition 6.9. Let $0 < a < b$, $0 < r \leq 1$, $\rho > 0$ and $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have

$$\left|\frac{1}{H((ra)^\rho, b^\rho)} - \frac{1}{L((ra)^\rho, b^\rho)}\right|$$

$$\leq \left(\frac{2}{\rho+1}\right)^{\frac{1}{q}} (b^\rho - (ra)^\rho)\left(\frac{1}{p\rho+1}\right)^{\frac{1}{p}}\left(\frac{1}{H\left((ra)^{2q\rho}, b^{2q\rho}\right)}\right)^{\frac{1}{q}}. \tag{6.9}$$

Proof. Consider the function $f(x) = \frac{1}{x}$, $x > 0$ but use Theorem 4.5 for $\alpha = 1$. Q.E.D.

Proposition 6.10. Let $0 < a < b$, $0 < r \leq 1$ and $q \geq 1$. Then, we have

$$\left|\frac{1}{H(ra,b)} - \frac{1}{L(ra,b)}\right| \leq \frac{(b-ra)}{4}\left(\frac{4}{3}\right)^{\frac{1}{q}} \tag{6.10}$$

$$\times\left\{\left(\frac{1}{H\left(2(ra)^{2q}, b^{2q}\right)}\right)^{\frac{1}{q}} + \left(\frac{1}{H\left((ra)^{2q}, 2b^{2q}\right)}\right)^{\frac{1}{q}}\right\}.$$

Proof. Consider the function $f(x)=\frac{1}{x}$, $x>0$ but use Theorem 4.6 for $\alpha=\rho=1$. Q.E.D.

Proposition 6.11. Let $0<a<b$, $0<r\leq 1$ and $q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$. Then, we have

$$\left|\frac{1}{L(ra,b)}-\frac{1}{A(ra,b)}\right|\leq (b-ra) \tag{6.11}$$

$$\times\left\{\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{H\left((ra)^{2q},b^{2q}\right)}\right)^{\frac{1}{q}}+\frac{3}{4}\frac{1}{H\left(3(ra)^{2q},b^{2q}\right)}\right\}.$$

Proof. Consider the function $f(x)=\frac{1}{x}$, $x>0$ but use Theorem 4.9 for $\alpha=\rho=1$. Q.E.D.

Proposition 6.12. Let $0<a<b$, $0<r\leq 1$ and $q\geq 1$. Then, we have

$$\left|\frac{1}{L(ra,b)}-\frac{1}{A(ra,b)}\right|\leq (b-ra) \tag{6.12}$$

$$\times\left\{\frac{1}{2}\left(\frac{1}{6}\right)^{\frac{1}{q}}\left(\frac{1}{H\left(2(ra)^{2q},b^{2q}\right)}\right)^{\frac{1}{q}}+\frac{3}{4}\frac{1}{H\left(3(ra)^{2q},b^{2q}\right)}\right\}.$$

Proof. Consider the function $f(x)=\frac{1}{x}$, $x>0$ but use Theorem 4.10 for $\alpha=\rho=1$. Q.E.D.

Next, we provide some new error estimates for the trapezoidal and midpoint formula.
Let P be the partition of the points $a=x_0<x_1<\ldots<x_n=b$ of the interval $[a,b]$. Let consider the following quadrature formulas:

$$\int_a^b f(x)dx=T(f,P)+E(f,P),\quad \int_a^b f(x)dx=M(f,P)+E^*(f,P),$$

where

$$T(f,P)=\sum_{i=0}^{n-1}\frac{f(x_i+x_{i+1})}{2}(x_{i+i}-x_i)$$

and

$$M(f,P)=\sum_{i=0}^{n-1}f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+i}-x_i)$$

are respectively, the trapezoidal version and midpont version, and $E(f,P),E^*(f,P)$ are denotes their associated approximation errors.

Proposition 6.13. Let $f:[a,b]\longrightarrow\mathbb{R}$ be a differentiable function on (a,b) with $0<a<b$. If $|f'|^q$ is convex on $[a,b]$ for $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:

$$\left|E(f,P)\right|\leq\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^2}{4}\left[|f'(x_i)|^q+|f'(x_{i+1})|^q\right]^{\frac{1}{q}}. \tag{6.13}$$

Proof. Applying Theorem 4.3 for $\alpha = r = \rho = 1$ on the subintervals $[x_i, x_{i+1}]$ $(i = 0, \ldots, n-1)$ of the partition P, we have

$$\left| \frac{f(x_i) + f(x_{i+1})}{2} - \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(x)dx \right|$$

$$\leq \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{x_{i+1} - x_i}{2} \right) \left[\frac{|f'(x_i)|^q + |f'(x_{i+1})|^q}{2} \right]^{\frac{1}{q}}. \tag{6.14}$$

Hence from (6.14), we get

$$\begin{aligned} \big|E(f,P)\big| &= \left| \int_a^b f(x)dx - T(f,P) \right| \\ &\leq \left| \sum_{i=0}^{n-1} \left\{ \int_{x_i}^{x_{i+1}} f(x)dx - \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i) \right\} \right| \\ &\leq \sum_{i=0}^{n-1} \left| \left\{ \int_{x_i}^{x_{i+1}} f(x)dx - \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i) \right\} \right| \\ &\leq \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^2}{4} \Big[|f'(x_i)|^q + |f'(x_{i+1})|^q \Big]^{\frac{1}{q}}. \end{aligned}$$

The proof of Proposition 6.13 is completed. Q.E.D.

Proposition 6.14. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a differentiable function on (a, b) with $0 < a < b$. If $|f'|^q$ is convex on $[a, b]$ for $q \geq 1$, then the following inequality holds:

$$\big|E(f,P)\big| \leq \left(\frac{1}{2} \right)^{2+\frac{1}{q}} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \Big[|f'(x_i)|^q + |f'(x_{i+1})|^q \Big]^{\frac{1}{q}}. \tag{6.15}$$

Proof. The proof is analogous as to that of Proposition 6.13 but use Theorem 4.4 for $\alpha = r = \rho = 1$. Q.E.D.

Proposition 6.15. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a differentiable function on (a, b) with $0 < a < b$. If $|f'|^q$ is convex on $[a, b]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:

$$\big|E(f,P)\big| \leq \left(\frac{1}{2} \right)^{\frac{1}{q}} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \Big[|f'(x_i)|^q + |f'(x_{i+1})|^q \Big]^{\frac{1}{q}}. \tag{6.16}$$

Proof. The proof is analogous as to that of Proposition 6.13 but use Theorem 4.5 for $\alpha = r = \rho = 1$. Q.E.D.

Proposition 6.16. Let $f:[a,b]\longrightarrow\mathbb{R}$ be a differentiable function on (a,b) with $0<a<b$. If $|f'|^q$ is convex on $[a,b]$ for $q\geq 1$, then the following inequality holds:

$$\left|E(f,P)\right|\leq\frac{1}{4}\left(\frac{1}{3}\right)^{\frac{1}{q}}$$

$$\times\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left\{\left[|f'(x_i)|^q+2|f'(x_{i+1})|^q\right]^{\frac{1}{q}}+\left[2|f'(x_i)|^q+|f'(x_{i+1})|^q\right]^{\frac{1}{q}}\right\}. \tag{6.17}$$

Proof. The proof is analogous as to that of Proposition 6.13 but use Theorem 4.6 for $\alpha=r=\rho=1$. Q.E.D.

Proposition 6.17. Let $f:[a,b]\longrightarrow\mathbb{R}$ be a differentiable function on (a,b) with $0<a<b$. If $|f'|^q$ is convex on $[a,b]$ for $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:

$$\left|E^*(f,P)\right|\leq\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left\{\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{|f'(x_i)|^q+|f'(x_{i+1})|^q}{2}\right]^{\frac{1}{q}}\right. \tag{6.18}$$

$$\left.+\frac{1}{8}\left[|f'(x_i)|^q+3|f'(x_{i+1})|^q\right]\right\}.$$

Proof. Applying Theorem 4.9 for $\alpha=r=\rho=1$ on the subintervals $[x_i,x_{i+1}]$ $(i=0,\ldots,n-1)$ of the partition P, we have

$$\left|f\left(\frac{x_i+x_{i+1}}{2}\right)-\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}f(x)dx\right|$$

$$\leq(x_{i+1}-x_i)^2\left\{\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{|f'(x_i)|^q+|f'(x_{i+1})|^q}{2}\right]^{\frac{1}{q}}\right. . \tag{6.19}$$

Hence from (6.19), we get

$$\left|E^*(f,P)\right|=\left|\int_a^b f(x)dx-M(f,P)\right|$$

$$\leq\left|\sum_{i=0}^{n-1}\left\{\int_{x_i}^{x_{i+1}}f(x)dx-f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)\right\}\right|$$

$$\leq\sum_{i=0}^{n-1}\left|\left\{\int_{x_i}^{x_{i+1}}f(x)dx-f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)\right\}\right|$$

$$\leq\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left\{\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{|f'(x_i)|^q+|f'(x_{i+1})|^q}{2}\right]^{\frac{1}{q}}\right.$$

$$\left.+\frac{1}{8}\left[|f'(x_i)|^q+3|f'(x_{i+1})|^q\right]\right\}.$$

The proof of Proposition 6.17 is completed. Q.E.D.

Proposition 6.18. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a differentiable function on (a, b) with $0 < a < b$. If $|f'|^q$ is convex on $[a, b]$ for $q \geq 1$, then the following inequality holds:

$$\left|E^*(f,P)\right| \leq \sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left\{\frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{q}}\left[|f'(x_i)|^q+2|f'(x_{i+1})|^q\right]^{\frac{1}{q}}\right. \tag{6.20}$$

$$\left.+\frac{1}{8}\left[|f'(x_i)|^q+3|f'(x_{i+1})|^q\right]\right\}.$$

Proof. The proof is analogous as to that of Proposition 6.17 but use Theorem 4.10 for $\alpha = r = \rho = 1$. Q.E.D.

7 Conclusion

Interested reader can obtain in a similar way new results for different operators such as the k-Riemann-Liouville fractional integral, Katugampola fractional integrals, Hadamard fractional integrals, generalized fractional integrals etc. This results can be applied in convex analysis, optimization and also different areas of pure and applied sciences.

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