Solution to time fractional non homogeneous first order PDE with non constant coefficients

Arman Aghili

University of Guilan, Faculty of Mathematical Sciences, Department of Applied Mathematics, Rasht P.O.BOX 1841, Iran
E-mail: arman.aghili@gmail.com

Abstract

In this study, the author used the joint Fourier- Laplace transform to solve non-homogeneous time fractional first order partial differential equation with non-constant coefficients. Constructive examples are also provided throughout the paper. It is a remarkable feature of the first order fractional differential equations that a procedure can be developed for solving this equation, regardless of its complexity.

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1 Introduction

In the literature, there are many application in engineering, applied mathematics and mathematical physics that involve the solution and formulation of the first order time fractional partial differential equations. From the mathematical point of view, the first order partial fractional differential equations have the advantage of providing a conceptual basis that can be used for higher order equations, but with the attractive feature that there are methods of solution which are particularly suited to first order equations. In this work, the author implemented the joint transform method for solving partial fractional differential equation which arise in applications. Several methods have been introduced to solve fractional differential equations, the popular Laplace transform method, [1],[2],[3], [4], [10],[11],[15],[20], the Fourier transform method [17], the iteration method [15] and operational method [10],[20]. However, most of these methods are suitable for special types of fractional differential equations, mainly the linear with constant coefficients. More detailed information about some of these results can be found in a paper by Kilbas and Trujillo [8].Atanackovic and Stankovic [6],[7]and Stankovic [19] used the Laplace transform in a certain space of distributions to solve a system of partial differential equations with fractional derivatives, and indicated that such a system may serve as a certain model for a viscoelastic rod. Oldham and Spanier I, [13] and [14], studied respectively, by reducing a boundary value problem involving Ficks second law in electroanalytic chemistry to a formulation based on the partial Riemann Liouville fractional with half derivative.Wyss [21] and Schneider[18] considered the time fractional diffusion and wave equations and obtained the solution in terms of Fox functions. Until now, two methods, have been more extensively used for solving PDEs, the Laplace and Fourier transforms on the one hand and separation of variables on the other hand. Let us mention also solution in the form of a series of functions. The Laplace and Fourier transforms are the best known and most widely used of the integral transforms, particularly in boundary value problems.

Definition 1.1. Laplace transform of function $f(t)$ is defined as follows [5]

$$\mathcal{L}\{\varphi(t)\} = \int_0^\infty e^{-st}\varphi(t)dt := \Phi(s). \qquad (1.1)$$

In 1916, T.J.Bromwich show that we can express the inverse Laplace transform in terms of a complex integral. If $\mathcal{L}\{\varphi(t)\} = \Phi(s)$, then $\mathcal{L}^{-1}\{\Phi(s)\}$, is given by

$$\varphi(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\Phi(s)ds, \qquad (1.2)$$

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the value of c must be greater than the real part of any of the singularties of $\Phi(s)$, or $\Phi(s)$, is analytic in the region $\mathrm{R}e(s) > c$.

Bessel's functions constitute a most recent set of mathematical tools, impacting many branches of mathematical physics and applied sciences. To illustrate inversion by Bromwich's integral when the Bessel's functions are present, let us consider the following example.

Example 1.1. By using an appropriate integral representation for the modified Bessel's functions of the second kind of order ν, $s^{-\frac{\nu}{2p}}K_{\frac{\nu}{p}}(2\sqrt{s})$, show that

$$\mathcal{L}^{-1}\{s^{-\frac{\nu}{2p}}K_{\frac{\nu}{p}}(2\sqrt{s})\} = \frac{e^{-\frac{1}{t}}}{2t^{1-\frac{\nu}{p}}}. \qquad (1.3)$$

Solution. By taking the inverse Laplace transform of the given $s^{-\frac{\nu}{2p}}K_{\frac{\nu}{p}}(2\sqrt{s})$, we have

$$\varphi(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}s^{-\frac{\nu}{2p}}K_{\frac{\nu}{p}}(2\sqrt{s})ds, \qquad (1.4)$$

at this stage, using the following integral representation for $s^{-\frac{\nu}{2p}}K_{\frac{\nu}{p}}(2\sqrt{s})$

$$s^{-\frac{\nu}{2p}}K_{\frac{\nu}{p}}(2\sqrt{s}) = \frac{p}{2}\int_0^\infty e^{-s\xi^p-\frac{1}{\xi^p}}\xi^{\nu-1}d\xi. \quad (1.5)$$

By setting relation (1.5) in (1.4), we have

$$\varphi(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}(\frac{p}{2}\int_0^\infty e^{-s\xi^p-\frac{1}{\xi^p}}\xi^{\nu-1}d\xi)ds, \qquad (1.6)$$

at this point we change the order of integration in relation (1.6), to get

$$\varphi(t) = \frac{p}{2}\int_0^\infty \xi^{\nu-1}e^{-\xi^{-p}}(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{s(t-\xi^p)}ds)d\xi, \qquad (1.7)$$

the value of the inner integral is $\delta(t-\xi^p)$, therefore

$$\varphi(t) = \frac{p}{2}\int_0^\infty e^{-\xi^{-p}}\xi^{\nu-1}\delta(t-\xi^p)d\xi, \qquad (1.8)$$

after making a change of variable $t-\xi^p = \eta$, and considerable algebra and elimination process, we obtain

$$\varphi(t) = \frac{p}{2}\int_{-\infty}^{t}\delta(\eta)e^{\frac{-1}{t-\eta}}\frac{d\eta}{p(t-\eta)^{1-\frac{\nu}{p}}} = \frac{e^{-\frac{1}{t}}}{2t^{1-\frac{\nu}{p}}}, \qquad (1.9)$$

Let us recall some important properties of the Laplace transform, useful Lemmas, that will be considered in the next part of this article.

Lemma 1.1. Let $\mathcal{L}\{f(t)\} = F(s)$, then, the following identities hold true.

1. $e^{-\omega s^\beta} = \frac{1}{\pi}\int_0^\infty e^{-r^\beta(\omega cos\beta\pi)}sin(\omega r^\beta sin\beta\pi)(\int_0^\infty e^{-s\tau-r\tau}d\tau)dr,$
2. $\mathcal{L}^{-1}(F(s^\alpha)) = \frac{1}{\pi}\int_0^\infty f(u)\int_0^\infty e^{-tr-ur^\alpha cos\alpha\pi}sin(ur^\alpha sin\alpha\pi)drdu,$
3. $\mathcal{L}^{-1}(F(\sqrt{s}) = \frac{1}{2t\sqrt{\pi t}}\int_0^\infty ue^{-\frac{u^2}{4t}}f(u)du.$

Proof. See [5].

Definition 1.2. The left Caputo fractional derivative of order α $(0 < \alpha < 1)$ of $\varphi(t)$, is defined as

$$D_a^{c,\alpha}\varphi(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{1}{(t-\xi)^\alpha}\varphi'(\xi)d\xi.$$

Let us recall some important properties of the Laplace transform, useful Lemmas, that will be considered in the next part of this article. The most important use of the Caputo fractional derivative is treated in boundary value problems where the initial conditions are expressed in terms of integer order derivatives. In this respect, it is interesting to know the Laplace transform of this type of derivatives, and in general [16]

$$\mathcal{L}\{D_a^{c,\alpha}f(t)\} = s^{\alpha-1}F(s) - \sum_{k=0}^{k=m-1-k} s^{\alpha-1-k}f^k(0+), m-1 < \alpha < m,$$

in special case, when $0 < \alpha < 1$,

$$\mathcal{L}\{D_a^{c,\alpha}f(t)\} = sF(s) - f(0+), 0 < \alpha < 1.$$

The geometrical and physical meaning of the fractional derivatives has been investigated by several authors [15],[16].
The Laplace transform provides a useful technique for the solution of fractional singular integro-differential equations.

Kelvin functions: Kelvin functions $ber(x)$ and $bei(x)$ of order zero, are defined as [5],[8]

$$ber(x) = \mathcal{R}eJ_0(i\sqrt{i}x)\ ,\ bei(x) = \mathcal{I}mJ_0(i\sqrt{i}x).$$

It is well - known that $\mathcal{L}\{J_0(2\sqrt{at}); t \to s\} = \frac{e^{-\frac{a}{s}}}{s}$ let $a = i^3$ to obtain
$\mathcal{L}\{bei(2\sqrt{t})\} = Im\left(\frac{e^{-\frac{i^3}{s}}}{s}\right) = \frac{1}{s}\sin\frac{1}{s}, \qquad \mathcal{L}\{ber(2\sqrt{t})\} = Re\left(\frac{e^{-\frac{i^3}{s}}}{s}\right) = \frac{1}{s}\cos\frac{1}{s}.$

Solution to singular integral equation with kernel of Kelvin's functions of order zero.
Problem 1.1. Let us consider the following singular integral equation

$$\sqrt{\frac{2}{\pi}}\int_0^{+\infty} \varphi(t)\ ber(2\sqrt{\xi t})dt = \frac{1}{\alpha}J_0(\frac{\xi}{\alpha}),$$

the formal solution is as follows

$$\varphi(t) = \sqrt{\frac{2}{\pi}}K_0(\alpha t).$$

Solution. By taking the Laplace transform term - wise in the above equation, we get

$$\sqrt{\frac{2}{\pi}}\int_0^{+\infty} \varphi(t)\ \mathcal{L}\{ber(2\sqrt{\xi t})dt = \frac{1}{\alpha}\mathcal{L}\{J_0(\frac{\xi}{\alpha})\},$$

$$\sqrt{\frac{2}{\pi}}\int_0^{+\infty} \varphi(t)\ \{\frac{1}{s}\cos\frac{t}{s}\}dt = \frac{1}{\alpha(\sqrt{s^2 + \frac{1}{\alpha^2}})}.$$

For the sake of simplicity, let us assume that

$$\frac{1}{s} = \omega.$$

Then we get

$$\sqrt{\frac{2}{\pi}}\int_0^{+\infty} \varphi(t)\ \cos\omega t\, dt = \frac{1}{\sqrt{\omega^2 + \alpha^2}},$$

the left hand side by definition is the Fourier- cosine transform of certain functions, so that

$$\mathcal{F}_c\{\varphi(t); t->\omega\} = \frac{1}{\sqrt{\omega^2+\alpha^2}}.$$

Using the inversion formula for the Fourier- cosine transform to obtain

$$\varphi(t) = \sqrt{\frac{2}{\pi}}\int_0^{+\infty} \frac{\cos(t\omega)}{\sqrt{\omega^2+\alpha^2}}d\omega,$$

or

$$\varphi(t) = \sqrt{\frac{2}{\pi}}K_0(\alpha t).$$

Corollary 1.1. From the above, we have the following integral representation for the Bessel's function of order zero.

$$J_0(\frac{\xi}{\alpha}) = \frac{2}{\pi}\int_0^{+\infty} K_0(\alpha t)\ ber(2\sqrt{\xi t})dt.$$

2 Solution to time fractional non-homogeneous partial differential equation with non - constant coefficients

In this section, we present an extension to the problem of evolution of disturbances at an inviscid interface. While in general the joint transform method works best on constant coefficient equations, there are some variable - coefficient equations which also lend themselves to the joint transform method. At this stage, the results which has been introduced are used to solve a time fractional partial differential equation. We implemented the joint Laplace - Fourier transform technique for solving time fractional partial differential equations, where the fractional derivative is in the Caputo sense. The differential equations defined in terms of Caputo derivatives require regular boundary conditions. For this reason, the Caputo fractional derivatives are popular among scientists and engineers.

Problem 2.1. Let us consider the following time fractional PDE, with initial condition

$$2x\frac{\partial^{c,\alpha}u(x,t)}{\partial t^\alpha} + \frac{\partial u(x,t)}{\partial x} + 2\beta x u(x,t) = \frac{t^{\nu\alpha-1}}{\Gamma(\nu\alpha)}f(x), \qquad 0<\alpha,\nu<1. \tag{2.1}$$

where $-\infty < x < \infty$, $t > 0$ and subject to the initial condition $u(x,0) = u(0,t) = 0$.

$$lim_{|x|->+\infty}u(x,t) = lim_{|x|->+\infty}u_x(x,t) = 0. \tag{2.2}$$

Note: Fractional derivative is in the Caputo sense.
Solution: Let us define the joint Laplace - Fourier transform as following

$$\mathcal{F}\{\mathcal{L}\{u(x,t)\} = (\tfrac{1}{\sqrt{2\pi}})\int_{-\infty}^{+\infty} e^{i\omega x}\int_0^{\infty} e^{-st}u(x,t)dt)dx := U(\omega,s),$$

application of the joint Laplace - Fourier transform to (2.1) - (2.2), leads to the transformed problem

$$-i\omega U(\omega,s) + 2(s^\alpha+\beta)(-i)U'_\omega(\omega,s) = \tfrac{F(\omega)}{s^{\nu\alpha}},$$

after simplifying we obtain the following first order ordinary differential equation

$$U'_\omega(\omega,s) + \tfrac{\omega}{2(s^\alpha+\beta)}U(\omega,s) = \tfrac{F(\omega)}{-2i(s^\alpha+\beta)s^{\nu\alpha}},$$

in order to solve the above first order ordinary differential equation, we should first find an integrating factor $\varphi(\omega)$ as follows $\varphi(\omega) = \exp(\frac{\omega^2}{4(s^\alpha+\beta)})$.
Multiplying the above equation by integrating factor and after simplifying we get the following equation

$$d(\exp(\tfrac{\omega^2}{4(s^\alpha+\beta)}U(\omega,s)) = -\tfrac{F(\omega)\exp(-\frac{\omega^2}{4(s^\alpha+\beta)})}{2is^{\nu\alpha}(s^\alpha+\beta)},$$

at this stage, by integrating the above equation from 0 to ω we get the following

$$\int_0^\omega d(\exp(\tfrac{k^2}{4(s^\alpha+\beta)}U(k,s))dk = -\int_0^\omega \tfrac{F(k)\exp(-\frac{k^2}{4(s^\alpha+\beta)})}{2is^{\nu\alpha}(s^\alpha+\beta)}dk,$$

after simplifying and using boundary condition we obtain

$$\exp(\tfrac{\omega^2}{4(s^\alpha+\beta)})U(\omega,s) = -\int_0^\omega \tfrac{F(k)\exp(-\frac{k^2}{4(s^\alpha+\beta)})}{2is^{\nu\alpha}(s^\alpha+\beta)}dk,$$

from the above relation we arrive at

$$U(\omega,s) = -\int_0^\omega \tfrac{F(k)\exp(-\frac{\omega^2-k^2}{4(s^\alpha+\beta)})}{2is^{\nu\alpha}(s^\alpha+\beta)}dk,$$

after taking the inverse joint Laplace - Fourier transform, we obtain

$$u(x,t) = -\tfrac{1}{2i}\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(-ix\omega)(\int_0^\omega F(k)L^{-1}(\tfrac{1}{s^{\nu\alpha}}\tfrac{\exp(-\frac{\omega^2-k^2}{4(s^\alpha+\beta)})}{(s^\alpha+\beta)})dk)d\omega. \quad (2.3)$$

Let us find the value of the inverse Laplace transform inside the inner integral. By convolution theorem for the Laplace transforms we have

$$g(t) = L^{-1}(\tfrac{1}{s^{\nu\alpha}}.\tfrac{\exp(-\frac{\omega^2-k^2}{4(s^\alpha+\beta)})}{(s^\alpha+\beta)}) = L^{-1}(\tfrac{1}{s^{\nu\alpha}}) * L^{-1}(\tfrac{\exp(-\frac{\omega^2-k^2}{4(s^\alpha+\beta)})}{(s^\alpha+\beta)}).$$

Using the table of Laplace transform, leads to

$$L^{-1}(\frac{1}{s^{\nu\alpha}}) = \frac{t^{\nu\alpha-1}}{\Gamma(\nu\alpha)},$$

in view of the part two of the Lemma 1.1 and table of the Laplace transform we have

$$L^{-1}(\frac{\exp(-\frac{\omega^2-k^2}{4(s^\alpha+\beta)})}{(s^\alpha+\beta)}) = \frac{1}{\pi}\int_0^{+\infty} e^{-\beta\eta}J_0(\sqrt{(\omega^2-k^2)}\eta)(\int_0^{+\infty} e^{-tr-\eta r^\alpha cos\pi\alpha}sin(\eta r^\alpha sin\pi\alpha)dr)d\eta,$$

consequently, we arrive at

$$g(t) = \int_0^t \frac{(t-\xi)^{\nu\alpha-1}}{\pi\Gamma(\nu\alpha)}(\frac{1}{\pi}\int_0^{+\infty} e^{-\beta\eta}J_0(\sqrt{(\omega^2-k^2)}\eta)(\int_0^{+\infty} e^{-tr-\eta r^\alpha cos\pi\alpha}sin(\eta r^\alpha sin\pi\alpha)dr)d\eta)d\xi,$$

by setting the above relation in (2.3), it follows that

$$u(x,t) = -\tfrac{1}{2i}\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ix\omega}(\int_0^\omega F(k)\int_0^t \tfrac{(t-\xi)^{\nu\alpha-1}}{\pi^2\Gamma(\nu\alpha)}(\int_0^{+\infty} e^{-\beta\eta}J_0(\sqrt{(\omega^2-k^2)}\eta)....$$
$$.....(\int_0^{+\infty} e^{-tr-\eta r^\alpha cos\pi\alpha}sin(\eta r^\alpha sin\pi\alpha)dr)d\eta)d\xi)dk)d\omega.$$

Let us consider the semi - derivative case $\alpha = 0.5$, where $f(x)$ is an odd function, we have

$$u(x,t) = -\tfrac{1}{2i}\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ix\omega}(\int_0^\omega F(k)\int_0^t \tfrac{(t-\xi)^{\nu\alpha-1}}{\pi^2\Gamma(\nu\alpha)}(\int_0^{+\infty} e^{-\beta\eta}J_0(\sqrt{(\omega^2-k^2)}\eta)....$$
$$.....(\int_0^{+\infty} e^{-tr}sin(\eta\sqrt{r})dr)d\eta)d\xi)dk)d\omega.$$

Let us evaluate the inner integral

$$\int_0^{+\infty} e^{-tr}sin(\eta\sqrt{r})dr = \frac{\eta}{2\xi}\sqrt{\frac{\pi}{\xi}}\exp(-\frac{\eta^2}{4\xi}).$$

After substitution of the value of integral and changing the order of integration, we arrive at the following

$$u(x,t) = -\frac{1}{2i\sqrt{\pi}\Gamma(\frac{\nu}{2})} \int_0^t \frac{(t-\xi)^{\frac{\nu}{2}-1}}{\xi\sqrt{\xi}} (\int_0^\infty \eta e^{-(\beta\eta+\frac{\eta^2}{4\xi})} (\frac{1}{\sqrt{2\pi}}) (\int_{-\infty}^{+\infty} e^{-ix\omega} (\int_0^\omega F(k) J_0(\sqrt{(\omega^2-k^2)}\eta) dk) d\omega) d\eta) d\xi.$$

Since $f(x)$ is an odd function, then $F(k)$ is odd. Therefore, we get the following solution

$$u(x,t) = \frac{1}{\sqrt{\pi}\Gamma(\frac{\nu}{2})} \int_0^t \frac{(t-\xi)^{\frac{\nu}{2}-1}}{\xi\sqrt{\xi}} (\int_0^\infty \eta e^{-(\beta\eta+\frac{\eta^2}{4\xi})} (\frac{1}{\sqrt{2\pi}}) (\int_0^{+\infty} sin(x\omega) (\int_0^\omega F(k) J_0(\sqrt{(\omega^2-k^2)}\eta) dk) d\omega) d\eta) d\xi.$$

3 Conclusions

The main objective of present work is to extend the application of the joint transform method to derive analytic solution for boundary value problem. The author implemented the joint Fourier - Laplace transforms for solving time fractional non homogeneous partial differential equation with non- constant coefficients. We develop a method for finding analytic solutions to partial fractional differential equation with non - constant coefficients. The article is intended for scientists and researchers of different disciplines of engineering and science dealing with the solutions of fractional integro - differential and fractional PDEs. The results reveal that the integral transforms method is very convenient and effective.

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