Results on entire and meromorphic functions that share small function with their homogeneous and linear differential polynomials

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Abstract

Using the results of S. S. Bhoosnurmath, we mainly study the uniqueness of entire and meromorphic functions that share small functions with their homogeneous and linear differential polynomials. In this paper, we obtain significant improvements and generalizations of the results of H. X. Yi.

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1 Introduction, definitons and results

For a meromorphic function f in the complex plane we assume familiarity with the standard notations of Nevanlinna theory such as, $T(r,f)$, $N(r,f)$ and $m(r,f)$ etc., as explained in [2,3].

We need the following definitions as explained in [2,3]:

Definition 1.1. Let $f(z)$ and $a(z)$ be meromorphic functions in the complex plane. If $T(r,a) = S(r,f)$, then $a(z)$ is called a small function of $f(z)$, where $S(r,f) = o(T(r,f))$ as $r \to \infty$, except possibly on a set of finite linear measure.

Definition 1.2. Two meromorphic functions f and g share a CM (counting multiplicity) or IM (ignoring multiplicity) if $f-a$ and $g-a$ have the same set of zeros counting multiplicities or ignoring multiplicities respectively.

As in [2], for a meromorphic function f, deficiency of $'a'$ and ramification index of $'a'$ are given by

$$\delta(a,f) = 1 - \limsup_{r\to\infty} \frac{N(r,\frac{1}{f-a})}{T(r,f)},$$

and

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,\frac{1}{f-a})}{T(r,f)}$$

respectively.

In $\{z : |z| < r\}$, the reduced counting function of those zeros of $F-1$ is denoted by $\overline{N}_L(r,\frac{1}{F-1})$ and is equal to $N(r,\frac{1}{F}) + \overline{N}(r,F) + S(r,F)$, the reduced counting function of common simple zeros

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of $F-1$ is denoted by $N_E^{1)}(r,\frac{1}{F-1})$ and common multiple zeros of $F-1$ is denoted by $\overline{N}_E^{(2}(r,\frac{1}{F-1})$.

Definition 1.3. ([1]). An expression of the form

$$H[f]=\sum_{i=1}^{n}a_i(z)f^{n_{i0}}(f')^{n_{i1}}...(f^{(k)})^{n_{ik}} \tag{1.1}$$

is called as differential polynomial in f of degree $\overline{d}(H)$, lower degree $\underline{d}(H)$ and weight Γ_H, where n_{i0}, $n_{i1},...,n_{ik}$ are non negative integers and $a_i(z)$ are meromorphic functions satisfying $T(r,a_i)=S(r,f)$ and

$$\overline{d}(H)=max\left\{\sum_{j=0}^{k}n_{ij};1\leq i\leq n\right\},$$

$$\underline{d}(H)=min\left\{\sum_{j=0}^{k}n_{ij};1\leq i\leq n\right\},$$

$$\Gamma_H=max\left\{\sum_{j=0}^{k}(j+1)n_{ij};1\leq i\leq n\right\},$$

If $\overline{d}(H)=\underline{d}(H)=n$(say), then the differential polynomial $H[f]$ is called a homogeneous differential polynomial in f of degree n.

In 1997, H.X.Yi proved the following results.

Theorem A (See [4]). Let f and g be two nonconstant meromorphic functions such that $f^{(n)}$ and $g^{(n)}$ share the value 1 CM. If $2\delta(0,f)+(n+4)\Theta(\infty,f)>n+5$ and $2\delta(0,g)+(n+4)\Theta(\infty,g)>n+5$ then either $f\equiv g$ or $f^{(n)}g^{(n)}\equiv 1$.

Theorem B (See [4]). Let f and g be two nonconstant meromorphic functions such that $f^{(n)}$ and $g^{(n)}$ share the value 1 IM. If $5\delta(0,f)+(4n+7)\Theta(\infty,f)>4n+11$ and $5\delta(0,g)+(4n+7)\Theta(\infty,g)>4n+11$ then either $f\equiv g$ or $f^{(n)}g^{(n)}\equiv 1$.

We obtain significant improvements and generalizations of the above mentioned results to general differential polynomials. Note that our results hold good for non-zero small functions as shared values.

Theorem 1.1. Let f and g be two nonconstant meromorphic functions, and $a(z)$ be a small meromorphic function such that $a(z)\not\equiv 0,\infty$. Let $P[f]$ and $P[g]$ be nonconstant differential polynomials in f and g respectively as defined in (1.1). If $P[f]$ and $P[g]$ share a IM and

$$5\underline{d}(P)\delta(0,f)+(2Q_1+2Q_2+7)\Theta(\infty,f)>2Q_1+2Q_2+4\overline{d}(P)+6 \tag{1.2}$$

and

$$5\underline{d}(P)\delta(0,g)+(2Q_1+2Q_2+7)\Theta(\infty,g)>2Q_1+2Q_2+4\overline{d}(P)+6 \tag{1.3}$$

then either $P[f]\equiv P[g]$ or $P[f]P[g]\equiv a^2$.

Theorem 1.2. Let f and g be two nonconstant meromorphic functions, and $a(z)$ be a small meromorphic function such that $a(z)\not\equiv 0,\infty$. Let $P[f]$ and $P[g]$ be nonconstant differential polynomials in f and g respectively as defined in (1.1). If $P[f]$ and $P[g]$ share a CM and

$$2\underline{d}(P)\delta(0,f)+(Q_1+4)\Theta(\infty,f)>Q_1+\overline{d}(P)+3 \tag{1.4}$$

and

$$2\underline{d}(P)\delta(0,g)+(Q_1+4)\Theta(\infty,g)>Q_1+\overline{d}(P)+3 \tag{1.5}$$

then either $P[f]\equiv P[g]$ or $P[f]P[g]\equiv a^2$.

Theorem 1.3. Let f and g be two nonconstant entire functions, and $a(z)$ be a small meromorphic function such that $a(z)\not\equiv 0,\infty$. Let $P[f]$ and $P[g]$ be nonconstant differential polynomials in f and g respectively as defined in (1.1). If $P[f]$ and $P[g]$ share a IM and

$$\underline{d}(P)\delta(0,f)>\frac{4}{5}\overline{d}(P)-1 \tag{1.6}$$

and

$$\underline{d}(P)\delta(0,g)>\frac{4}{5}\overline{d}(P)-1 \tag{1.7}$$

then either $P[f]\equiv P[g]$ or $P[f]P[g]\equiv a^2$.

Theorem 1.4. Let f and g be two nonconstant entire functions, and $a(z)$ be a small meromorphic function such that $a(z)\not\equiv 0,\infty$. Let $P[f]$ and $P[g]$ be nonconstant differential polynomials in f and g respectively as defined in (1.1). If $P[f]$ and $P[g]$ share a CM and

$$\underline{d}(P)\delta(0,f)>\frac{\overline{d}(P)}{2}-1 \tag{1.8}$$

and

$$\underline{d}(P)\delta(0,g)>\frac{\overline{d}(P)}{2}-1 \tag{1.9}$$

then either $P[f]\equiv P[g]$ or $P[f]P[g]\equiv a^2$.

Suppose $P[f]=f^{(k)}$ and $P[g]=g^{(k)}$, then $Q_1=Q_2=k$ and $\overline{d}(P)=\underline{d}(P)=1$.
We get the following results which improve and extend results given by H.X.Yi to more general differential polynomials.

Corollary 1.1. Let f and g be two nonconstant meromorphic functions, and $a(z)$ be a small meromorphic function such that $a(z)\not\equiv 0,\infty$. Let $f^{(k)}$ and $g^{(k)}$ share a IM and $5\delta(0,f)+(4k+7)\Theta(\infty,f)>4k+10$ and $5\delta(0,g)+(4k+7)\Theta(\infty,g)>4k+10$ then either $f\equiv g$ or $f^{(k)}g^{(k)}\equiv a^2$.
Corollary 1.1 significantly improves Theorem B.

Corollary 1.2. Let f and g be two nonconstant meromorphic functions, and $a(z)$ be a small meromorphic function such that $a(z)\not\equiv 0,\infty$. Let $f^{(k)}$ and $g^{(k)}$ share a CM and $2\delta(0,f)+(k+4)\Theta(\infty,f)>k+4$ and $2\delta(0,g)+(k+4)\Theta(\infty,g)>k+4$ then either $f\equiv g$ or $f^{(k)}g^{(k)}\equiv a^2$.
Corollary 1.2 significantly improves Theorem A.

2 Some Lemmas.

Lemma 2.1. ([4]). Let F and G be two nonconstant meromorphic functions such that F and G share 1 IM. Then

$$\begin{aligned}T(r,F)\le{}&\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+N_E^{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)\\&-N_0\left(r,\frac{1}{F'}\right)-N_0\left(r,\frac{1}{G'}\right)+S(r,F)+S(r,G),\end{aligned} \tag{2.1}$$

where $N_0\left(r,\frac{1}{F'}\right)$ denotes the counting function corresponding to the zeros of F' that are not zeros of F and $F-1$, $N_0\left(r,\frac{1}{G'}\right)$ denotes the counting function corresponding to the zeros of G' that are not zeros of G, G-1.

Lemma 2.2. ([1]). Let f be a transcendental meromorphic function. Let $P[f]$ be as defined in (1.1). If $P[f]\neq 0$, we have

$$N\left(r,\frac{1}{P[f]}\right)\leq Q\overline{N}(r,F)+[\overline{d}(P)-\underline{d}(P)]m\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{\overline{d}(P)}}\right)+S(r,f)$$

and

$$T(r,P[f])\leq \overline{d}(P)T(r,f)+Q\overline{N}(r,F)+S(r,f).$$

3 Proof of the Theorems.

Proof of Theorem 1.1.

Let

$$F=\frac{P[f]}{a}\ and\ G=\frac{P[g]}{a}. \tag{3.1}$$

Suppose F and G share 1 IM, then we have by Lemma 2.1,

$$\begin{aligned}T(r,F)\leq{}&\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+N_E^{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)\\&-N_0\left(r,\frac{1}{F'}\right)-N_0\left(r,\frac{1}{G'}\right)+S(r,F)+S(r,G).\end{aligned} \tag{3.2}$$

Let H be defined as follows

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right). \tag{3.3}$$

Suppose that $H\not\equiv 0$. If $F-1$ and $G-1$ has common simple zero z_0, by a simple computation, we get that z_0 is a zero of H. Thus we have

$$N_E^{1)}\left(r,\frac{1}{F-1}\right)\leq\overline{N}(r,\frac{1}{H})\leq T(r,H)+O(1)\leq N(r,H)+S(r,F)+S(r,G). \tag{3.4}$$

By simple computation we obtain

$$\begin{aligned}N(r,H)\leq{}&\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)\\&+\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)\\&+S(r,F)+S(r,G).\end{aligned} \tag{3.5}$$

We have

$$\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)\leq N\left(r,\frac{1}{F}\right), \tag{3.6}$$

and

$$\overline{N}_L\left(r,\frac{1}{F-1}\right)\leq N\left(r,\frac{1}{F-1}\right)-\overline{N}\left(r,\frac{1}{F-1}\right)\leq N\left(r,\frac{1}{F'}\right). \tag{3.7}$$

Using (3.4)-(3.7) in (3.2) we get,

$$T(r,F)\leq 4\overline{N}(r,F)+3\overline{N}(r,G)+3N\left(r,\frac{1}{F}\right)+2N\left(r,\frac{1}{G}\right)+S(r,F)+S(r,G). \tag{3.8}$$

From Lemma 2.2 and $N\left(r,\frac{1}{f^{\overline{d}(P)}}\right)\leq \overline{d}(P)N\left(r,\frac{1}{f}\right)$ we get

$$\begin{aligned} N\left(r,\frac{1}{P[f]}\right)&\leq Q\overline{N}(r,F)+[\overline{d}(P)-\underline{d}(P)]m\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{\overline{d}(P)}}\right)+S(r,f)\\ &\leq Q\overline{N}(r,F)+[\overline{d}(P)-\underline{d}(P)]\left\{T\left(r,\frac{1}{f}\right)-N\left(r,\frac{1}{f}\right)\right\}\\ &+\overline{d}(P)N\left(r,\frac{1}{f}\right)+S(r,f),\\ &\leq Q\overline{N}(r,F)+[\overline{d}(P)-\underline{d}(P)]T(r,f)+\underline{d}(P)N\left(r,\frac{1}{f}\right), \end{aligned} \tag{3.9}$$

and

$$\overline{N}(r,P[f])\leq \overline{N}(r,f). \tag{3.10}$$

Substituting (3.9) and (3.10) in (3.8), we get

$$\begin{aligned} [3\underline{d}(P)-2\overline{d}(P)]T(r,f)-[2\overline{d}(P)-2\underline{d}(P)]T(r,g)&\leq (4+2Q_1)\overline{N}(r,f)+(3+2Q_2)\overline{N}(r,g)\\ &+3\underline{d}(P)N\left(r,\frac{1}{f}\right)+2\underline{d}(P)N\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g). \end{aligned} \tag{3.11}$$

Similarly

$$\begin{aligned} [3\underline{d}(P)-2\overline{d}(P)]T(r,g)-[2\overline{d}(P)-2\underline{d}(P)]T(r,f)&\leq (4+2Q_1)\overline{N}(r,g)+(3+2Q_2)\overline{N}(r,f)\\ &+3\underline{d}(P)N\left(r,\frac{1}{g}\right)+2\underline{d}(P)N\left(r,\frac{1}{f}\right)+S(r,f)+S(r,g). \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12), we get

$$\begin{aligned} [5\overline{d}(P)-4\underline{d}(P)]\{T(r,f)+T(r,g)\}&\leq (2Q_1+2Q_2+7)\overline{N}(r,f)+(2Q_1+2Q_2+7)\overline{N}(r,g)\\ &+5\underline{d}(P)N\left(r,\frac{1}{f}\right)+5\underline{d}(P)N\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g). \end{aligned} \tag{3.13}$$

From the above equation we obtain

$$\begin{aligned} &[5\underline{d}(P)\delta(0,f)+(2Q_1+2Q_2+7)\Theta(\infty,f)]T(r,f)+[5\underline{d}(P)\delta(0,g)+(2Q_1+2Q_2+7)\Theta(\infty,g)]T(r,g)\\ &\leq Q_1+2Q_2+4\overline{d}[P]+7)T(r,f)+(2Q_1+2Q_2+4\overline{d}[P]+6)T(r,g)+S(r,f)+S(r,g). \end{aligned} \tag{3.14}$$

This contradicts our assumptions of (1.1) and (1.2). Therefore $H \equiv 0$.
Integrating (3.3), we get

$$\frac{1}{G-1} = \frac{C_1}{F-1} + C_2,$$

where, $C_1 \neq 0$ and C_2 are constants. Thus

$$G = \frac{(C_2+1)F + (C_1 - C_2 - 1)}{C_2F + (C_1 - C_2)} \tag{3.15}$$

$$F = \frac{(C_2 - C_1)G + (C_1 - C_2 - 1)}{C_2G - (C_2+1)}. \tag{3.16}$$

We consider the following three cases.
Case 1: If $C_2 \neq 0, -1$, then from (3.16), we have

$$\overline{N}\left(r, \frac{1}{G - (C_2+1)/C_2}\right) = \overline{N}(r, F). \tag{3.17}$$

Using this and second fundamental theorem, we have

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,G) + N\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{G - (C_2+1)/C_2}\right) + S(r,G) \\ &\leq \overline{N}(r,G) + N\left(r, \frac{1}{G}\right) + \overline{N}(r,F) + S(r,G). \end{aligned}$$

Using Lemma 2.2, (3.9) and (3.10), we get

$$\underline{d}(P)T(r,g) \leq \overline{N}(r,g) + \overline{N}(r,f) + \underline{d}(P)N\left(r, \frac{1}{g}\right) + S(r,g). \tag{3.18}$$

If $C_1 - C_2 - 1 \neq 0$, from (3.15) we have $N\left(r, \frac{1}{F - \frac{C_2+1-C_1}{C_2+1}}\right) = N\left(r, \frac{1}{G}\right)$.
Hence by the second fundamental theorem, we get

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,F) + N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{F - \frac{C_2+1-C_1}{C_2+1}}\right) + S(r,F) \\ &\leq \overline{N}(r,F) + N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{G}\right) + S(r,F). \end{aligned}$$

Using Lemma 2.2, (3.9) and (3.10), we get

$$\begin{aligned} \underline{d}(P)T(r,f) - (\overline{d}(P) - \underline{d}(P))T(r,g) \leq Q_1\overline{N}(r,f) + Q_2\overline{N}(r,g) + \underline{d}(P)N\left(r, \frac{1}{f}\right) \\ + \underline{d}(P)N\left(r, \frac{1}{g}\right) + S(r,f). \end{aligned} \tag{3.19}$$

From (3.18) and (3.19), we get

$$\begin{aligned}\underline{d}(P)T(r,f)+(2\underline{d}(P)-\overline{d}(P))T(r,g)\leq 2\overline{N}(r,f)+(Q_2+1)\overline{N}(r,g)+\underline{d}(P)N\left(r,\frac{1}{f}\right)\\ +2\underline{d}(P)N\left(r,\frac{1}{g}\right)+S(r,f)\end{aligned}\tag{3.20}$$

which contradicts the assumption of (1.2) and (1.3). Thus $C_1-C_2-1=0$. From (3.15),

$$G=\frac{(C_2+1)F}{C_2F+1}.$$

Hence $\overline{N}\left(r,\frac{1}{F+1/C_2}\right)=\overline{N}(r,G)$.
From the second fundamental theorem, we get

$$\begin{aligned}T(r,F) &= \overline{N}(r,F)+N\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F+1/C_2}\right)+S(r,F)\\ &\leq \overline{N}(r,F)+N\left(r,\frac{1}{F}\right)+\overline{N}(r,G)+S(r,F).\end{aligned}$$

Using Lemma 2.2, (3.9) and (3.10), we get

$$\underline{d}(P)T(r,f)+\leq\overline{N}(r,f)+\overline{N}(r,g)+\underline{d}(P)N\left(r,\frac{1}{f}\right)+S(r,f).\tag{3.21}$$

Combining (3.18) and (3.21) we get

$$\underline{d}(P)T(r,f)+\underline{d}(P)T(r,g)\leq 2\overline{N}(r,f)+2\overline{N}(r,g)+\underline{d}(P)N\left(r,\frac{1}{f}\right)+\underline{d}(P)N\left(r,\frac{1}{g}\right)+S(r,f).\tag{3.22}$$

Which contradicts the assumption of (1.2) and (1.3).
Case 2: Suppose that $C_2=0$. From (3.15) and (3.16), we get

$$G=\frac{F+C_1-1}{C_1},\quad and\ \ F=C_1G+1-C_1$$

If $C_1-1\neq 0$, then it follows that

$$N\left(r,\frac{1}{F-(C_1-1)}\right)=N\left(r,\frac{1}{G}\right)\quad and\quad N\left(r,\frac{1}{G-(C_1-1)/C_1}\right)=N\left(r,\frac{1}{F}\right)$$

Using the similar argument of Case(1) we arrive at a contradiction. Therefore $C_1-1=0$ and so $P[f]\equiv P[g]$.
Case 3: Suppose that $C_2=-1$. From (3.15) and (3.16), we get

$$G=\frac{C_1}{C_1+1-F},\quad and\ \ F=\frac{(C_1+1)G-C_1}{G}.$$

If $C_1+1\neq 0$, then it follows that

$$\overline{N}\left(r,\frac{1}{F-(C_1+1)}\right)=\overline{N}(r,G)\ \ and\ \ N\left(r,\frac{1}{G-C_1/(C_1+1)}\right)=N\left(r,\frac{1}{F}\right).$$

Using the similar argument of Case(1) we arrive at a contradiction. Therefore $C_1+1=0$ and so $P[f]P[g]\equiv a^2$.
This proves Theorem 1.1.

Proof of Theorem 1.2.
Let F and G be given by (3.1). Suppose that F and G share a CM, then $\overline{N}_L(r,\frac{1}{F-1})=\overline{N}_L(r,\frac{1}{G-1})=0$.
Hence we get from (3.2)-(3.6)

$$T(r,F)\leq 2\overline{N}(r,F)+2\overline{N}(r,G)+N\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{G}\right)+S(r,F)+S(r,G). \tag{3.23}$$

Using Lemma 2.2, (3.9) and (3.10), we get

$$\begin{aligned}\underline{d}(P)T(r,f)-[\overline{d}(P)-\underline{d}(P)]T(r,g)\leq\ & 2\overline{N}(r,f)+(Q_2+2)\overline{N}(r,g)\\ &+\underline{d}(P)N\left(r,\frac{1}{f}\right)+\underline{d}(P)N\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g).\end{aligned} \tag{3.24}$$

Similarly we get

$$\begin{aligned}\underline{d}(P)T(r,g)-[\overline{d}(P)-\underline{d}(P)]T(r,f)\leq\ & 2\overline{N}(r,g)+(Q_1+2)\overline{N}(r,f)\\ &+\underline{d}(P)N\left(r,\frac{1}{f}\right)+\underline{d}(P)N\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g).\end{aligned} \tag{3.25}$$

Combining (3.24) and (3.25), we get

$$\begin{aligned}(2\underline{d}(P)-\overline{d}(P))\{T(r,f)+T(r,g)\}\ &\leq\ (Q_1+4)\overline{N}(r,g)+(Q_2+4)\overline{N}(r,f)\\ &+\ 2\underline{d}(P)N\left(r,\frac{1}{f}\right)+2\underline{d}(P)N\left(r,\frac{1}{g}\right)\\ &+\ S(r,f)+S(r,g).\end{aligned}$$

From the above equation we obtain

$$\begin{aligned}&[2\underline{d}(P)\delta(0,f)+(Q_1+4)\Theta(\infty,f)]T(r,f)+[2\underline{d}(P)\delta(0,g)+(Q_2+4)\Theta(\infty,g)]T(r,g)\\ &\qquad\leq(Q_1+\overline{d}[P]+4)T(r,f)+(Q_2+\overline{d}[P]+4)T(r,g)+S(r,f)+S(r,g).\end{aligned} \tag{3.26}$$

This contradicts our assumptions of (1.4) and (1.5). Therefore $H\equiv 0$. Proceeding as in Theorem 1.1, we can prove Theorem 1.2.

Proof of Theorem 1.3.
Let F and G be given by (3.1). Suppose that F and G share a IM. If f and g are non-constant entire functions then

$$\overline{N}(r,f)=\overline{N}(r,g)=0 \tag{3.27}$$

Hence we get from (3.8)

$$T(r,F) \leq 3N\left(r,\frac{1}{F}\right) + 2N\left(r,\frac{1}{G}\right) + S(r,F) + S(r,G). \tag{3.28}$$

Using Lemma 2.2, (3.9) and (3.10), we get

$$\begin{aligned}(3\underline{d}(P) - 2\overline{d}(P))T(r,f) - 2(\underline{d}(P) - \overline{d}(P))T(r,g) \leq 3\underline{d}(P)N\left(r,\frac{1}{f}\right) + 2\underline{d}(P)N\left(r,\frac{1}{g}\right) \\ + S(r,f) + S(r,g).\end{aligned} \tag{3.29}$$

Similarly, we get

$$\begin{aligned}(3\underline{d}(P) - 2\overline{d}(P))T(r,g) - 2(\underline{d}(P) - \overline{d}(P))T(r,f) \leq 3\underline{d}(P)N\left(r,\frac{1}{g}\right) + 2\underline{d}(P)N\left(r,\frac{1}{f}\right) \\ + S(r,f) + S(r,g).\end{aligned} \tag{3.30}$$

Combining (3.29) and (3.30), we get

$$(5\underline{d}(P) - 4\overline{d}(P))\{T(r,f) + T(r,g)\} \leq 5\underline{d}(P)N\left(r,\frac{1}{g}\right) + 5\underline{d}(P)N\left(r,\frac{1}{f}\right) + S(r,f) + S(r,g). \tag{3.31}$$

From the above equation we obtain

$$\begin{aligned}[5\underline{d}(P)\delta(0,f)]T(r,f) + [5\underline{d}(P)\delta(0,g)]T(r,g) \leq 4\overline{d}[P]T(r,f) + 4\overline{d}[P]T(r,g) \\ + S(r,f) + S(r,g).\end{aligned} \tag{3.32}$$

This contradicts our assumptions of (1.6) and (1.7). Therefore $H \equiv 0$. Proceeding as in Theorem 1.1, we can prove Theorem 1.3.

Proof of Theorem 1.4.

Let F and G be given by (3.1). Suppose that F and G share a CM. We get from (3.23) and (3.27)

$$T(r,F) \leq N\left(r,\frac{1}{F}\right) + N\left(r,\frac{1}{G}\right) + S(r,F) + S(r,G). \tag{3.33}$$

Using Lemma 2.2, (3.9) and (3.10), we get

$$\underline{d}(P)T(r,f) - (\overline{d}(P) - \underline{d}(P))T(r,g) \leq \underline{d}(P)N\left(r,\frac{1}{f}\right) + \underline{d}(P)N\left(r,\frac{1}{g}\right) + S(r,f) + S(r,g). \tag{3.34}$$

Similarly, we get

$$\underline{d}(P)T(r,g) - (\overline{d}(P) - \underline{d}(P))T(r,f) \leq \underline{d}(P)N\left(r,\frac{1}{g}\right) + \underline{d}(P)N\left(r,\frac{1}{f}\right) + S(r,f) + S(r,g). \tag{3.35}$$

Combining (3.34) and (3.35), we get

$$(2\underline{d}(P) - \overline{d}(P))\{T(r,f) + T(r,g)\} \leq 2\underline{d}(P)N\left(r,\frac{1}{g}\right) + 2\underline{d}(P)N\left(r,\frac{1}{f}\right) + S(r,f) + S(r,g). \tag{3.36}$$

From the above equation we obtain

$$[\underline{d}(P)\delta(0,f)]T(r,f)+[\underline{d}(P)\delta(0,g)]T(r,g)\leq\frac{1}{2}\overline{d}(P)T(r,f)+\frac{1}{2}\overline{d}(P)T(r,g)+S(r,f)+S(r,g). \tag{3.37}$$

This contradicts our assumptions of (1.8) and (1.9). Therefore $H\equiv 0$. Proceeding as in Theorem 1.1, we can prove Theorem 1.4.

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