A note on the spherical images of W-partially null curves in Minkowski space-time $\mathbb{E}_1^4$

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Abstract

In this study, we investigate the tangent, principal normal and trinormal spherical images of a W-partially null curve in pseudohyperbolic space $\mathbb{H}_0^3$ of Minkowski space time $\mathbb{E}_1^4$. The tangent, principal normal spherical images of a W-partially null curve occur as spacelike curves lying in pseudosphere $\mathbb{S}_0^3$, then the Frenet-Serret invariants of the mentioned image curves are obtained in terms of the invariants of W-partially null curve. The trinormal spherical images of a W-partially null curve occur as spacelike curves lying in pseudohyperbolic space $\mathbb{H}_0^3$, then the Frenet-Serret invariants of the mentioned image curves are obtained in terms of the invariants of W-partially null curve. Finally, we give some characterizations of the spherical images being helices.

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1 Introduction

Minkowski space-time has a special place in differential geometry since the theory of relativity is based on this space. This space-time is a semi-Riemannian manifold, on which the metric tensor g induced, with index 1 and dimension 4 [7]. As known, the behaviour of a differentiable curve is basically studied by constructing a frame about neighbourhood of a point of the curve. This proces is also similar in Minkowski space-time [4, 7].

The frame constructed in any point of the curve in Minkowski space-time has a tetrad of mutually orthogonal unit vectors [2]. These mutually orthogonal unit vectors are called tangent, normal, binormal and trinormal vector fields. The rates of change belonging to these vectors along the curve define the curvatures of the curve in the space $\mathbb{E}_1^4$ [13, 14, 15].

Spherical image is a well-known concept in classical differential geometry of curves [5]. The basic idea of spherical images is that translating one of the frame vector fields of the curve to the sphere as it becomes its radius vector, the trajectory of the end points of the vectors which are on the sphere will form spherical image curve. There exists a literature on this subject, for instance [3, 5, 13, 16].

In this work, we study spherical images of a W-partially null curve lying on the pseudosphere $\mathbb{S}_0^3$ and pseudohyperbolic space $\mathbb{H}_0^3$ in Minkowski space-time. We investigate relations between Frenet-Serret invariants of these spherical images and their base curve. Additionally, some characterizations of spherical images being general helices are presented in Minkowski space-time $\mathbb{E}_1^4$.

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2 Preliminaries

Minkowski space-time $\mathbb{E}_1^4$ is the real vector 4-dimensional space $\mathbb{R}^4$ provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in $\mathbb{R}^4$ [7]. Since g is an indefinite metric, recall that a vector $v \in \mathbb{E}_1^4$ can have one of the three causal characters; it can be spacelike if $g(v,v) > 0$ or $v = 0$, timelike if $g(v,v) < 0$ and null (lightlike) if $g(v,v)=0$ and $v \neq 0$. Similary, an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{E}_1^4$ can be locally spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null. Also, recall the norm of a vector v given by

$$\|v\| = \sqrt{|g(v,v)|}.$$

Therefore, v is a unit vector if $g(v,v) = \pm 1$. Next, vectors v, w in $\mathbb{E}_1^4$ are said to be orthogonal if $g(v,w) = 0$. The velocity of the curve $\alpha(s)$ is given by $\|\alpha'(s)\|$ [4]. Let a and b be two spacelike vectors in $\mathbb{E}_1^4$, then there is a unique real number $0 \leq \theta \leq \pi$, called as angle between a and b, such that $g(a,b) = \|a\| \, . \, \|b\| \cos\theta$ [9].

The pseudosphere with center $m = (m_1, m_2, m_3, m_4) \in \mathbb{E}_1^4$ and radius $r \in R^+$ in the space-time $\mathbb{E}_1^4$ is a hyperquadric

$$\mathbb{S}_0^3(r) = \{a = (a_1, a_2, a_3, a_4) \in \mathbb{E}_1^4 \mid g(a-m, a-m) = r^2\}$$

with dimension 3 and index 0. The pseudohyperbolic space with center $m = (m_1, m_2, m_3, m_4) \in \mathbb{E}_1^4$ and radius $r \in R^+$ in the space-time $\mathbb{E}_1^4$ is a hyperquadric

$$\mathbb{H}_0^3(r) = \{a = (a_1, a_2, a_3, a_4) \in \mathbb{E}_1^4 \mid g(a-m, a-m) = -r^2\}$$

with dimension 3 and index 0 [7].

Let $\vartheta = \vartheta(s)$ be a curve in $\mathbb{E}_1^4$. If the tangent vector field of this curve forms a constant angle with a constant vector field U, then this curve is called a general helix. Recall that, if a regular curve in $\mathbb{E}_1^4$ has constant Frenet-Serret curvatures ratios, (i.e., $\frac{\tau}{\kappa}$ and $\frac{\sigma}{\tau}$ are constants), then it is called a *ccr-curve* [6], [8]. Also, if these curvatures are non-zero constants, curve is said to be *W-curve* (or helix) [10].

Denote by $\{T(s), N(s), B_1(s), B_2(s)\}$ the moving Frenet-Serret frame along the curve $\alpha(s)$ in the space $\mathbb{E}_1^4$. Then T, N, B_1, B_2 are, respectively, the tangent, the principal normal, the binormal (the first binormal) and the trinormal (the second binormal) vector fields. A spacelike or timelike curve $\alpha(s)$ is said to be parametrized by arclength function s, if $g(\alpha'(s), \alpha'(s)) = \pm 1$.

Let $\alpha(s)$ be a partially null curve which is a spacelike curve where N is spacelike and B_1 is null in the space-time $\mathbb{E}_1^4$, parametrized by arclength function s. Then, the following Frenet-Serret equations are given in [12]:

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & -\tau & 0 & -\sigma \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where T, N, B_1 and B_2 are mutually orthogonal vectors satisfying equations

$$g(T,T) = g(N,N) = 1, g(B_1,B_1) = g(B_2,B_2) = 0,$$
$$g(T,N) = g(T,B_1) = g(T,B_2) = g(N,B_1) = g(N,B_2) = 0,\ g(B_1,B_2) = 1,$$

where, κ, τ and σ are first, second and third curvature of the curve α, respectively.

In the same space, in [1], the authors express a characterization of spacelike curves lying on $\mathbb{H}_0^3$ by the following theorem:

Theorem 2.1. Let $\alpha(s)$ be a unit speed spacelike curve in $\mathbb{E}_1^4$, with spacelike N, B_1 and curvatures $\kappa \neq 0, \tau \neq 0, \sigma \neq 0$ for each $s \in I \subset R$. Then, α lies on pseudohyperbolic space if and only if

$$\begin{aligned} \frac{\sigma}{\tau}\frac{d\rho}{ds} &= \frac{d}{ds}\left[\frac{1}{\sigma}\left(\rho\tau + \frac{d}{ds}(\frac{1}{\tau}\frac{d\rho}{ds})\right)\right], \\ \left\{\frac{1}{\sigma}\left[\rho\tau + \frac{d}{ds}(\frac{1}{\tau}\frac{d\rho}{ds})\right]\right\}^2 &> \rho^2 + (\frac{1}{\tau}\frac{d\rho}{ds})^2, \end{aligned} \tag{2.1}$$

where $\rho = \dfrac{1}{\kappa}$.

In the same space, in [14] authors defined a vector product and gave a method to determine the Frenet-Serret invariants for an arbitrary curve by following definition and theorem:

Definition 2.2. Let $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$ and $c = (c_1, c_2, c_3, c_4)$ be vectors in $\mathbb{E}_1^4$. The vector product in Minkowski space-time $\mathbb{E}_1^4$ is defined by the determinant

$$a \wedge b \wedge c = -\begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix},$$

where e_1, e_2, e_3 and e_4 are mutually orthogonal vectors (coordinate direction vectors) satisfying equations

$e_1 \wedge e_2 \wedge e_3 = e_4$, $e_2 \wedge e_3 \wedge e_4 = e_1$, $e_3 \wedge e_4 \wedge e_1 = e_2$, $e_4 \wedge e_1 \wedge e_2 = -e_3$.

Theorem 2.3. Let $\alpha = \alpha(t)$ be an arbitrary spacelike curve in Minkowski space-time $\mathbb{E}_1^4$. The Frenet-Serret apparatus of α can be written as follows;

$$T = \frac{\alpha'}{\|\alpha'\|},$$

$$N = \frac{\|\alpha'\|^2 \alpha'' - g(\alpha', \alpha'')\alpha'}{\left\|\|\alpha'\|^2 \alpha'' - g(\alpha', \alpha'')\alpha'\right\|}, \tag{2.2}$$

$$B_1 = \mu N \wedge T \wedge B_2,$$

$$B_2 = \mu \frac{T \wedge N \wedge \alpha'''}{\|T \wedge N \wedge \alpha'''\|}, \tag{2.3}$$

$$\kappa = \frac{\left\| \|\alpha'\|^2 \alpha'' - g(\alpha', \alpha'')\alpha' \right\|}{\|\alpha'\|^4}$$

$$\tau = \frac{\|T \wedge N \wedge \alpha'''\| \, \|\alpha'\|}{\left\| \|\alpha'\|^2 \alpha'' - g(\alpha', \alpha'')\alpha' \right\|}$$

and

$$\sigma = \frac{g(\alpha^{(IV)}, B_2)}{\|T \wedge N \wedge \alpha'''\| \, \|\alpha'\|}, \tag{2.4}$$

where μ is taken -1 or $+1$ to make $+1$ the determinant of $[T, N, B_1, B_2]$ matrix.

Here, we shall use Frenet-Serret invariants of a W-partially null curve.

3 The tangent spherical image of a W-partially null curve in $\mathbb{E}_1^4$

By the spirit of the paper [14], first we adapt the definition of tangent spherical image to W-partially null curves of Minkowski space-time $\mathbb{E}_1^4$ as follows:

Let $\beta = \beta(s)$ be a unit speed W-partially null curve in Minkowski space-time. If we translate the tangent vector to the center O of the pseudosphere $\mathbb{S}_0^3$, we obtain a curve $\delta = \delta(s_\delta)$. This curve is called the tangent spherical image of the curve β in $\mathbb{E}_1^4$.

Theorem 3.1. Let $\beta = \beta(s)$ be a unit speed W-partially null curve and let $\delta = \delta(s_\delta)$ be its tangent spherical image. Then;

i) $\delta = \delta(s_\delta)$ is a space-like curve,

ii) The spherical image curve $\delta = \delta(s_\delta)$ is a circle lying in the osculating plane $T_\delta N_\delta$.

Proof. Let $\beta = \beta(s)$ be a unit speed W-partially-null curve and let $\delta = \delta(s_\delta)$ be its tangent spherical image. It can be written as

$$\delta = T(s). \tag{3.1}$$

Differentiating (3.1) with respect to s, we find

$$\delta' = \dot{\delta}\frac{ds_\delta}{ds} = \kappa N.$$

Here we shall denote differentiation according to s by a dash, and differentiation according to s_δ by a dot. Thus, we obtain the unit tangent vector of the tangent spherical image curve δ as

$$T_\delta = N \tag{3.2}$$

and

$$\|\delta'\| = \frac{ds_\delta}{ds} = \kappa.$$

The causal character of the the tangent spherical image curve $\delta(s_\delta)$ is determined by the following inner product:

$$g(\delta', \delta') = \kappa^2 > 0. \tag{3.3}$$

By (3.3), it is seen that the spherical image curve is a spacelike one.

Considering the previous method and using the property of the curve to be W-curve, we form the following differentiations with respect to s:

$$\begin{cases} \delta'' = -\kappa^2 T + \kappa\tau B_1, \\ \delta''' = -\kappa^3 N + \kappa\tau\sigma B_1, \\ \delta^{(IV)} = \kappa^4 T + (\kappa^3\tau + \kappa\tau\sigma^2) B_1. \end{cases}$$

By equation (2.2) we arrive

$$\|\delta'\|^2 \delta'' - g(\delta', \delta'')\delta' = -\kappa^4 T + \kappa^3\tau B_1.$$

Then by taking norm of the above equation, we can get the principal normal vector as

$$N_\delta = -T + \frac{\tau}{\kappa} B_1 \tag{3.4}$$

and the first curvature as

$$\kappa_\delta = 1. \tag{3.5}$$

Now let us calculate the vector $T_\delta \wedge N_\delta \wedge \delta'''$, that is,

$$T_\delta \wedge N_\delta \wedge \delta''' = - \begin{vmatrix} -T & N & B_1 & B_2 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & \frac{\tau}{\kappa} & -\frac{\tau}{\sqrt{|\tau\sigma|}} \\ 0 & -\kappa^3 & \kappa\tau\sigma & 0 \end{vmatrix}.$$

This product yields

$$T_\delta \wedge N_\delta \wedge \delta''' = \kappa\tau\sigma B_2. \tag{3.6}$$

Hence, since the norm of (3.6) vanishes, we could not obtain the trinormal (second binormal) vector field of the curve $\delta(s_\delta)$. Thus $B_{2\delta}$ becomes undefined vector field, therefore the vector field $B_{1\delta}$ remains undefined. It is also seen that the second curvature τ is zero. Finally the third curvature function σ beomes undefined. All of these mean that the spherical image curve $\delta = \delta(s_\delta)$ is a circle lying in the osculating plane $T_\delta N_\delta$. Q.E.D.

Theorem 3.2. Let $\beta = \beta(s)$ be a unit speed W-partially null curve and $\delta(s_\delta)$ be its tangent spherical image. Then, the fixed direction U of δ is composed as

$$U = (c_1\kappa s + c_2)T + c_1 N + \frac{e^{-\sigma s - \sigma c_3} - \tau}{\sigma} B_1 + e^{\sigma s + c_4} B_2,$$

where c_1 is a nonzero-constant, and c_2, c_3, c_4 are constants.

Proof. Let $\beta = \beta(s)$ be a unit speed W-partially null curve and $\delta = \delta(s_\delta)$ be its space-like tangent spherical image. Since $\delta = \delta(s_\delta)$ is a general helix, there is a constant space-like vector U, we may express

$$g(T_\delta, U) = \cos\theta, \tag{3.7}$$

where θ is a constant angle. The equation (3.7) is also congruent to

$$g(N, U) = \cos\theta.$$

One can form constant vector U according to $\{T, N, B_1, B_2\}$ as the following

$$U = \varepsilon_1 T + \varepsilon_2 N + \varepsilon_3 B_1 + \varepsilon_4 B_2. \tag{3.8}$$

Differentiating (3.8) with respect to s, we have the following system of ordinary differential equations

$$\begin{cases} \varepsilon_1' - \varepsilon_2 \kappa = 0, \\ \varepsilon_1 \kappa - \varepsilon_4 \tau = 0, \\ \varepsilon_3' + \varepsilon_3 \sigma + \varepsilon_2 \tau = 0, \\ \varepsilon_4' - \varepsilon_4 \sigma = 0. \end{cases} \tag{3.9}$$

We know that $\varepsilon_2 = c_1 \neq 0$ is a constant. Also since the curve $\beta = \beta(s)$ is a W-curve, its curvature funcions are constants. Then the solution of the system (3.9) can be obtained as:

$$\begin{aligned} \varepsilon_1 &= c_1 \kappa s + c_2, \\ \varepsilon_2 &= c_1, \\ \varepsilon_3 &= \frac{e^{-\sigma s - \sigma c_3} - \tau}{\sigma}, \\ \varepsilon_4 &= e^{\sigma s + c_4}, \end{aligned}$$

where c_2, c_3, c_4 are constants. Q.E.D.

4 The principal normal spherical image of a W-partially null curve in $\mathbb{E}_1^4$

By the spirit of the paper [14], first we adapt the definition of principal normal spherical image to W-partially null curves of Minkowski space-time $\mathbb{E}_1^4$.

Let $\beta = \beta(s)$ be a unit speed W-partially null curve in Minkowski space-time. If we translate the principal normal vector to the center O of the pseudosphere $\mathbb{S}_0^3$, we obtain a curve $\psi = \psi(s_\psi)$. This curve is called the principal normal spherical image of the curve β in $\mathbb{E}_1^4$.

Theorem 4.1. Let $\beta = \beta(s)$ be a unit speed W-partially null curve and $\psi = \psi(s_\psi)$ be its principal normal spherical image. Then;

i) $\psi = \psi(s_\psi)$ is a space-like curve.

ii) The spherical image curve $\psi = \psi(s_\psi)$ is a circle lying in the osculating plane $T_\psi N_\psi$.

Proof. Let $\beta = \beta(s)$ be a unit speed W-partially-null curve and $\psi = \psi(s_\psi)$ be its principal normal spherical image. It can be written as

$$\psi = N(s). \tag{4.1}$$

Differentiating (4.1) with respect to s, we find

$$\psi' = \dot{\psi} \frac{ds_\psi}{ds} = -\kappa T + \tau B_1.$$

Here we shall denote differentiation according to s by a dash, and differentiation according to s_ψ by a dot. Thus, we obtain the unit tangent vector of the principal normal spherical image curve ψ as

$$T_\psi = -T + \frac{\tau}{\kappa} B_1, \tag{4.2}$$

and

$$\|\psi'\| = \frac{ds_\psi}{ds} = \kappa.$$

The causal character of the the principal normal spherical image curve $\psi(s_\psi)$ is determined by the following inner product:

$$g(\psi', \psi') = \kappa^2 > 0. \tag{4.3}$$

By (4.3), we see that the spherical image curve is a spacelike one.

Considering the previous method and using the property of the curve to be W-curve, we form the following differentiations with respect to s:

$$\begin{cases} \psi'' = -\kappa^2 N + \tau\sigma B_1, \\ \psi''' = \kappa^3 T + \tau(\sigma^2 - \kappa^2) B_1, \\ \psi^{(IV)} = \kappa^4 N + \tau\sigma(\sigma^2 - \kappa^2) B_1. \end{cases}$$

By equation (2.2) we arrive

$$\|\psi'\|^2 \psi'' - g(\psi', \psi'')\psi' = -\kappa^4 N + \kappa^2 \tau\sigma B_1.$$

Then by taking norm of the above equation, we can get the principal normal vector as

$$N_\psi = -N - \frac{\tau\sigma}{\kappa^2} B_1 \tag{4.4}$$

and the first curvature as

$$\kappa_\psi = 1. \tag{4.5}$$

Now let us calculate the vector $T_\psi \wedge N_\psi \wedge \psi'''$, that is,

$$T_\psi \wedge N_\psi \wedge \psi''' = - \begin{vmatrix} -T & N & B_1 & B_2 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{\sigma}{\sqrt{|\tau\sigma|}} & 0 & -\frac{\tau}{\sqrt{|\tau\sigma|}} \\ \kappa\tau^2 & 0 & 2\tau^2\sigma & 0 \end{vmatrix}.$$

This product yields

$$T_\psi \wedge N_\psi \wedge \psi''' = -\tau\sigma^2 B_2. \tag{4.6}$$

Hence, since the norm of (4.6) vanishes, we could not obtain the trinormal (second binormal) vector field of the curve $\psi(s_\psi)$. Thus $B_{2\psi}$ becomes undefined vector field, therefore the vector field $B_{1\psi}$ remains undefined. It is also seen that the second curvature τ is zero. Finally the third curvature function σ beomes undefined. All of these mean that the spherical image curve $\psi = \psi(s_\psi)$ is a circle lying in the osculating plane $T_\psi N_\psi$. Q.E.D.

Theorem 4.2. Let $\beta = \beta(s)$ be a unit speed W-partially null curve and $\psi(s_\psi)$ be its principal normal spherical image. Then, the fixed direction U of ψ is composed as

$$U = (-\tfrac{1}{2} c_1 \kappa s^2 + c_2 \kappa s + c_3) T + (-c_1 s + c_2) N + (\frac{c_1 \tau}{\sigma} s + c_4 e^{-\sigma s} + c_3) B_1 + e^{\sigma s + c_5} B_2,$$

where c_1 is a nonzero-constant, and c_2, c_3, c_4, c_5 are constants.

Proof. Let $\beta = \beta(s)$ be a W-partially null unit speed curve and $\psi = \psi(s_\psi)$ be its space-like principal normal spherical image. Since $\psi = \psi(s_\psi)$ is a general helix, there is a constant space-like vector U, we may express

$$g(T_\psi, U) = \cos\theta, \tag{4.7}$$

where θ is a constant angle. The equation (4.7) is also congruent to

$$g(-T + \frac{\tau}{\kappa}B_1, U) = \cos\theta.$$

One can form constant vector U according to $\{T, N, B_1, B_2\}$ as the following

$$U = \varepsilon_1 T + \varepsilon_2 N + \varepsilon_3 B_1 + \varepsilon_4 B_2. \tag{4.8}$$

Differentiating (4.8) with respect to s, we have the following system of ordinary differential equations

$$\begin{cases} \varepsilon_1' - \varepsilon_2\kappa = 0, \\ \varepsilon_1\kappa + \varepsilon_2' - \varepsilon_4\tau = 0, \\ \varepsilon_2\tau + \varepsilon_3' + \varepsilon_3\sigma = 0, \\ \varepsilon_4' - \varepsilon_4\sigma = 0. \end{cases} \tag{4.9}$$

We know that $\varepsilon_1\kappa - \varepsilon_4\tau = c_1 \neq 0$ is a constant. Also since the curve $\beta = \beta(s)$ is a W-curve, its curvature functions are constants. Then the solution of the system (4.9) can be obtained as:

$$\begin{aligned} \varepsilon_1 &= -\tfrac{1}{2}c_1\kappa s^2 + c_2\kappa s + c_3, \\ \varepsilon_2 &= -c_1 s + c_2, \\ \varepsilon_3 &= \frac{c_1\tau}{\sigma}s + c_4 e^{-\sigma s} + c_3, \\ \varepsilon_4 &= e^{\sigma s + c_5}, \end{aligned}$$

where c_2, c_3, c_4, c_5 are constants. Q.E.D.

5 The trinormal spherical image of a W-partially null curve lying on $\mathbb{H}_0^3$

By the spirit of the paper [14], first we adapt the definition of trinormal spherical image to W-partially null curves of Minkowski space-time.

Let $\beta = \beta(s)$ be a unit speed W-partially null curve in Minkowski space-time. If we translate the trinormal vector to the center O of the pseudohyperbolic space $\mathbb{H}_0^3$, we obtain a curve $\varphi = \varphi(s_\varphi)$. This curve $\varphi = \varphi(s_\varphi)$ is called the trinormal spherical image of the curve β in $\mathbb{E}_1^4$.

Theorem 5.1. Let $\beta = \beta(s)$ be an W-partially null unit speed curve and $\varphi = \varphi(s_\varphi)$ be its trinormal spherical image. Then;

i) $\varphi = \varphi(s_\varphi)$ is a space-like curve.

ii) The Frenet-Serret apparatus of φ, $\{T_\varphi, N_\varphi, B_{1\varphi}, B_{2\varphi}, \kappa_\varphi, \tau_\varphi, \sigma_\varphi\}$ can be formed by the apparatus of β, $\{T, N, B_1, B_2, \kappa, \tau, \sigma\}$.

Proof. Let $\beta = \beta(s)$ be a unit speed W-partially-null curve and $\varphi = \varphi(s_\varphi)$ be its trinormal spherical image. It can be written as

$$\varphi = B_2. \tag{5.1}$$

Differentiating (5.1) with respect to s, we find

$$\varphi' = \dot{\varphi}\frac{ds_\varphi}{ds} = -\tau N - \sigma B_2.$$

Here we shall denote differentiation according to s by a dash, and differentiation according to s_φ by a dot. Thus we obtain the unit tangent vector of the trinormal spherical image curve φ as

$$T_\varphi = -N - \frac{\sigma}{\tau}B_2 \tag{5.2}$$

and

$$\|\varphi'\| = \frac{ds_\varphi}{ds} = \tau.$$

The causal character of the the trinormal spherical image curve $\varphi(s_\varphi)$ is determined by the following inner product:

$$g(\varphi', \varphi') = \tau^2 > 0. \tag{5.3}$$

By (5.3), we will take the spherical image curve as spacelike one.

Considering the previous method and using the property of the curve to be W-curve, we form the following differentiations with respect to s:

$$\begin{cases} \varphi'' = \kappa\tau T + \tau\sigma N - \tau^2 B_1 + \sigma^2 B_2, \\ \varphi''' = -\kappa\tau\sigma T + (\kappa^2 - \sigma^2)\tau N - \sigma^3 B_2, \\ \varphi^{(IV)} = -\kappa\tau(\kappa^2 - \sigma^2)T - \tau\sigma(\kappa^2 - \sigma^2)N + \tau(\kappa^2 - \sigma^2)B_1 + \sigma^4 B_2. \end{cases}$$

By equation (2.2) we arrive

$$\|\varphi'\|^2 \varphi'' - g(\varphi', \varphi'')\varphi' = \tau^3\kappa T + \tau^3\sigma N - \tau^4 B_1 + \tau^2\sigma^2 B_2.$$

Then by taking norm of the above equation, we can get the principal normal vector as

$$N_\varphi = \frac{\kappa}{\sqrt{|\kappa^2 - \sigma^2|}}T + \frac{\sigma}{\sqrt{|\kappa^2 - \sigma^2|}}N - \frac{\tau}{\sqrt{|\kappa^2 - \sigma^2|}}B_1 + \frac{\sigma^2}{\tau\sqrt{|\kappa^2 - \sigma^2|}}B_2 \tag{5.4}$$

and the first curvature as

$$\kappa_\varphi = \frac{\sqrt{|\kappa^2 - \sigma^2|}}{\tau}. \tag{5.5}$$

Now let us calculate the vector $T_\varphi \wedge N_\varphi \wedge \varphi'''$, that is,

$$T_\varphi \wedge N_\varphi \wedge \varphi''' = -\begin{vmatrix} -T & N & B_1 & B_2 \\ 0 & -1 & 0 & -\frac{\sigma}{\tau} \\ \frac{\kappa}{\sqrt{|\kappa^2 - \sigma^2|}} & \frac{\sigma}{\sqrt{|\kappa^2 - \sigma^2|}} & -\frac{\tau}{\sqrt{|\kappa^2 - \sigma^2|}} & \frac{\sigma^2}{\tau\sqrt{|\kappa^2 - \sigma^2|}} \\ -\kappa\tau\sigma & (\kappa^2 - \sigma^2)\tau & 0 & -\sigma^3 \end{vmatrix}.$$

This product yields

$$T_\varphi \wedge N_\varphi \wedge \varphi''' = -\frac{\kappa^2\tau\sigma}{\sqrt{|\kappa^2 - \sigma^2|}}T + \frac{\kappa\tau\sigma^2}{\sqrt{|\kappa^2 - \sigma^2|}}N + \frac{\kappa\sigma((1-\tau)\sigma^2 + \kappa^2)}{\sqrt{|\kappa^2 - \sigma^2|}}B_1 - \frac{\kappa\tau^2\sigma}{\sqrt{|\kappa^2 - \sigma^2|}}B_2. \tag{5.6}$$

Hence, by taking norm of (5.6) and using it in (2.3), we obtain the trinormal (second binormal) vector field of the curve $\varphi(s_\varphi)$ as follows:

$$\begin{aligned} B_{2\varphi} = &\frac{\tau^2-\sigma}{\sqrt{|2\tau-1-\kappa^2|\,|\kappa^2-\sigma^2|}}T + \frac{\kappa\sigma((1-\tau)\sigma^2+\kappa^2-1)}{\sqrt{|2\tau-1-\kappa^2|\,|\kappa^2-\sigma^2|}}N \\ &-\frac{\kappa(\tau^2+\sigma^2)}{\tau\sqrt{|2\tau-1-\kappa^2|\,|\kappa^2-\sigma^2|}}B_1 + \frac{\kappa\tau(1-(1-\tau)\sigma^2-\kappa^2)}{\sqrt{|2\tau-1-\kappa^2|\,|\kappa^2-\sigma^2|}}B_2. \end{aligned} \tag{5.7}$$

By the norm of (5.6), we find the second curvature

$$\tau_\varphi = \frac{\kappa\sigma\sqrt{|2\tau-1-\kappa^2|}}{\tau(\kappa^2-\sigma^2)}. \tag{5.8}$$

Finding the binormal vector field, we express

$$N_\varphi \wedge T_\varphi \wedge B_{2\varphi} = -\mu \begin{vmatrix} -T & N & B_1 & B_2 \\ \dfrac{\kappa}{\sqrt{|\kappa^2-\sigma^2|}} & \dfrac{\sigma}{\sqrt{|\kappa^2-\sigma^2|}} & -\dfrac{\tau}{\sqrt{|\kappa^2-\sigma^2|}} & \dfrac{\sigma^2}{\tau\sqrt{|\kappa^2-\sigma^2|}} \\ 0 & -1 & 0 & -\dfrac{\sigma}{\tau} \\ -\dfrac{\kappa}{\sqrt{|2\tau-1-\kappa^2|}} & \dfrac{\sigma}{\sqrt{|2\tau-1-\kappa^2|}} & \dfrac{\tau((1-\tau)\sigma^2+\kappa^2)}{\sqrt{|2\tau-1-\kappa^2|}} & \dfrac{-\tau}{\sqrt{|2\tau-1-\kappa^2|}} \end{vmatrix}. \tag{5.9}$$

Calculating (5.9), we have

$$\begin{aligned} N_\varphi \wedge T_\varphi \wedge B_{2\varphi} = \mu\{&\frac{\tau^2-\sigma}{\sqrt{|2\tau-1-\kappa^2|\,|\kappa^2-\sigma^2|}}T + \frac{\kappa\sigma((1-\tau)\sigma^2+\kappa^2-1)}{\sqrt{|2\tau-1-\kappa^2|\,|\kappa^2-\sigma^2|}}N \\ &-\frac{\kappa(\tau^2+\sigma^2)}{\tau\sqrt{|2\tau-1-\kappa^2|\,|\kappa^2-\sigma^2|}}B_1 + \frac{\kappa\tau(1-(1-\tau)\sigma^2-\kappa^2)}{\sqrt{|2\tau-1-\kappa^2|\,|\kappa^2-\sigma^2|}}B_2\}. \end{aligned}$$

So we obtain the binormal vector as

$$\begin{aligned} B_{1\varphi} = &\frac{\tau^2-\sigma}{\sqrt{|2\tau-1-\kappa^2|\,|\kappa^2-\sigma^2|}}T + \frac{\kappa\sigma((1-\tau)\sigma^2+\kappa^2-1)}{\sqrt{|2\tau-1-\kappa^2|\,|\kappa^2-\sigma^2|}}N \\ &-\frac{\kappa(\tau^2+\sigma^2)}{\tau\sqrt{|2\tau-1-\kappa^2|\,|\kappa^2-\sigma^2|}}B_1 + \frac{\kappa\tau(1-(1-\tau)\sigma^2-\kappa^2)}{\sqrt{|2\tau-1-\kappa^2|\,|\kappa^2-\sigma^2|}}B_2. \end{aligned} \tag{5.10}$$

Finally, using (2.4) and the obtained equations, we arrive the third curvature as

$$\sigma_\varphi = \frac{(\sigma^2-\kappa^2)\sqrt{|\kappa^2-\sigma^2|}}{\kappa\tau\sigma\sqrt{|2\tau-1-\kappa^2|}}. \tag{5.11}$$

Q.E.D.

Corollary 5.2. $\{T_\varphi, N_\varphi, B_{1\varphi}, B_{2\varphi}\}$ is an orthonormal frame of Minkowski space-time.

Proof. It can be straightforwardly seen by using the equations (5.2), (5.4), (5.7) and (5.10). Q.E.D.

Considering above theorem, we also give:

Corollary 5.3. Let $\beta = \beta(s)$ be a W-partially null unit speed curve and $\varphi(s_\varphi)$ be its trinormal spherical image. Then, φ is also a helix.

Proof. Let $\beta = \beta(s)$ be a unit speed W-partially null curve. Then we know that the curvature functions are constants. Therefore, we straightforwardly see that the curvature functions of trinormal spherical image $\varphi(s_\varphi)$ are constants by means of the equations (5.5), (5.8) and (5.11). Hence the curve $\varphi(s_\varphi)$ becomes W-curve which is the special case of general helix.

Q.E.D.

Theorem 5.4. Let $\beta = \beta(s)$ be a W-partially null unit speed curve and $\varphi(s_\varphi)$ be its trinormal spherical image. Then, the fixed direction U of φ is composed as

$$\begin{aligned} U = &\{c_4 \sin(\kappa s) - c_5 \cos(\kappa s) + \frac{\tau - \sigma^2}{\kappa^2} s + c_6\} T \\ &+ \{c_4 \cos(\kappa s) + c_5 \sin(\kappa s) + \frac{\tau - \sigma^2}{\kappa^2}\} N \\ &+ \{-c_1 s + c_2\} B_1 + e^{\sigma s + c_3} B_2, \end{aligned}$$

where c_1 is a nonzero-constant, and c_2, c_3, c_4, c_5, c_6 are constants.

Proof. Let $\beta = \beta(s)$ be a unit speed W-partially null curve and $\varphi = \varphi(s_\varphi)$ be its space-like trinormal spherical image. Since $\varphi = \varphi(s_\varphi)$ is a general helix, there is a constant space-like vector U, we may express

$$g(T_\varphi, U) = \cos\theta, \tag{5.12}$$

where θ is a constant angle. The equation (5.12) is also congruent to

$$g(-N - \frac{\sigma}{\tau} B_2, U) = \cos\theta.$$

One can form constant vector U according to $\{T, N, B_1, B_2\}$ as the following

$$U = \varepsilon_1 T + \varepsilon_2 N + \varepsilon_3 B_1 + \varepsilon_4 B_2. \tag{5.13}$$

Differentiating (5.13) with respect to s, we have the following system of ordinary differential equations

$$\begin{cases} \varepsilon_1' - \varepsilon_2 \kappa = 0, \\ \varepsilon_1 \kappa + \varepsilon_2' - \varepsilon_4 \tau = 0, \\ \varepsilon_2 \tau + \varepsilon_3 \sigma + \varepsilon_3' = 0, \\ \varepsilon_4' - \varepsilon_4 \sigma = 0. \end{cases} \tag{5.14}$$

We know that $\varepsilon_2 \tau + \varepsilon_3 \sigma = c_1 \neq 0$ is a constant. Also since the curve $\beta = \beta(s)$ is a W-curve, its curvature functions are constants. Then the solution of the system (5.14) can be found as:

$$\begin{aligned} \varepsilon_1 &= c_4 \sin(\kappa s) - c_5 \cos(\kappa s) + \frac{\tau - \sigma^2}{\kappa^2} s + c_6, \\ \varepsilon_2 &= c_4 \cos(\kappa s) + c_5 \sin(\kappa s) + \frac{\tau - \sigma^2}{\kappa^2}, \\ \varepsilon_3 &= -c_1 s + c_2, \\ \varepsilon_4 &= e^{\sigma s + c_3}, \end{aligned}$$

where c_1 is a nonzero-constant, and c_2, c_3, c_4, c_5, c_6 are constants. Q.E.D.

6 Conclusion

In this work, we extend spherical image concept to the W-partially null curves of Minkowski space-time. We investigate the tangent, principal normal and trinormal spherical images of a W-partially null curve and observe that all of these spherical curves are space-like curves. Thereafter, we determine relations among Frenet-Serret invariants of spherical images and base curve. Finally, we give some characterizations of the spherical images being helices relative to choosing the base curve as a special helix. As a further research, one can study the spherical images of partially null curves without putting any condition for the base curve.

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