Some properties of q-Bernstein-Durrmeyer operators

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Abstract

In the present paper we shall investigate the pointwise approximation properties of the q analogue of the Bernstein-Durrmeyer operators and estimate the rate of pointwise convergence of these operators to the functions f whose q-derivatives are bounded variation on the interval $[0,1]$. We give an estimate for the rate of convergence of the operator $(L_{n,q}f)$ at those points x at which the one sided q-derivatives $D_q^+ f(x), D_q^- f(x)$exist. We shall also prove that the operators $L_{n,q}f$ converges to the limit $f(x)$. To the best of my knowledge, the present study will be the first study on the approximation of q- operators in the space of D_qBV.

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1 Introduction

The theory of quantum calculus (or q analysis) has attracted the attention of several researchers (see [1], [5]-[10], [16]-[18] and the references therein), because of its potential for applications, since this theory can be used to investigate black holes, quantum mechanics, nuclear and high energy physics, mathematical physics, functional analysis and especially in the last decades approximation theory and so on. The subject of q-analysis concerns mainly the properties of the so-called q-special functions, which are the extensions of the classical special functions based on a parameter, or the base, q.

Due to this relation, in the year 1997, Phillips [16] introduced the generalization of Bernstein polynomials based on q-integers to prove the approximation (or superposition) problem of Weierstrass. It is useful to point out that, the first person to define the q-analog of the Bernstein operators was Lupas [12]. Recently Derriennic [1] defined a q-Bernstein Durrmeyer polynomials and studied some approximation properties of this operators. Later Gupta [5] defined another q-analogue of the well-known Durrmeyer operators and established some approximation properties of his q-Durrmeyer operators.

For a function defined on the interval [0,1], the Durrmeyer operators (M_nf) (see e.g [2]), are defined by

$$(M_nf)(x) = (n+1)\sum_{k=0}^{n} P_{n,k}(x) \int_0^1 f(t)P_{n,k}(t)\ dt,\ \ 0 \le x \le 1, \tag{1}$$

where $P_{n,k}(x) = \left(\begin{array}{c} n \\ k \end{array} \right) x^k(1-x)^{n-k}$. Lupas [11] also introduced these operators later in some different form.

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These operators are the integral modification of Bernstein polynomials so as to approximate Lebesgue integrable functions on the interval [0,1]. Zeng-Chen [19] and Guo [4] recently estimated the convergence rate of $(M_n f)$ operators for functions of bounded variation on the interval [0,1].

We now deal with the Gupta's q-analogue of Durrmeyer operators $(L_{n,q})$, defined as

$$(L_{n,q}f)(x) = [n+1]_q \sum_{k=0}^{n} q^{-k} P_{n,k,q}(x) \int_0^1 f(t) P_{n,k,q}(qt)\, d_q t, \;\; 0 \le x \le 1, \tag{2}$$

where $p_{k,n,q}(x) = \left[\begin{array}{c} n \\ k \end{array} \right]_q x^k \prod_{s=0}^{n-k-1}(1-q^s x)$ is the Bernstein basis $(0 \le x \le 1)$.
It can be easily verified that in case $q = 1$, the operators defined by (2) reduce to the well known Durrmeyer operators (1).

Actually several researchers have defined the q analogue of only discrete type operators. Here our aim is to study the q analogue of summation-integral type operators. Here we estimate the rate of convergence of operators $(L_{n,q}f)$ which have derivative of bounded variation on the interval $[0,1]$. At the point x, which is a discontinuity of the first kind of the $q-$*derivative*, we shall prove that the operator $(L_{n,q}f)$ being defined in (2) converge to the limit $f(x)$. To the best of my knowledge, this study is the first study on the approximation of $q-$operators in this space.

Before getting onto the main subject, we first give definitions of $q-$*number*, $q-$*binomial* coefficient, $q-$*derivative* and $q-$*integral*, together with their properties of q calculus, which are required in this paper. For any fixed real number $q > 0$ and non-negative integer r the $q-$*integer* of the number r is defined by

$$[r]_q = \begin{cases} (1-q^r)/(1-q) & , \;\; q \neq 1 \\ r & , \;\; q = 1 \end{cases}.$$

The $q-$*factorial* is defined by

$$[r]_q! = \begin{cases} [r]_q[r-1]_q.....[1]_q & , \;\; r = 1,2,3,..... \\ 1 & , \;\; r = 0. \end{cases},$$

and $q-$*Binomial* coefficient is defined as

$$\left[\begin{array}{c} n \\ r \end{array} \right]_q = \frac{[n]_q!}{[r]_q![n-r]_q!},$$

for integers $n \ge r \ge 0$.

Let $q \in (0,1)$ and let I be a real interval containing 0.

Definition 1. Let $f : I \to R$ be a function and let $x \in I$. The Jackson's $q-$derivative D_q (see [8]), or Jackson's difference operator, of a function f at x is given by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \;\; if \;\; x \neq 0, \qquad D_q f(0) := \lim_{x \to 0} D_q f(x). \tag{3}$$

Here $D_q f(x)$ tends to $f'(x)$ as q tends to 1, provided $f'(x)$ exists.

Definition 2. Let $f : I \to R$ be a function and let $x \in I$. The left (backward), right (forward) and symmetric $q-derivatives$ of a function f are defined as, respectively,

$$D_q^- f(x) := \frac{f(x) - f(qx)}{(1-q)x}, \tag{4}$$

$$D_q^+ f(x) := \frac{f\left(\frac{x}{q}\right) - f(x)}{(1-q)x} \tag{5}$$

and

$$D_q^s f(x) := \frac{f(qx) - f\left(\frac{x}{q}\right)}{\left(q - \frac{1}{q}\right)x}. \tag{6}$$

provided that $x \neq 0$. (See [10]).

Since $q-$calculus is an interdisciplinary subject, many different definitions of the above derivatives have been used. As an example, the left (backward) $q-derivative$ of $f : I \to R$ at $x \neq 0$ is defined by the quotient

$$D_q^- f(x) := \frac{f(x) - f\left(\frac{x}{q}\right)}{\left(1 - \frac{1}{q}\right)x}. \tag{7}$$

(See [10]).

Remark 1. Note that the Jackson's $q-derivative$ is sometimes called the left (backward) $q-derivative$. This means that

$$D_q f(x) = D_q^- f(x).$$

It is clear that if a function is f differentiable (in the classical sense) at x, then

$$\lim_{q \to 1^-} D_q^- f(x) = \lim_{q \to 1^-} D_q^+ f(x) = f'(x),$$

holds true, where f'denotes the usual derivative.

The generalized form of the backward and forward $q-derivative$ operators, together with some relations, can be found in [13].

Remark 2. We note that, a continuous function on an interval, which does not include zero, is continuous $q-$differentiable.

Remark 3. Although the difference operators convey the same idea, it turns out that their proper choice in constructing the Fourier transform between configuration and momentum space [[3], p. 1797] and solving $q-$diffusion equations by Laplace and Mellin transform [[6], sections 2-3].

Example 1. We consider the following $q-$diffusion equation

$$D_t^q y(x,t) = \frac{\partial^2}{\partial x^2} y(x,t), \qquad (-\infty < x < \infty, \quad t > 0)$$

with the initial condition

$$y(x,0)=f(x).$$

The question now is to choose an appropriate integral transform to remove the $q-derivative$. In view of the positivity of the time variable, the two most natural choices are the Laplace and the Mellin transform.

If one use the Laplace transform of the $q-derivatives$ given in (4) and (7), and trying to solve this $q-$diffusion equation, then Ho [[6], sections 2] informed us that the Laplace transform is not useful in solving equations involving $q-derivatives$.

Moreover, if we use the Mellin transform of the $q-derivatives$ given is (4) and (7), and trying to solve the $q-$diffusion equation, then Ho [[6], sections 3] showed that the $q-derivative$ given in (7) more useful and simpler than the use $q-derivative$ given in (4).

From the definition of the $q-derivative$ one can further obtain the $q-derivative$ of a product as follows;

$$D_q\left(f(x)g(x)\right)=D_q\left(f(x)\right)g(x)+f(qx)D_q\left(g(x)\right).$$

Now, we mention about $q-integral$ or the so called Jackson $q-integral$.

The $q-analog$ of integration, independently discovered by Thomae [17] and Jackson [7], in the interval $[0,a]$ is defined by

$$\int_0^a f(t)d_qt:=a(1-q)\sum_{n=0}^{\infty}f(aq^n)q^n \qquad 0<q<1, \tag{8}$$

and over a general interval:

$$\int_c^d f(t)d_qt:=\int_0^d f(t)d_qt-\int_0^c f(t)d_qt.$$

It is worthy to mention some important properties of the $q-integration$. For example:

1- From $q-derivative$ of a product, one can also obtain the following formula of $q-integration$ by parts

$$\int_c^d f(t)D_qg(t)d_qt=f(t)g(t)\,\big|_c^d-\int_c^d g(qt)D_qf(t)d_qt$$

provided f and g are continuous on $[c,d]$.

2- For continuous functions $f(t)$ one can easily see that $q-integral$ approaches the Riemann integral as $q\to 1$, and also that the operators of $q-differentiation$ and indefinite $q-integration$ are inverse to each other, namely

$$D_{q,t}\int f(t)d_qt=f(t)=\int D_{q,t}f(t)d_qt.$$

3- If F is any anti $q-derivative$ of the function f, namely $D_qF=f$, continuous at $t=0$, then

$$\int_0^a f(t)d_qt = F(a)-F(0).$$

Let $D_qBV[a,b]$ denotes the class of real valued $q-differentiable$ functions defined on a set $[a,b]$, whose $q-derivatives$ are bounded variation on $[a,b]$, which can be written as

$$f(x)=C+\int_a^x \Psi(t)d_qt,\ -\infty<a\le x\le b$$

where C is a constant and $\Psi\in BV[a,b]$. It is clear that

$$D_qBV[a,b]:=\{f:D_qf=\Psi\in BV[a,b]\}.$$

The first theorem of this paper is stated as:

Theorem 1. *Let f be a bounded function on $[0,1]$. Suppose that the right and left q-derivatives exist at a point $x\in(0,1)$. Then at this point $x\in(0,1)$, and n sufficiently large, we have*

$$\begin{aligned}|(L_{n,q}f)(x)-f(x)| &\le \left|\frac{D_q^+f(x)+D_q^-f(x)}{2}\right|\left(\left(\frac{1}{q}-1\right)+\frac{1}{[n+2]_q}\right)\\ &+\left|\frac{D_q^+f(x)-D_q^-f(x)}{2}\right|\sqrt{A_{n,q}(x)}+(1-q)\sum_{l=0}^{\infty}q^l\left[\frac{1}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{1-x}{\sqrt{n}}}(D_qf_x)+\frac{A_{n,q}(x)}{x(1-x)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{k}}(D_qf_x)\right]\end{aligned} \tag{9}$$

where

$$D_qf_x(t)=\begin{cases} D_qf(t)-D_q^+f(x) & ,\ x<t\le 1\\ 0 & ,\ t=x\\ D_qf(t)-D_q^-f(x) & ,\ 0\le t<x\end{cases}, \tag{10}$$

$\bigvee_a^b(f)$ *is the total variation (or Jordan Variation) of f on $[a,b]$ and*

$$\begin{aligned}A_{n,q}(x) &:= x^2\left(\frac{[n]_q^2(1-q)^2-2q[n]_q+6}{[n+3]_q[n+2]_q}\right)\\ &+x\left(\frac{q[n]_q(1+q)^2}{[n+3]_q[n+2]_q}-\frac{2}{[n+2]_q}\right)+\frac{2}{[n+3]_q[n+2]_q}.\end{aligned}$$

In view of Theorem 1 and Remark 2, we get the following corollary:

Corollary 1. *Let $f \in C\,[0,1]$. Then for every $x \in (0,1)$, and n sufficiently large, we have*

$$|(L_{n,q}f)(x) - f(x)| \leq |D_q f(x)| \left(\left(\frac{1}{q} - 1 \right) + \frac{1}{[n+2]_q} \right)$$

$$+(1-q)\sum_{l=0}^{\infty} q^l \left[\frac{1}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{1-x}{\sqrt{n}}} (D_q f_x) + \frac{A_{n,q}(x)}{x(1-x)} \sum_{k=1}^{\left[\sqrt{n}\right]} \bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{k}} (D_q f_x) \right]$$

2 Auxiliary results

In this section, our goal is to present some basic concepts concerning the theory of quantum calculus and some lemmas about aforementioned operators, which are necessary to prove our theorems.

Lemma 1 [5]. *For $s \in N$ and $s \leq n$*

$$(L_{n,q}t^s)(x) = \frac{[n+1]_q!}{[n+s+1]_q!} \sum_{k=0}^{n} P_{n,k,q}(x) \frac{[k+s]_q!}{[k]_q!}.$$

In particular one has

$$(L_{n,q}1)(x) = 1, \quad (L_{n,q}t)(x) = \frac{1 + xq\,[n]_q}{[n+2]_q}$$

and

$$\left(L_{n,q}t^2\right)(x) = \frac{x^2 q^3\,[n]_q \left([n]_q - 1\right) + xq\,[n]_q\,(1+q)^2 + 1 + q}{[n+3]_q\,[n+2]_q}.$$

Remark 4. *By elemantary computation, we get*

$$\begin{aligned} \left(L_{n,q}(t-x)^2\right)(x) &\leq x^2 \left(\frac{[n]_q^2\,(1-q)^2 - 2q\,[n]_q + 6}{[n+3]_q\,[n+2]_q} \right) \\ &\quad + x \left(\frac{q\,[n]_q\,(1+q)^2}{[n+3]_q\,[n+2]_q} - \frac{2}{[n+2]_q} \right) + \frac{2}{[n+3]_q\,[n+2]_q} \\ &= : A_{n,q}(x). \end{aligned} \tag{11}$$

Lemma 2. *For all $x \in (0,1)$, and n sufficiently large, then for $0 < t < x$, we have*

$$\lambda_{n,q}(x,t) = \int_0^t K_{n,q}(x,u)\,d_q u \leq \frac{A_{n,q}(x)}{(x-t)^2}, \tag{12}$$

where

$$K_{n,q}(x,u) = \begin{cases} [n+1]_q \sum\limits_{k \leq nu} P_{n,k,q}(x) P_{n,k,q}(u) & , 0 < u \leq 1 \\ 0 & , \ u = 0 \end{cases}$$

Proof. Clearly

$$\lambda_{n,q}(x,t)=\int_0^t K_{n,q}(x,u)d_qu\leq\int_0^t K_{n,q}(x,u)(\frac{x-u}{x-t})^2d_qu$$

$$=\frac{1}{(x-t)^2}\int_0^t K_{n,q}(x,u)(x-u)^2d_qu=\frac{\left(L_{n,q}(u-x)^2\right)(x)}{(x-t)^2}.$$

By (11), one can easily obtain

$$\lambda_{n,q}(x,t)\leq\frac{1}{(x-t)^2}A_{n,q}(x).$$

Remark 5. *By Cauchy-Schwarz-Bunyakowsky inequality, one has*

$$\left(L_{n,q}\left|t-x\right|\right)(x)\leq\left(\left(L_{n,q}(t-x)^2\right)(x)\right)^{\frac{1}{2}}\leq\sqrt{A_{n,q}(x)}. \tag{13}$$

Remark 6. *By Lemma 1, we have*

$$\left(L_{n,q}\left(t-x\right)\right)(x)=\left|\frac{1+xq\left[n\right]_q}{\left[n+2\right]_q}-x\right|\leq\left(\left(\frac{1}{q}-1\right)+\frac{1}{\left[n+2\right]_q}\right). \tag{14}$$

3 Main results

Proof of Theorem 1. We write the difference between $\left(L_{n,q}f\right)(x)$ and $f(x)$ as follows

$$\begin{aligned}\left(L_{n,q}f\right)(x)-f(x) &= \left[n+1\right]_q\sum_{k=0}^{n}P_{n,k,q}\left(x\right)\int_0^1 f(t)P_{n,k,q}\left(t\right)d_qt-f(x)\\ &= \int_0^1\left[f(t)-f(x)\right]K_{n,q}(x,t)d_qt,\end{aligned}$$

where $K_{n,q}(x,t)$ being defined in Lemma 2.

Since $D_qf=\Psi\in BV[0,1]$,

$$\left(L_{n,q}f\right)(x)-f(x)=\int_0^x\left[f(t)-f(x)\right]K_n(x,t)d_qt+\int_x^1\left[f(t)-f(x)\right]K_n(x,t)d_qt$$

$$\begin{aligned}&= -\int_0^x\left[\int_t^x D_qf(u)\,d_qu\right]K_{n,q}(x,t)\,d_qt+\int_x^1\left[\int_x^t D_qf(u)\,d_qu\right]K_{n,q}(x,t)d_qt\\ &= -E_{1,q}(x)+E_{2,q}(x)\end{aligned}$$

where

$$E_{1,q}(x) := \int_0^x \left[\int_t^x D_q f(u)\, d_q u \right] K_{n,q}(x,t)\, d_q t \tag{15}$$

and

$$E_{2,q}(x) := \int_x^1 \left[\int_x^t D_q f(u)\, d_q u \right] K_{n,q}(x,t)\, d_q t. \tag{16}$$

Considering the definitions (4)-(3) and (10), for any $D_q f = \Psi \in BV[0,1]$,we decompose $D_q f(t)$ into four parts as

$$\begin{aligned} D_q f(t) &= \frac{D_q^+ f(x) + D_q^- f(x)}{2} + D_q f_x(t) + \frac{D_q^+ f(x) - D_q^- f(x)}{2} sgn(t-x) \\ &\quad + \delta_x(t) \left[D_q f(x) - \frac{D_q^+ f(x) + D_q^- f(x)}{2} \right] \end{aligned} \tag{17}$$

where

$$\delta_x(t) = \begin{cases} 1\ , & x = t \\ 0\ , & x \neq t. \end{cases} .$$

If we use (17) in (15) and (16), we have the following expressions.

$$\begin{aligned} E_{1,q}(x) &= \int_0^x \left\{ \int_t^x \frac{D_q^+ f(x) + D_q^- f(x)}{2} + D_q f_x(u) + \frac{D_q^+ f(x) - D_q^- f(x)}{2} sgn(u-x) \right. \\ &\quad \left. + \delta_x(u) \left[D_q f(x) - \frac{D_q^+ f(x) + D_q^- f(x)}{2} \right] d_q u \right\} K_{n,q}(x,t)\, d_q t \end{aligned}$$

and

$$\begin{aligned} E_{2,q}(x) &= \int_x^1 \left\{ \int_x^t \frac{D_q^+ f(x) + D_q^- f(x)}{2} + D_q f_x(u) + \frac{D_q^+ f(x) - D_q^- f(x)}{2} sgn(u-x) \right. \\ &\quad \left. + \delta_x(u) \left[D_q f(x) - \frac{D_q^+ f(x) + D_q^- f(x)}{2} \right] d_q u \right\} K_{n,q}(x,t)\, d_q t. \end{aligned}$$

Firstly, we evaluate $E_{1,q}(x)$.

$$\begin{aligned} E_{1,q}(x) &= \frac{D_q^+ f(x) + D_q^- f(x)}{2} \int_0^x (x-t) K_{n,q}(x,t)\, d_q t \\ &+ \int_0^x \left[\int_x^t D_q f_x(u)\, d_q u \right] K_{n,q}(x,t)\, d_q t \\ &- \frac{D_q^+ f(x) - D_q^- f(x)}{2} \int_0^x (x-t) K_{n,q}(x,t)\, d_q t \\ &+ \left[D_q f(x) - \frac{D_q^+ f(x) + D_q^- f(x)}{2} \right] \int_0^x \left[\int_x^t \delta_x(u)\, d_q u \right] K_{n,q}(x,t)\, d_q t \end{aligned}$$

Since $\int_x^t \delta_x(u)\, d_q u = 0$, one has

$$\begin{aligned} E_{1,q}(x) &= \frac{D_q^+ f(x) + D_q^- f(x)}{2} \int_0^x (x-t) K_{n,q}(x,t)\, d_q t \\ &+ \int_0^x \left[\int_x^t D_q f_x(u)\, d_q u \right] K_{n,q}(x,t)\, d_q t \\ &- \frac{D_q^+ f(x) - D_q^- f(x)}{2} \int_0^x (x-t) K_{n,q}(x,t)\, d_q t. \end{aligned} \tag{18}$$

Using similar method, for evaluating $E_{2,q}(x)$, we find that

$$\begin{aligned} E_2(x) &= \frac{D_q^+ f(x) + D_q^- f(x)}{2} \int_x^1 (t-x) K_{n,q}(x,t)\, d_q t \\ &+ \int_x^1 \left[\int_x^t D_q f_x(u)\, d_q u \right] K_{n,q}(x,t)\, d_q t \\ &- \frac{D_q^+ f(x) - D_q^- f(x)}{2} \int_x^1 (t-x) K_{n,q}(x,t)\, d_q t. \end{aligned} \tag{19}$$

Combining (18) and (19),

$$\begin{aligned} -E_{1,q}(x)+E_{2,q}(x) &= \frac{D_q^+f(x)+D_q^-f(x)}{2}\int_0^1 (t-x)K_{n,q}(x,t)\,d_qt \\ &\quad +\frac{D_q^+f(x)-D_q^-f(x)}{2}\int_0^1 |t-x|\,K_{n,q}(x,t)\,d_qt \\ &\quad -\int_0^x \left[\int_t^x D_qf_x(u)\,d_qu\right] K_{n,q}(x,t)\,d_qt \\ &\quad +\int_x^1 \left[\int_x^t D_qf_x(u)\,d_qu\right] K_{n,q}(x,t)\,d_qt. \end{aligned} \tag{20}$$

From (20), we can rewrite the difference between $(L_{n,q}f)(x)$ and $f(x)$,

$$\begin{aligned} (L_{n,q}f)(x)-f(x) &= \frac{D_q^+f(x)+D_q^-f(x)}{2}\int_0^1 (t-x)K_{n,q}(x,t)\,d_qt \\ &\quad +\frac{D_q^+f(x)-D_q^-f(x)}{2}\int_0^{b_n} |t-x|\,K_{n,q}(x,t)\,d_qt \\ &\quad -\int_0^x \left[\int_t^x D_qf_x(u)\,d_qu\right] K_{n,q}(x,t)\,d_qt \\ &\quad +\int_x^1 \left[\int_x^t D_qf_x(u)\,d_qu\right] K_{n,q}(x,t)\,d_qt. \end{aligned} \tag{21}$$

On the other hand, since

$$\int_0^1 |t-x|\,K_{n,q}(x,t)\,d_qt = (L_{n,q}\,|t-x|)\,(x) \tag{22}$$

and

$$\int_0^1 (t-x)K_{n,q}(x,t)\,d_qt = (L_{n,q}\,(t-x))\,(x), \tag{23}$$

then using (22) and (23) in (21), we get

$$\begin{aligned}|(L_{n,q}f)(x)-f(x)| &\leq \left|\frac{D_q^+f(x)+D_q^-f(x)}{2}\right||(L_{n,q}(t-x))(x)| \\ &+\left|\frac{D_q^+f(x)-D_q^-f(x)}{2}\right||(L_{n,q}|t-x|)(x)| \\ &+\left|-\int_0^x\left[\int_t^x D_qf_x(u)\,d_qu\right]K_{n,q}(x,t)\,d_qt\right| \\ &+\left|\int_x^1\left[\int_x^t D_qf_x(u)\,d_qu\right]K_{n,q}(x,t)\,d_qt.\right|\end{aligned} \tag{24}$$

From the definition $\lambda_{n,q}(x,t)$, we write

$$\int_0^x\left[\int_t^x D_qf_x(u)\,d_qu\right]K_{n,q}(x,t)\,d_qt=\int_0^x\left[\int_t^x D_qf_x(u)\,d_qu\right]D_{q,t}\lambda_{n,q}(x,t)\,d_qt. \tag{25}$$

Using partial $q-integration$, the right hand side of (25), we obtain

$$\int_0^x\left[\int_t^x D_qf_x(u)\,d_qu\right]D_{q,t}\lambda_{n,q}(x,t)\,d_qt=\int_0^x D_qf_x(t)\,\lambda_{n,q}(x,t)\,d_qt.$$

Thus

$$\left|-\int_0^x\left[\int_x^t D_qf_x(u)\,d_qu\right]K_{n,q}(x,t)\,d_qt\right|\leq\int_0^x|D_qf_x(t)|\;\lambda_{n,q}(x,t)\,d_qt$$

and

$$\begin{aligned}\left|-\int_0^x\left[\int_x^t D_qf_x(u)\,d_qu\right]K_{n,q}(x,t)\,d_qt\right| &\leq \int_0^{x-\frac{x}{\sqrt{n}}}|D_qf_x(t)|\;\lambda_{n,q}(x,t)\,d_qt \\ &+\int_{x-\frac{x}{\sqrt{n}}}^x|D_qf_x(t)|\;\lambda_{n,q}(x,t)\,d_qt.\end{aligned}$$

Since $D_qf_x(x)=0$ and $\lambda_{n,q}(x,t)\leq 1$,

$$\begin{aligned}\int_{x-\frac{x}{\sqrt{n}}}^x|D_qf_x(t)|\;\lambda_{n,q}(x,t)\,d_qt &= \int_{x-\frac{x}{\sqrt{n}}}^x|D_qf_x(t)-D_qf_x(x)|\;\lambda_{n,q}(x,t)\,d_qt \\ &\leq \int_{x-\frac{x}{\sqrt{n}}}^x\bigvee_t^x(D_qf_x)\,d_qt.\end{aligned}$$

Besides from (12), we get

$$\begin{aligned}\int_0^{x-\frac{x}{\sqrt{n}}} |D_q f_x(t)|\, \lambda_{n,q}(x,t)\, d_q t &\leq A_{n,q}(x) \int_0^{x-\frac{x}{\sqrt{n}}} |D_q f_x(t)| \frac{d_q t}{(x-t)^2} \\ &= A_{n,q}(x) \int_0^{x-\frac{x}{\sqrt{n}}} |D_q f_x(t) - D_q f_x(x)| \frac{d_q t}{(x-t)^2} \\ &\leq A_{n,q}(x) \int_0^{x-\frac{x}{\sqrt{n}}} \bigvee_t^x (D_q f_x) \frac{d_q t}{(x-t)^2}.\end{aligned}$$

Make the change of variables $t = x - \frac{x}{u}$, then one has from (8)

$$\begin{aligned}\int_{x-\frac{x}{\sqrt{n}}}^{x} \bigvee_t^x (D_q f_x)\, d_q t &\leq \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} (D_q f_x) \int_{x-\frac{x}{\sqrt{n}}}^{x} d_q t \\ &= \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} (D_q f_x) \left[x(1-q) - (x - \frac{x}{\sqrt{n}})(1-q)\right] \sum_{l=0}^{\infty} q^l \\ &= \frac{x}{\sqrt{n}}(1-q) \sum_{l=0}^{\infty} q^l \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} (D_q f_x)\end{aligned}$$

and

$$\begin{aligned}A_{n,q}(x) \int_0^{x-\frac{x}{\sqrt{n}}} \bigvee_t^x (D_q f_x) \frac{d_q t}{(x-t)^2} &= A_{n,q}(x) \int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^{x} (D_q f_x) \frac{(\frac{x}{u^2})\, d_q u}{(-\frac{x}{u})^2} \\ &= \frac{A_{n,q}(x)}{x} \int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^{x} (D_q f_x)\, d_q u \\ &= \frac{A_{n,q}(x)}{x} \sum_{k=1}^{[\sqrt{n}]} \int_k^{k+1} \bigvee_{x-\frac{x}{u}}^{x} (D_q f_x)\, d_q u \\ &\leq \frac{A_{n,q}(x)}{x} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x} (D_q f_x)\, (1-q) \sum_{l=0}^{\infty} q^l.\end{aligned}$$

Consequently

$$\left|-\int_0^x\left[\int_t^x D_qf_x(u)\,du\right]K_{n,q}(x,t)\,d_qt\right| \leq \frac{x}{\sqrt{n}}(1-q)\sum_{l=0}^{\infty}q^l\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(D_qf_x) + A_{n,q}(x)\frac{1}{x}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}(D_qf_x)\,(1-q)\sum_{l=0}^{\infty}q^l. \tag{26}$$

By the same way,

$$\int_x^1\left[\int_x^t D_qf_x(u)\,d_qu\right]K_{n,q}(x,t)\,d_qt \leq \int_x^1|D_qf_x(t)|\;D_{q,t}\lambda_{n,q}(x,t)\,d_qt$$

$$= \int_x^{x+\frac{1-x}{\sqrt{n}}}|D_qf_x(t)|\;\lambda_{n,q}(x,t)\,d_qt + \int_{x+\frac{1-x}{\sqrt{n}}}^1|D_qf_x(t)|\;\lambda_{n,q}(x,t)\,d_qt$$

$$\leq \int_x^{x+\frac{1-x}{\sqrt{n}}}|D_qf_x(t)|\;dt + A_{n,q}(x)\int_{x+\frac{1-x}{\sqrt{n}}}^1|D_qf_x(t)|\,\frac{d_qt}{(x-t)^2}$$

$$= \int_x^{x+\frac{1-x}{\sqrt{n}}}|D_qf_x(t)-D_qf_x(x)|\;d_qt + A_{n,q}(x)\int_{x+\frac{1-x}{\sqrt{n}}}^1|D_qf_x(t)-D_qf_x(x)|\,\frac{d_qt}{(x-t)^2}$$

$$\leq \bigvee_x^{x+\frac{1-x}{\sqrt{n}}}(D_qf_x)\frac{1-x}{\sqrt{n}}(1-q)\sum_{l=0}^{\infty}q^l + A_{n,q}(x)\int_{x+\frac{1-x}{\sqrt{n}}}^1\bigvee_x^t(D_qf_x)\frac{d_qt}{(x-t)^2}$$

Make the change of variables $t = x+\frac{1-x}{u}$, again from (8)

$$\leq \bigvee_x^{x+\frac{1-x}{\sqrt{n}}}(D_qf_x)\frac{1-x}{\sqrt{n}}(1-q)\sum_{l=0}^{\infty}q^l + A_{n,q}(x)\int_{\sqrt{n}}^1\bigvee_{x+\frac{1-x}{u}}^{x}(D_qf_x)\frac{-(\frac{1-x}{u^2})\,d_qu}{(\frac{x-1}{u})^2}$$

$$= \bigvee_x^{x+\frac{1-x}{\sqrt{n}}}(D_qf_x)\frac{1-x}{\sqrt{n}}(1-q)\sum_{l=0}^{\infty}q^l + \frac{A_{n,q}(x)}{(1-x)}\int_1^{\sqrt{n}}\bigvee_x^{x+\frac{1-x}{u}}(D_qf_x)\,d_qu$$

$$\leq \bigvee_{x}^{x+\frac{1-x}{\sqrt{n}}} (D_q f_x) \frac{1-x}{\sqrt{n}} (1-q) \sum_{l=0}^{\infty} q^l + \frac{A_{n,q}(x)}{(1-x)} \sum_{k=1}^{[\sqrt{n}]} \int_{k}^{k+1} \bigvee_{x}^{x+\frac{1-x}{u}} (D_q f_x)\, d_q u$$

$$\leq \bigvee_{x}^{x+\frac{1-x}{\sqrt{n}}} (D_q f_x) \frac{1-x}{\sqrt{n}} (1-q) \sum_{l=0}^{\infty} q^l + \frac{A_{n,q}(x)}{(1-x)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x}^{x+\frac{1-x}{k}} (D_q f_x) (1-q) \sum_{l=0}^{\infty} q^l.$$

Consequently

$$\begin{aligned} \int_{x}^{1} \left[\int_{x}^{t} D_q f_x(u)\, d_q u \right] K_{n,q}(x,t)\, d_q t \quad \leq \quad & \bigvee_{x}^{x+\frac{1-x}{\sqrt{n}}} (D_q f_x) \frac{1-x}{\sqrt{n}} (1-q) \sum_{l=0}^{\infty} q^l \\ & + \frac{A_{n,q}(x)}{(1-x)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x}^{x+\frac{1-x}{k}} (D_q f_x) (1-q) \sum_{l=0}^{\infty} q^l. \end{aligned} \tag{27}$$

Combining (13), (14), (26) and (27) in (24), we get the desired result (9).

Thus the proof is completed.

In order to obtain an approximation theorem, we use the following way. In connection with this problem we introduce a sequence. Let (q_n) be a sequence of real numbers such that $0 < q_n < 1$ and

$$\lim_{n\to\infty} q_n = 1.$$

If we replace q by q_n, we have immediately

$$[n]_{q_n} \to \infty \qquad (n \to \infty).$$

It is worthy to mention that similar approachs can be found [18] and [9].

On convergence formaIly, Theorem 2 reads:

Theorem 2. *Let (q_n) be a sequence of real numbers such that $0 < q_n < 1$ and $\lim_{n\to\infty} q_n = 1$. Let f be a bounded function on $[0,1]$. Suppose that the right and left q-derivatives exist at a point $x \in (0,1)$. Then at this point $x \in (0,1)$, and n sufficiently large, we have*

$$\begin{aligned} |(L_{n,q_n} f)(x) - f(x)| \quad \leq \quad & \left| \frac{D^+_{q_n} f(x) + D^-_{q_n} f(x)}{2} \right| \left(\left(\frac{1}{q_n} - 1 \right) + \frac{1}{[n+2]_{q_n}} \right) \\ & + \left| \frac{D^+_{q_n} f(x) - D^-_{q_n} f(x)}{2} \right| \sqrt{A_{n,q_n}(x)} \\ & + (1-q_n) \sum_{l=0}^{\infty} q_n^l \left[\frac{1}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{1-x}{\sqrt{n}}} (D_{q_n} f_x) + \frac{A_{n,q_n}(x)}{x(1-x)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{k}} (D_{q_n} f_x) \right]. \end{aligned}$$

It is useful to point out that

$$A_{n,q_n}(x) \to 0 \qquad (n \to \infty).$$

In view of Theorem 2 and Remark 2, we get the following corollary:

Corollary 2. *Let $f \in C[0,1]$. Then for every $x \in (0,1)$, and n sufficiently large, we have*

$$|(L_{n,q_n} f)(x) - f(x)| \leq |D_{q_n} f(x)| \left(\left(\frac{1}{q_n} - 1 \right) + \frac{1}{[n+2]_{q_n}} \right)$$

$$+(1-q_n) \sum_{l=0}^{\infty} q_n^l \left[\frac{1}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{1-x}{\sqrt{n}}} (D_{q_n} f_x) + \frac{A_{n,q_n}(x)}{x(1-x)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{k}} (D_{q_n} f_x) \right]$$

The proof of Theorem 2 is the same as Theorem 1, so we omit it.

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