Stabilities of various multiplicative inverse functional equations

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Abstract

This study deals with the validation of classical stabilities of various multiplicative inverse functional equations pertinent to Ulam stability theory of functional equations in the setting of non-Archimedean fields. Suitable counter-examples are demonstrated to disprove the invalidity of stability results of these equations for singular cases.

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1 Introduction

The motivation for many researchers to study stability of functional equations is due to the query framed by Ulam in [25]. Ulam's problem was solved by many mathematicians via different versions in [8] (stability involving a positive constant, U-H stability), [14] (stability incorporated with sum of powers of norms, U-H-T stability), [13] (stability comprising product of different powers of norms, U-H-J stability), [7] (stability controlled by a general control function, U-H-G stability) and [15] (stability including mixed product-sum of powers of norms, J. Rassias product-sum stability). For the past 4 decades, there are several published papers with interesting, motivating and novel results concerning stability of various forms of functional equations, one can see ([1], [2], [3], [4], [5], [9], [10], [11], [12], [16], [17], [18], [19], [20], [21], [22], [23], [24]).

In this study, we propose a new multiplicative inverse functional equation

$$m_r\left(\frac{pq}{p+q}\right) = m_r(p) + m_r(q), \tag{1.1}$$

a generalized multiplicative inverse functional equation

$$m_r\left(\frac{pq}{\alpha_1 p+\alpha_2 q}\right) + m_r\left(\frac{pq}{\alpha_2 p+\alpha_1 q}\right) = (\alpha_1+\alpha_2)\left[m_r(p)+m_r(q)\right] \tag{1.2}$$

with $\alpha_1+\alpha_2 \neq 1$ and an n-dimensional multiplicative inverse functional equation

$$m_r\left(\frac{\prod_{\ell=1}^{n} p_\ell}{\sum_{\ell=1}^{n}\mu_\ell\left(\prod_{m=1,m\neq\ell}^{n} p_m\right)}\right) = \sum_{\ell=1}^{n}\mu_\ell m_r\left(p_\ell\right). \tag{1.3}$$

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with $\sum_{\ell=1}^{n} \mu_\ell \neq 1$. We solve Ulam stability problem for the above equations (1.1), (1.2) and (1.3) pertinent to U-H-G stability in the framework of non-Archimedean fields. Then, we extend their results concerning U-H stability, U-H-T stability, U-H-J and J. Rassias product-sum stability. We also provide counter-examples that the equations (1.1), (1.2) and (1.3) are not stable for singular values. It can be verified that the multiplicative inverse function $m_r(p) = \frac{k}{p}$ is a solution of the equations (1.1), (1.2) and (1.3).

Remark 1.1. Equation (1.3) produces various functional equations for different values of n. For instance, when $n = 2$, it generates the following equation:

$$m_r\left(\frac{p_1p_2}{\mu_2p_1 + \mu_1p_2}\right) = \mu_1 m_r\left(p_1\right) + \mu_2 m_r\left(p_2\right).$$

Similarly, for $n = 3$, it induces the ensuing equation:

$$m_r\left(\frac{p_1p_2p_3}{\mu_3p_1p_2 + \mu_2p_1p_3 + \mu_3p_1p_2}\right) = \mu_1 m_r\left(p_1\right) + \mu_2 m_r\left(p_2\right) + \mu_3 m_r\left(p_3\right).$$

2 Preliminaries

In order to achieve our desired results, let us introduce basic definition and notions associated with non-Archimedean fields.

Definition 2.1. By a non-Archimedean field, we mean a field $\mathbb{T}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{T}$ into $[0, \infty)$ such that $|k| = 0$ if and only if $k = 0$, $|k\ell| = |k||\ell|$ and $|k+\ell| \leq sup\{|k|, |\ell|\}$ for all $k, \ell \in \mathbb{T}$.

Clearly $|1| = |-1| = 1$ and $|k| \leq 1$ for all positive integers k. Further, we presume that $|\cdot|$ is non-trivial, that is, there exists a $\beta_0 \in \mathbb{T}$ such that $|\beta_0| \notin \{0, 1\}$. In lieu of the ensuing inequality: $|y_k - y_\ell| \leq \sup\{|y_{j+1} - y_j| : \ell \leq j \leq k-1\} \quad (k > \ell)$, a sequence $\{y_k\}$ is Cauchy if and only if $\{y_{k+1} - y_k\}$ converges to zero in a non-Archimedean field. If every Cauchy sequence is convergent in the field, then it is called a complete non-Archimedean field.

The mapping $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$ is an example of a non-Archimedean valuation. This valuation is called obviously no non-trivial. The following is one more exemplar of a non-Archimedean valuation on a field $\mathbb{T}$ is the mapping

$$|p| = \begin{cases} 0 & \text{if } p = 0 \\ \frac{1}{p} & \text{if } p > 0 \\ -\frac{1}{p} & \text{if } p < 0 \end{cases}$$

for any $p \in \mathbb{T}$.

Let p be a prime number. For any non-zero rational number $u = p^s \frac{\ell}{k}$ in which ℓ and k are coprime to the prime number p. Consider the p-adic absolute value $|u|_p = p^{-s}$ on $\mathbb{Q}$. It is easy to check that $|\cdot|$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|$ which is denoted by $\mathbb{Q}_p$ is said to be the p-adic number field. Note that if $p > 2$, then $\left|2^k\right| = 1$ for all integers k.

Throughout this paper, let us presume that $\mathbb{G}$ and $\mathbb{H}$ is a non-Archimedean field and a complete non-Archimedean field, respectively. Hereafter, we use the notation $\mathbb{H}^* = \mathbb{H}\backslash\{0\}$. For the purpose of simplification, let us define the difference operators $\Gamma_1 m_r, \Gamma_2 m_r : \mathbb{G}^* \times \mathbb{G}^* \longrightarrow \mathbb{H}$ and $\Gamma_3 m_r : \underbrace{\mathbb{G}^* \times \mathbb{G}^* \times \ldots \mathbb{G}^\star}_{n \quad \text{times}} \longrightarrow \mathbb{H}$ via

$$\Gamma_1 m_r(p,q) = m_r\left(\frac{pq}{p+q}\right) - m_r(p) - m_r(q),$$

$$\Gamma_2 m_r(p,q) = m_r\left(\frac{pq}{\alpha_1 p + \alpha_2 q}\right) + m_r\left(\frac{pq}{\alpha_2 p + \alpha_1 q}\right) - (\alpha_1+\alpha_2)\left[m_r(p) + m_r(q)\right]$$

and

$$\Gamma_3 m_r(p_1, p_2, \ldots, p_n) = m_r\left(\frac{\prod_{\ell=1}^n p_\ell}{\sum_{\ell=1}^n \mu_\ell \left(\prod_{m=1, m\neq\ell}^n p_m\right)}\right) - \sum_{\ell=1}^n \mu_\ell m_r\left(p_\ell\right)$$

for all $p, q, p_i \in \mathbb{G}^*$, $i = 1, 2, ...n$.

3 Classical stabilities of equation (1.1)

In the ensuing theorem and corollaries, we prove the fundamental stabilities of equation (1.1) pertinent to U-H-G, U-H, U-H-T, U-H-J and J. Rassias product-sum stabilities.

Theorem 3.1. Let $\alpha \neq \pm 1$ be fixed, and let $U : \mathbb{G}^* \times \mathbb{G}^* \longrightarrow \infty$ be a mapping such that

$$\lim_{k\to\infty} \left|\frac{1}{2}\right|^{\alpha k} U\left(\frac{p}{2^{\alpha k + \frac{\alpha+1}{2}}}, \frac{q}{2^{\alpha k + \frac{\alpha+1}{2}}}\right) = 0 \tag{3.1}$$

for all $p, q \in \mathbb{G}^*$. Suppose that $m_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ is a mapping satisfying the inequality

$$|\Gamma_1 m_r(p,q)| \leq U(p,q) \tag{3.2}$$

for all $p, q \in \mathbb{G}^*$. Then, a unique multiplicative inverse mapping $M_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ exists and satisfies

$$|m_r(p) - M_r(p)| \leq \sup\left\{\left|\frac{1}{2}\right|^{s\alpha + \frac{\alpha-1}{2}} U\left(\frac{p}{2^{s\alpha+\frac{\alpha+1}{2}}}, \frac{p}{2^{s\alpha+\frac{\alpha+1}{2}}}\right) : s \in \mathbb{N} \cup \{0\}\right\} \tag{3.3}$$

for all $p \in \mathbb{G}^*$.

Proof. First, let us show that there exists a unique multiplicative inverse mappping M_r. For this, let us plug (p,q) into $\left(\frac{p}{2}, \frac{p}{2}\right)$ in (3.2), we obtain

$$\left|m_r(p) - \frac{1}{2^\alpha} m_r\left(\frac{p}{2^\alpha}\right)\right| \leq |2|^{\frac{|\alpha-1|}{2}} U\left(\frac{p}{2^{\frac{\alpha+1}{2}}}, \frac{p}{2^{\frac{\alpha+1}{2}}}\right) \tag{3.4}$$

for all $p \in \mathbb{G}^*$. Now, reinstating p by $\frac{p}{2^{\alpha k}}$ in (3.5) and multiplying by $\left|\frac{1}{2}\right|^{\alpha k}$, we have

$$\left|\frac{1}{2^{\alpha k}} m_r\left(\frac{p}{2^{\alpha k}}\right) - \frac{1}{2^{(k+1)\alpha}} m_r\left(\frac{p}{2^{(k+1)\alpha}}\right)\right| \leq \left|\frac{1}{2}\right|^{\alpha k+\frac{\alpha-1}{2}} U\left(\frac{p}{2^{\alpha k+\frac{\alpha+1}{2}}}, \frac{p}{2^{\alpha k+\frac{\alpha+1}{2}}}\right) \tag{3.5}$$

for all $p \in \mathbb{G}^*$. By the applications of the relations (3.1) and (3.5), we observe that the sequence $\left\{\frac{1}{2^{\alpha k}} m_r\left(\frac{p}{2^{\alpha k}}\right)\right\}$ is Cauchy. By the completeness of $\mathbb{H}$, this sequence converges to a mapping $M_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ defined by

$$M_r(p) = \lim_{k\to\infty} \frac{1}{2^{\alpha k}} m_r\left(\frac{p}{2^{\alpha k}}\right). \tag{3.6}$$

Also, for each $p \in \mathbb{G}^*$ and non-negative integers k, we have

$$\begin{aligned}
&\left|\frac{1}{2^{\alpha k}} m_r\left(\frac{p}{2^{\alpha k}}\right) - m_r(p)\right| \\
&\quad = \left|\sum_{s=0}^{k-1}\left\{\frac{1}{2^{(s+1)\alpha}} m_r\left(\frac{p}{2^{(s+1)\alpha}}\right) - \frac{1}{2^{s\alpha}} c\left(\frac{p}{2^{s\alpha}}\right)\right\}\right| \\
&\quad \leq \sup\left\{\left|\frac{1}{2^{(s+1)\alpha}} m_r\left(\frac{p}{2^{(s+1)\alpha}}\right) - \frac{1}{2^{s\alpha}} m_r\left(\frac{p}{2^{s\alpha}}\right)\right| : 0 \leq s < k\right\} \\
&\quad \leq \sup\left\{\left|\frac{1}{2}\right|^{s\alpha+\frac{\alpha-1}{2}} U\left(\frac{p}{2^{s\alpha+\frac{\alpha+1}{2}}}, \frac{p}{2^{s\alpha+\frac{\alpha+1}{2}}}\right) : 0 \leq s < k\right\}.
\end{aligned} \tag{3.7}$$

Applying (3.6) and letting $k \to \infty$ in the inequality (3.7), we find that the inequality (3.3) holds. Using (3.1), (3.2) and (3.6), for all $p, q \in \mathbb{G}^*$ we have

$$|\Gamma_1 m_r(p,q)| = \lim_{k\to\infty}\left|\frac{1}{2}\right|^{\alpha k}\left|\Gamma_1 m_r\left(\frac{p}{2^{\alpha k}}, \frac{q}{2^{\alpha k}}\right)\right| \leq \lim_{k\to\infty}\left|\frac{1}{2}\right|^{\alpha k} U\left(\frac{p}{2^{\alpha k}}, \frac{q}{2^{\alpha k}}\right) = 0.$$

Thus, the mapping M_r satisfies (1.1) and hence it is multiplicative inverse mapping. In order to prove the uniqueness of M_r, let us consider $M_r' : \mathbb{G}^* \longrightarrow \mathbb{H}$ be another multiplicative inverse mapping satisfying (3.3). Then

$$\begin{aligned}
&|M_r(p) - M_r'(p)| \\
&= \lim_{m\to\infty}\left|\frac{1}{2}\right|^{\alpha m}\left|M_r\left(\frac{p}{2^{\alpha m}}\right) - M_{r'}\left(\frac{p}{2^{\alpha m}}\right)\right| \\
&\leq \lim_{m\to\infty}\left|\frac{1}{2}\right|^{\alpha m}\sup\left\{\left|M_r\left(\frac{p}{2^{\alpha m}}\right) - m_r\left(\frac{p}{2^{\alpha m}}\right)\right|, \left|m_r\left(\frac{p}{2^{\alpha m}}\right) - M_{r'}\left(\frac{p}{2^{\alpha m}}\right)\right|\right\} \\
&\leq \lim_{m\to\infty}\lim_{k\to\infty}\sup\Big\{\sup\Big\{\left|\frac{1}{2}\right|^{(s+m)\alpha+\frac{\alpha-1}{2}} U\left(\frac{p}{2^{(s+m)\alpha+\frac{\alpha+1}{2}}}, \frac{p}{2^{(s+m)\alpha+\frac{\alpha+1}{2}}}\right) : \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad m \leq s \leq k+m\Big\}\Big\} \\
&= 0
\end{aligned}$$

for all $p \in \mathbb{G}^*$, which shows that M_r is unique. This completes the proof. Q.E.D.

The following corollaries are immediate consequences of Theorem 3.1 concerning the stabilities of equation (1.1).

Corollary 3.2. Let $\eta > 0$ be a constant. If $m_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ satisfies

$$|\Gamma_1 m_r(p,q)| \leq \eta$$

for all $p, q \in \mathbb{G}^*$, then a unique multiplicative inverse mapping $M_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ exists, satisfying (1.1) and

$$|m_r(p) - M_r(p)| \leq \eta$$

for all $p \in \mathbb{G}^*$.

Proof. The proof follows by defining $U(p,q) = \eta$ and employing Theorem 3.1. Q.E.D.

Corollary 3.3. Let $\eta \geq 0$ and $\lambda \neq -1$, be fixed constants. If $m_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ satisfies

$$|\Gamma_1 m_r(p,q)| \leq \eta \left(|p|^\lambda + |q|^\lambda \right)$$

for all $p, q \in \mathbb{G}^*$, then a unique multiplicative inverse mapping $M_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ exists, satisfying (1.1) and

$$|m_r(p) - M_r(p)| \leq \begin{cases} \frac{\eta}{|2|^{\lambda-1}} |p|^\lambda, & \lambda > -1 \\ |2|^2 \eta |p|^\lambda, & \lambda < -1 \end{cases}$$

for all $p \in \mathbb{G}^*$.

Proof. The proof is obtained by letting $U(p,q) = \eta \left(|p|^\lambda + |q|^\lambda \right)$ in Theorem 3.1. Q.E.D.

Corollary 3.4. Let $m_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ be a mapping and let there exist real numbers $a, b, \lambda = a + b \neq -1$ and $\eta \geq 0$ such that

$$|\Gamma_1 m_r(p,q)| \leq \eta |p|^a |q|^b$$

for all $p, q \in \mathbb{G}^*$. Then, a unique multiplicative inverse mapping $M_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ exists, satisfying (1.1) and

$$|m_r(p) - M_r(p)| \leq \begin{cases} \frac{\eta}{|2|^\lambda} |p|^\lambda, & \lambda > -1 \\ \eta |2| |p|^\lambda, & \lambda < -1 \end{cases}$$

for all $p \in \mathbb{G}^*$.

Proof. The desired results are achieved by taking $U(p,q) = \eta |p|^a |q|^b$, for all $p, q \in \mathbb{G}^*$ in Theorem 3.1. Q.E.D.

Corollary 3.5. Let $\eta \geq 0$ and $\lambda \neq -1$ be real numbers, and $m_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ be a mapping satisfying the inequality

$$|\Gamma_1 m_r(p,q)| \leq \eta \left(|p|^{\frac{\lambda}{2}} |q|^{\frac{\lambda}{2}} + \left(|p|^{\lambda} + |q|^{\lambda} \right) \right)$$

for all $p, q \in \mathbb{G}^*$. Then, a unique multiplicative inverse mapping $M_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ exists, satisfying (1.1) and

$$|m_r(p) - M_r(p)| \leq \begin{cases} \frac{|3|\eta}{|2|^{\lambda}} |p|^{\lambda}, & \lambda > -1 \\ |3|\eta|2| |p|^{\lambda}, & \lambda < -1 \end{cases}$$

for all $p \in \mathbb{G}^*$.

Proof. The proof follows by selecting $U(p,q) = \eta \left(|p|^{\frac{\lambda}{2}} |q|^{\frac{\lambda}{2}} + \left(|p|^{\lambda} + |q|^{\lambda} \right) \right)$ in Theorem 3.1. Q.E.D.

Inspired by the counter-example in [6], we prove that the equation (1.1) is unstable for $\lambda = -1$ in Corollary 3.3 in the setting of non-zero real numbers. For this, we consider the following function:

$$\nu(p) = \begin{cases} \frac{\beta}{p}, & \text{for } p \in (1, \infty) \\ \beta, & \text{otherwise} \end{cases} \tag{3.8}$$

where $\nu : \mathbb{R}^* \longrightarrow \mathbb{R}$. Let $m_r : \mathbb{R}^* \longrightarrow \mathbb{R}$ be defined by

$$m_r(p) = \sum_{k=0}^{\infty} 2^{-k} \nu(2^{-k} p) \tag{3.9}$$

for all $p \in \mathbb{R}^\star$. Then the function m_r turns into a suitable example to illustrate the equation (1.1) is unstable for $\lambda = -1$ in Corollary 3.3 in the next theorem.

Theorem 3.6. If the function $m_r : \mathbb{R}^* \longrightarrow \mathbb{R}$ defined in (3.8) satisfies the inequality

$$|\Gamma_1 m_r(p,q)| \leq 12\beta \left(|p|^{-1} + |q|^{-1} \right) \tag{3.10}$$

for all $p, q \in \mathbb{R}^\star$, then a multiplicative inverse mapping $M_r : \mathbb{R}^* \longrightarrow \mathbb{R}$ and a constant $C > 0$ do not exist such that

$$|m_r(p) - M_r(p)| \leq C |p|^{-1} \tag{3.11}$$

for all $p \in \mathbb{R}^*$.

Proof. First, let us prove that the mapping m_r satisfies (3.10). Using the definition of m_r, we find $|m_r(p)| = \left|\sum_{k=0}^{\infty} 2^{-k}\nu(2^{-k}p)\right| \leq \sum_{k=0}^{\infty} \frac{\beta}{2^k} = 2\beta$. This implies that m_r is bounded by 2β on $\mathbb{R}$. If $|p|^{-1} + |q|^{-1} \geq 1$, then the left hand side of (3.10) is less than 12β. Now, suppose that $0 < |p|^{-1} + |q|^{-1} < 1$. Hence, there exists a positive integer j such that

$$\frac{1}{2^{j+1}} \leq |p|^{-1} + |q|^{-1} < \frac{1}{2^j}. \tag{3.12}$$

Hence, the relation (3.12) produces $2^j\left(|p|^{-1} + |q|^{-1}\right) < 1$, or equivalently;

$$2^j p^{-1} < 1, 2^j q^{-1} < 1.$$

So, $\frac{p}{2^j} > 1, \frac{q}{2^j} > 1$. From the last inequalities, we observe that

$$\frac{p}{2^{j-1}} > 2 > 1, \frac{q}{2^{j-1}} > 2 > 1$$

and as a consequence, we have

$$\frac{1}{2^{j-1}}(p) > 1, \frac{1}{2^{j-1}}(q) > 1, \frac{1}{2^{j-1}}(pq) > 1, \frac{1}{2^{j-1}}(p+q) > 1.$$

Hence, for every $k = 0, 1, 2, \ldots, j-1$, we obtain

$$\frac{1}{2^k}(p) > 1, \frac{1}{2^k}(q) > 1, \frac{1}{2^k}(pq) > 1, \frac{1}{2^k}(p+q) > 1$$

and $\Gamma_1\nu(2^{-k}p, 2^{-k}q) = 0$ for $k = 0, 1, 2, \ldots, j-1$. Using (3.8) and the definition of m_r, we obtain

$$|\Gamma_1 m_r(p,q)| \leq \sum_{k=j}^{\infty} \frac{\beta}{2^k} + \sum_{k=j}^{\infty} \frac{\beta}{2^k} + \sum_{k=j}^{\infty} \frac{\beta}{27^k} \leq 3\beta \sum_{k=j}^{\infty} \frac{1}{2^k} \leq \frac{12\beta}{2^{k+1}} \leq 12\beta\left(|p|^{-1} + |q|^{-1}\right)$$

for all $p, q \in \mathbb{R}^*$. Hence, the inequality (3.10) holds. Now, we assert that the equation (1.1) is unstable for $\lambda = -1$ in Corollary 3.3. For this, let us consider a multiplicative inverse mapping $m_r : \mathbb{R}^* \to \mathbb{R}$ exists and satisfies (3.11). Then, we have

$$|m_r(p)| \leq (C+1)|p|^{-1}. \tag{3.13}$$

However, we can choose a positive integer m with $m\beta > C+1$. If $p \in \left(1, 2^{m-1}\right)$ then $2^{-k}p \in (1, \infty)$ for all $k = 0, 1, 2, \ldots, m-1$ and thus

$$|m_r(p)| = \sum_{k=0}^{\infty} \frac{\nu(2^{-k}p)}{2^k} \geq \sum_{k=0}^{m-1} \frac{\frac{2^k\beta}{p}}{2^k} = \frac{m\beta}{p} > (C+1)p^{-1}$$

which contradicts (3.13). This concludes that the equation (1.1) is unstable for $\lambda = -1$ in Corollary 3.3. Q.E.D.

4 Fundamental stabilities of equation (1.2)

In the upcoming results, we prove various fundamental stabilities of equation (1.2). The results are obtained similar to the arguments as in the stability results of equation (1.1). Hence, we present only the essential part of the proofs.

Theorem 4.1. Let $\alpha \in \{1, -1\}$ be fixed, and let $U : \mathbb{G}^* \times \mathbb{G}^* \longrightarrow \infty$ be a mapping such that

$$\lim_{k\to\infty} \left|\frac{1}{\gamma}\right|^{\alpha k} U\left(\frac{p}{\gamma^{\alpha k+\frac{\alpha+1}{2}}}, \frac{q}{\gamma^{\alpha k+\frac{\alpha+1}{2}}}\right) = 0 \tag{4.1}$$

for all $p, q \in \mathbb{G}^*$, where $\gamma = \alpha_1 + \alpha_2 \neq 1$. Suppose that $m_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ is a mapping satisfying the inequality

$$|\Gamma_2 m_r(p, q)| \leq U(p, q) \tag{4.2}$$

for all $p, q \in \mathbb{G}^*$. Then, a unique multiplicative inverse mapping $M_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ exists and satisfies

$$|m_r(p) - M_r(p)| \leq \sup\left\{\left|\frac{1}{\gamma}\right|^{s\alpha+\frac{\alpha-1}{2}} U\left(\frac{p}{\gamma^{s\alpha+\frac{\alpha+1}{2}}}, \frac{p}{\gamma^{s\alpha+\frac{\alpha+1}{2}}}\right) : s \in \mathbb{N} \cup \{0\}\right\} \tag{4.3}$$

for all $p \in \mathbb{G}^*$.

Proof. Let us replace (p, q) by $\left(\frac{p}{\gamma}, \frac{p}{\gamma}\right)$ in (4.2), we obtain

$$\left|m_r(p) - \frac{1}{\gamma^\alpha} m_r\left(\frac{p}{\gamma^\alpha}\right)\right| \leq |\gamma|^{\frac{|\alpha-1|}{2}} U\left(\frac{p}{\gamma^{\frac{\alpha+1}{2}}}, \frac{p}{\gamma^{\frac{\alpha+1}{2}}}\right) \tag{4.4}$$

for all $p \in \mathbb{G}^*$. Now, reinstating p by $\frac{p}{\gamma^{\alpha k}}$ in (4.4) and multiplying by $\left|\frac{1}{\gamma}\right|^{\alpha k}$, we have

$$\begin{aligned}\left|\frac{1}{\gamma^{\alpha k}} m_r\left(\frac{p}{\gamma^{\alpha k}}\right) - \frac{1}{\gamma^{(k+1)\alpha}} m_r\left(\frac{p}{\gamma^{(k+1)\alpha}}\right)\right| \\ \leq \left|\frac{1}{\gamma}\right|^{\alpha k+\frac{\alpha-1}{2}} U\left(\frac{p}{\gamma^{\alpha k+\frac{\alpha+1}{2}}}, \frac{p}{\gamma^{\alpha k+\frac{\alpha+1}{2}}}\right)\end{aligned} \tag{4.5}$$

for all $p \in \mathbb{G}^*$. In lieu of (4.1) and (4.5), the sequence $\left\{\frac{1}{\gamma^{\alpha k}} m_r\left(\frac{p}{\gamma^{\alpha k}}\right)\right\}$ is Cauchy. Since $\mathbb{H}$ is complete, this sequence converges to a mapping $M_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ defined by $M_r(p) = \lim_{k\to\infty} \frac{1}{\gamma^{\alpha k}} m_r\left(\frac{p}{\gamma^{\alpha k}}\right)$. The enduring part of the proof is akin to Theorem 3.1. Q.E.D.

The following corollaries are immediate consequence of Theorem 3.1 concerning various other stabilities of (1.2). We present only the statements of cororllaries since the proofs are similar to the results of equations (1.1).

Corollary 4.2. Let $\eta > 0$ be a constant. If $m_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ satisfies $|\Gamma_2 m_r(p,q)| \leq \eta$ for all $p, q \in \mathbb{G}^*$, then a unique multiplicative inverse mapping $M_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ exists satisfying *(1.2)* and $|m_r(p) - M_r(p)| \leq \eta$ for all $p \in \mathbb{G}^*$.

Corollary 4.3. Let $\eta \geq 0$ and $\lambda \neq -1$, be fixed constants. If $m_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ satisfies $|\Gamma_2 m_r(p,q)| \leq \eta \left(|p|^\lambda + |q|^\lambda\right)$ for all $p, q \in \mathbb{G}^*$, then there exists a unique multiplicative inverse mapping $M_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ satisfying (1.2) and

$$|m_r(p) - M_r(p)| \leq \begin{cases} \frac{\eta}{|\gamma|^{\lambda-1}} |p|^\lambda, & \lambda > -1 \\ |2||\gamma|\eta\, |p|^\lambda, & \lambda < -1 \end{cases}$$

for all $p \in \mathbb{G}^*$.

Corollary 4.4. Let $m_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ be a mapping and let there exist real numbers $a, b, \lambda = a + b \neq -1$ and $\eta \geq 0$ such that $|\Gamma_2 m_r(p,q)| \leq \eta\, |p|^a\, |q|^b$ for all $p, q \in \mathbb{G}^*$. Then, there exists a unique multiplicative inverse mapping $M_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ satisfying *(1.2)* and

$$|m_r(p) - M_r(p)| \leq \begin{cases} \frac{\eta}{|\gamma|^\lambda} |p|^\lambda, & \lambda > -1 \\ \eta|\gamma|\, |p|^\lambda, & \lambda < -1 \end{cases}$$

for all $p \in \mathbb{G}^*$.

Corollary 4.5. Let $\eta \geq 0$ and $\lambda \neq -1$ be real numbers, and $m_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ be a mapping satisfying the inequality

$$|\Gamma_2 m_r(p,q)| \leq \eta \left(|p|^{\frac{\lambda}{2}} |q|^{\frac{\lambda}{2}} + \left(|p|^\lambda + |q|^\lambda\right)\right)$$

for all $p, q \in \mathbb{G}^*$. Then, a unique multiplicative inverse mapping $M_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ exists satisfying (1.2) and

$$|m_r(p) - M_r(p)| \leq \begin{cases} \frac{|3|\eta}{|\gamma|^\lambda} |p|^\lambda, & \lambda > -1 \\ |3|\eta|\gamma|\, |p|^\lambda, & \lambda < -1 \end{cases}$$

for all $p \in \mathbb{G}^*$.

The function ν defined in the following is used to disprove the stability of equation (1.2) for the critical case $\lambda = -1$.

$$\nu(p) = \begin{cases} \frac{\beta}{p}, & \text{for } p \in (1, \infty) \\ \beta, & \text{otherwise} \end{cases} \tag{4.6}$$

where $\nu : \mathbb{R}^* \longrightarrow \mathbb{R}$. Let $m_r : \mathbb{R}^* \longrightarrow \mathbb{R}$ be defined by

$$m_r(p) = \sum_{k=0}^{\infty} \gamma^{-k} \nu(\gamma^{-k} p) \tag{4.7}$$

for all $p \in \mathbb{R}^\star$. In the following theorem, the function m_r acts as an example to show that the equation (1.2) is unstable for $\lambda = -1$ in Corollary 4.3.

Theorem 4.6. If the function $m_r : \mathbb{R}^* \longrightarrow \mathbb{R}$ defined in (4.7) satisfies the inequality

$$|\Gamma_2 m_r(p,q)| \leq (\gamma+2)\beta\gamma^2\beta \left(|p|^{-1} + |q|^{-1} \right)$$

for all $p, q \in \mathbb{R}^\star$, then a multiplicative inverse mapping $M_r : \mathbb{R}^* \longrightarrow \mathbb{R}$ and a constant $C > 0$ do not exist such that

$$|m_r(p) - M_r(p)| \leq C\,|p|^{-1}$$

for all $p \in \mathbb{R}^*$.

Proof. The proof is obtained similar to Theorem 3.6. Q.E.D.

5 Various stabilities of equation (1.3)

In the ensuing outcomes, we establish various stabilities of equation (1.3). Since the arguments for proving the main results are similar to those of equation (1.1), we furnish below only the principal part of the results.

Theorem 5.1. Let $\alpha \in \{1, -1\}$ be fixed, and let $U : \underbrace{\mathbb{G}^* \times \mathbb{G}^* \times \cdots \times \mathbb{G}^*}_{n \text{ times}} \longrightarrow \infty$ be a mapping such that

$$\lim_{k\to\infty} \left|\frac{1}{\mu}\right|^{\alpha k} U\left(\frac{p_1}{\mu^{\alpha k + \frac{\alpha+1}{2}}}, \frac{p_2}{\mu^{\alpha k + \frac{\alpha+1}{2}}}, \ldots, \frac{p_n}{\mu^{\alpha k + \frac{\alpha+1}{2}}} \right) = 0 \tag{5.1}$$

for all $p_\ell \in \mathbb{G}^*, \ell = 1, 2, \ldots, n$, where $\mu = \sum_{\ell=1}^{n} \mu_\ell \neq 1$. Suppose that $m_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ is a mapping satisfying the inequality

$$|\Gamma_3 m_r(p_1, p_2, \ldots, p_n)| \leq U(p_1, p_2, \ldots, p_n) \tag{5.2}$$

for all $p_\ell \in \mathbb{G}^*, \ell = 1, 2, \ldots, n$. Then, a unique multiplicative inverse mapping $M_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ exists and satisfies

$$\begin{aligned} &|m_r(p) - M_r(p)| \\ &\quad \leq \sup\left\{ \left|\frac{1}{\mu}\right|^{s\alpha + \frac{\alpha-1}{2}} U\left(\frac{p}{\mu^{s\alpha + \frac{\alpha+1}{2}}}, \frac{p}{\mu^{s\alpha + \frac{\alpha+1}{2}}}, \ldots, \frac{p}{\mu^{s\alpha + \frac{\alpha+1}{2}}} \right) : s \in \mathbb{N} \cup \{0\} \right\} \end{aligned} \tag{5.3}$$

for all $p \in \mathbb{G}^*$.

Proof. First, let us change $(p_1, p_2, \ldots, p_n)$ into $\left(\frac{p}{\mu}, \frac{p}{\mu}, \ldots, \frac{p}{\mu}\right)$ in (5.2), we obtain

$$\left| m_r(p) - \frac{1}{\mu^{\alpha}} m_r\left(\frac{p}{\mu^{\alpha}}\right) \right| \leq |\mu|^{\frac{|\alpha-1|}{2}} U\left(\frac{p}{\mu^{\frac{\alpha+1}{2}}}, \frac{p}{\mu^{\frac{\alpha+1}{2}}}, \ldots, \frac{p}{\mu^{\frac{\alpha+1}{2}}}\right) \tag{5.4}$$

for all $p \in \mathbb{G}^*$. Now, replacing p by $\frac{p}{\mu^{\alpha k}}$ in (5.4) and multiplying by $\left|\frac{1}{\mu}\right|^{\alpha k}$, we have

$$\begin{aligned} &\left| \frac{1}{\mu^{\alpha k}} m_r\left(\frac{p}{\mu^{\alpha k}}\right) - \frac{1}{\mu^{(k+1)\alpha}} m_r\left(\frac{p}{\mu^{(k+1)\alpha}}\right) \right| \\ &\qquad \leq \left|\frac{1}{\mu}\right|^{\alpha k + \frac{\alpha-1}{2}} U\left(\frac{p}{\mu^{\alpha k + \frac{\alpha+1}{2}}}, \frac{p}{\mu^{\alpha k + \frac{\alpha+1}{2}}}, \ldots, \frac{p}{\mu^{\alpha k + \frac{\alpha+1}{2}}}\right) \end{aligned} \tag{5.5}$$

for all $p \in \mathbb{G}^*$. By the virtue of (5.1) and (5.5), the sequence $\left\{\frac{1}{\mu^{\alpha k}} m_r\left(\frac{p}{\mu^{\alpha k}}\right)\right\}$ turns out to be Cauchy. Since $\mathbb{H}$ is complete, this sequence converges to a mapping $M_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ defined by $M_r(p) = \lim_{k\to\infty} \frac{1}{\mu^{\alpha k}} m_r\left(\frac{p}{\mu^{\alpha k}}\right)$. The remaining part can be proved with similar lines of Theorem 3.1. Q.E.D.

Corollary 5.2. Let $\eta > 0$ be a constant. If a mapping $m_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ satisfies $|\Gamma_3 m_r(p_1, p_2, \ldots, p_n)| \leq \eta$ for all $p_\ell \in \mathbb{G}^*, \ell = 1, 2, \ldots, n$, then a unique multiplicative inverse mapping $M_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ exists satisfying (1.3) and $|m_r(p) - M_r(p)| \leq \eta$ for all $p \in \mathbb{G}^*$.

Corollary 5.3. Let $\eta \geq 0$ and $\lambda \neq -1$, be fixed constants. If $m_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ satisfies $|\Gamma_3 m_r(p_1, p_2, \ldots, p_n)| \leq \eta \sum_{\ell=1}^{n} |p_\ell|^{\lambda}$ for all $p_\ell \in \mathbb{G}^*, \ell = 1, 2, \ldots, n$, then there exists a unique multiplicative inverse mapping $M_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ satisfying (1.3) and

$$|m_r(p) - M_r(p)| \leq \begin{cases} \frac{\eta}{|\mu|^{\lambda-1}} |p|^{\lambda}, & \lambda > -1 \\ |n||\mu|\eta\, |p|^{\lambda}, & \lambda < -1 \end{cases}$$

for all $p \in \mathbb{G}^*$.

Corollary 5.4. Let $m_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ be a mapping and let there exist real numbers $a_1, a_1, \ldots, a_n, \lambda = \sum_{\ell=1}^{n} a_\ell \neq -1$ and $\eta \geq 0$ such that

$$|\Gamma_3 m_r(p_1, p_2, \ldots, p_n)| \leq \eta \prod_{\ell=1}^{n} |p_\ell|^{a_\ell}$$

for all $p_\ell \in \mathbb{G}^*, \ell = 1, 2, \ldots, n$. Then, a unique multiplicative inverse mapping $M_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ exists, satisfying *(1.3)* and

$$|m_r(p) - M_r(p)| \leq \begin{cases} \frac{\eta}{|\mu|^{\lambda}} |p|^{\lambda}, & \lambda > -1 \\ \eta|\mu|\, |p|^{\lambda}, & \lambda < -1 \end{cases}$$

for all $p \in \mathbb{G}^*$.

Corollary 5.5. Let $\eta \geq 0$ and $\lambda \neq -1$ be real numbers, and $m_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ be a mapping satisfying the inequality

$$|\Gamma_3 m_r(p_1, p_2, \ldots, p_n)| \leq \eta \left(\prod_{\ell=1}^{n} |p_\ell|^{\frac{\lambda}{\ell}} + \left(\sum_{\ell=1}^{n} |p_\ell|^{\lambda} \right) \right)$$

for all $p_\ell \in \mathbb{G}^*, \ell = 1, 2, \ldots, n$. Then, a unique multiplicative inverse mapping $M_r : \mathbb{G}^* \longrightarrow \mathbb{H}$ exists, satisfying (1.3) and

$$|m_r(p) - M_r(p)| \leq \begin{cases} \frac{|n+1|\eta}{|\mu|^{\lambda}} |p|^{\lambda}, & \lambda > -1 \\ |n+1|\eta|\mu| \, |p|^{\lambda}, & \lambda < -1 \end{cases}$$

for all $p \in \mathbb{G}^*$.

The function ν defined below is used to invalidate the stability result of equation (1.3) for the critical case $\lambda = -1$.

$$\nu(p) = \begin{cases} \frac{\beta}{p}, & \text{for } p \in (1, \infty) \\ \beta, & \text{otherwise} \end{cases} \tag{5.6}$$

where $\nu : \mathbb{R}^* \longrightarrow \mathbb{R}$. Let $m_r : \mathbb{R}^* \longrightarrow \mathbb{R}$ be defined by

$$m_r(p) = \sum_{k=0}^{\infty} \mu^{-k} \nu(\mu^{-k} p) \tag{5.7}$$

for all $p \in \mathbb{G}^\star$. In the following theorem, the function m_r acts as an example to show that the equation (1.3) is unstable for $\lambda = -1$ in Corollary 5.3.

Theorem 5.6. If the function $m_r : \mathbb{R}^* \longrightarrow \mathbb{R}$ defined in (5.7) satisfies the inequality

$$|\Gamma_3 m_r(p_1, p_2, \ldots, p_n)| \leq (n+1)\beta\mu \left(\sum_{\ell=1}^{n} |p_\ell|^{-1} \right) \tag{5.8}$$

for all $p_\ell \in \mathbb{R}^\star, \ell = 1, 2, \ldots, n$, then a multiplicative inverse mapping $M_r : \mathbb{R}^* \longrightarrow \mathbb{R}$ and a constant $C > 0$ do not exist such that

$$|m_r(p) - M_r(p)| \leq C \, |p|^{-1} \tag{5.9}$$

for all $p \in \mathbb{R}^*$.

6 Conclusion

There are many multiplicative inverse functional equations consisting of rational expressions of functions only, dealt in this field of study to prove their stabilities. This is the first attempt that

we have considered functional equations involving functions with rational expressions as arguments to investigate their stability results. There are many rational functions which play significant role in various fields such as physics, engineering and medicine. Hence the equations dealt in this study could be modelled in many physical problems. Also, we infer that the stability results of this type of multiplicative inverse functional equations are still valid in non-Archimedean fields.

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