Sufficient conditions for infinite series by absolute φ-product summable factor

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Abstract

Absolute Cesàro-Riesz summability method has been introduced and later on used to established a theorem containing least set of sufficient conditions for absolute φ-Cesàro-Riesz summable factor of an infinite series. The result has also been validated with the published literature on $\varphi - |\bar{N}, p_n|_k$ summable factor by reducing certain conditions in the presented result.

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1 Introduction

Let $\{s_n\}$ be the sequence of partial sums of the series $\sum_{n=0}^{\infty} a_n$. For any infinite matrix $T = (t_{nk})$, the n^{th} term of the matrix transform of a sequence is given by

$$t_n = \sum_{k=0}^{\infty} t_{nk} s_k. \tag{1.1}$$

A sequence of partial sums $\{s_n\} = \{s_0, s_1, ..., s_n\}$ is said to be of bounded variation if the series $|s_1 - s_0| + |s_2 - s_1| + ... + |s_n - s_{n-1}|$ converges; or equivalently,

$$\sum_n |\Delta s_n| < \infty. \tag{1.2}$$

Definition 1. If the sequence of means $\{t_n\}$ satisfies the conditions

$$\lim_{n\to\infty} t_n = s, \tag{1.3}$$

and

$$\sum_{n=1}^{\infty} |\, t_n - t_{n-1}| < \infty,$$

then the series $\sum_{n=0}^{\infty} a_n$ is said to be absolute summable by T.

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Definition 2. [1] The n^{th} $(C,1)$ mean of a sequence $\{s_n\}$ is given by t_n, where

$$t_n = \frac{1}{(n+1)} \sum_{k=0}^{n} s_k.$$

A series $\sum_{n=0}^{\infty} a_n$ is said to be $|C,1|_k$ summable for $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |t_n - t_{n-1}|^k < \infty. \tag{1.4}$$

Definition 3. Let $\{\varphi_n\}$ be a sequence of positive real number. If the sequence $\{t_n\}$ satisfies

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} |t_n - t_{n-1}|^k < \infty, \tag{1.5}$$

then the series $\sum_{n=0}^{\infty} a_n$ is said to be $\varphi - |C,\, 1|_k$ summable.

Definition 4. The n^{th} $(\bar{N}, p_n)$ mean of a sequence $\{s_n\}$, denoted by $t_n^{\bar{N}}$, is given by

$$t_n^{\bar{N}} = \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k.$$

A series $\sum_{n=0}^{\infty} a_n$ is said to be $\varphi - |\bar{N}, p_n|_k$ summable for $k \geq 1$, if

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} |\Delta t_n^{\bar{N}}|^k < \infty. \tag{1.6}$$

If $\varphi_n = n$, then $\varphi_n - |\bar{N}, p_n|_k$ summability reduces to $|\bar{N}, p_n|_k$ summability.

Definition 5. The n^{th} $C_1\bar{N}_p$ mean of a sequence $\{s_n\}$ is denoted by $t_n^{C\bar{N}}$, where

$$\begin{aligned} t_n^{C\bar{N}} &= \frac{1}{(n+1)} \sum_{k=0}^{n} t_k^{\bar{N}} \\ &= \frac{1}{(n+1)} \sum_{k=0}^{n} \frac{1}{P_k} \sum_{i=0}^{k} p_i s_i \\ &= \frac{1}{(n+1)} \sum_{k=0}^{n} \frac{1}{P_k} \left[\sum_{i=0}^{k} (P_k - P_{i-1}) a_i \right]. \end{aligned} \tag{1.7}$$

A series $\sum_{n=0}^{\infty} a_n$ is said to be $\varphi - |C_1\bar{N}_p|_k$ summable for $k \geq 1$, if

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} |\Delta t_n^{C\bar{N}}|^k < \infty. \tag{1.8}$$

In 1890 Cesáro [2] defined a new class of methods, known as the Cesáro summability methods, to deal with non-converging series. After his pioneer work, summability became a topic of interest for many mathematicians. Bor obtained several theorems [3, 4, 5] on absolute summability, using a generalization of the results of Mazhar [6]. In 1993 and 1994, Bor [7, 8] utilized $|\bar{N}, p_n|_k$ summability for the enhancement of some of the results of Mazhar [6] and applied it to Fourier series.

In 1980, Balci [9] defined absolute φ-summability factors and obtained very interesting results. Özarslan generalized the result of Bor [10] by using the more general absolute summability $\varphi - |C, \alpha|_k$. In 2016, Sonker and Munjal [11] determined sufficient conditions for an infinite series to be generalized absolute Cesáro summable.

To the best of the authors' knowledge, only a few mathematicians [12, 13, 14] have studied absolute product summability, and no work has been done to determine sufficient conditions for an infinite series to be generalized absolute product $\varphi - |C_1\bar{N}_p|_k$, $k \geq 1$ summable. In this paper, a study has been done in this direction.

Dikshit [12] established theorems concerning the absolute $(C, 1)(N, p_n)$ sum- mable factor of a Fourier series and its conjugate series. These theorems include previous results [15, 16] on the $|C|$-summability of Fourier series and its conjugate series as special cases. In 1970, Chandra [17, 18] wrote two papers on absolute Riesz summability. In [17] he established the absolute Riesz summability of Lebesgue-Fourier series by weakening the conditions imposed upon the generating function of Lebesgue-Fourier series and by using a more general type of Riesz means. In [18] he obtained theorems on the absoute Riesz summability of Fourier series, its conjugate series and their derived series. After his work on Riesz summability, Chandra [13] worked with Jain on absolute product $|(R, exp(logw)^{\Delta}, \alpha)(C, 1)|(\Delta > 0, \alpha > 0)$ summability of the Fourier series, its factored conjugate series and their derived series. It has also been verified that the sequence of factors $1/log(n+1)$, used for the conjugate series and its derived series cannot be dropped. Sulaiman [14] has applied absolute product $|(R, q_n)(R, p_n)|_k$ summability on infinite series to established the theorems.

2 Known results

Using $|C, 1|_k$ summability, Mazhar [6] proved the following theorem involving a minimal set of sufficient conditions for an infinite series to be absolute Cesáro summable.

Theorem 2.1. [6] If

$$\lambda_m = O(1),\ m \to \infty, \tag{2.1}$$

$$\sum_{n=1}^{m} n\ \log n\, |\Delta^2 \lambda_n| = O(1),\ m \to \infty, \tag{2.2}$$

and

$$\sum_{v=1}^{m} \frac{|t_v|^k}{v} = O(\log m),\ \ as\ m \to \infty, \tag{2.3}$$

then the series $\sum a_n\lambda_n$ is summable $|C,1|_k,\ k\geq 1$.
Özarslan [19] proved the following theorem dealing with the $\varphi-|C,1|_k$ summability factors of infinite series.

Theorem 2.2. [19] Let $\{\varphi_n\}$ be a sequence of positive real numbers for which the following conditions are satisfied:

$$\lambda_m = O(1),\ m\to\infty, \tag{2.4}$$

$$\sum_{n=1}^{m} n\ \log n\,|\Delta^2\lambda_n| = O(1),\ m\to\infty, \tag{2.5}$$

$$\sum_{v=1}^{m}\frac{\varphi_v^{k-1}}{v^k}|t_v|^k = O(\log m),\ \ as\ m\to\infty, \tag{2.6}$$

and

$$\sum_{v=1}^{m}\frac{\varphi_n^{k-1}}{n^k} = O(\frac{\varphi_v^{k-1}}{v^{k-1}}). \tag{2.7}$$

Then the series $\sum a_n\lambda_n$ is summable $\varphi-|C,1|_k,\ k\geq 1$.

3 Main results

The purpose of this study is to determine sufficient conditions for an infinite series to be generalized absolute Cesáro-Riesz $\varphi-|C_1\bar{N}_p|_k$ summable.

Theorem 3.1. Let $\{\chi_n\}$ be a positive non-decreasing real sequence satisfying the conditions,

$$\lambda_m = O(1),\ m\to\infty, \tag{3.1}$$

$$\beta_{n+1} = O(\beta_n), \tag{3.2}$$

$$\Delta\beta_n = O(n^{-1}\beta_n),\ \ as\ n\to\infty, \tag{3.3}$$

$$\sum_{n=1}^{\infty} n\,\chi_n|\Delta^2\lambda_n| = O(1), \tag{3.4}$$

$$\sum_{n=1}^{m}\frac{\varphi_n^{k-1}|\beta_n|^k|s_n|^k}{n^k\chi_n^{k-1}} = O(\chi_m),\ m\to\infty, \tag{3.5}$$

$$np_n = O(P_n),\ P_n = O(np_n),\ as\ n\to\infty, \tag{3.6}$$

$$\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k}|t_{C\bar{N}}|^k = O(1),\ m\to\infty, \tag{3.7}$$

$$\sum_{n=v}^{m}\frac{\varphi_n^{k-1}}{n^kP_{n-1}} = O\left(\frac{\varphi_v^{k-1}}{v^{k-1}P_v}\right),\ m\to\infty, \tag{3.8}$$

$$\sum_{n=v+1}^{m}\frac{\varphi_n^{k-1}}{n^k} = O\left(\frac{\varphi_v^{k-1}}{v^{k-1}}\right),\ m\to\infty, \tag{3.9}$$

and

$$\sum_{n=v+1}^{m} \frac{\varphi_n^{k-1}}{n} = O(\varphi_v^{k-1}), \ \ as \ \ m \to \infty. \tag{3.10}$$

Then the series $\sum a_n\lambda_n\beta_n$ is $\varphi - |C_1\bar{N}_p|_k$ summable for $k \geq 1$.

4 Lemma

In order to prove theorem 3.1, the following Lemma [20] is essential.

Lemma 4.1. [20] Conditions (3.1) and (3.4) imply that

$$\sum_{n=1}^{\infty} \chi_n|\Delta\lambda_n| = O(1), \tag{4.1}$$

$$n\chi_n|\Delta\lambda_n| = O(1), \, as \, n \to \infty, \tag{4.2}$$

and

$$\chi_n|\lambda_n| = O(1), \, as \, n \to \infty. \tag{4.3}$$

5 Proof of the Theorem

The $C_1\bar{N}_p$ transform mean $t_n^{C\bar{N}}$, or T_n, of the series $\sum\limits_{n=1}^{\infty} a_n\lambda_n\beta_n$, is given by

$$\begin{aligned} T_n = t_n^{C\bar{N}}(f) &= \frac{1}{(n+1)} \sum_{k=0}^{n} t_k^{\bar{N}} = \frac{1}{(n+1)} \sum_{k=0}^{n} \frac{1}{P_k} \sum_{v=0}^{k} p_v s_v \\ &= \frac{1}{(n+1)} \sum_{k=0}^{n} \frac{1}{P_k} \sum_{v=0}^{k} p_v \sum_{i=0}^{v} a_i\lambda_i\beta_i \\ &= \frac{1}{(n+1)} \sum_{k=0}^{n} \frac{1}{P_k} \sum_{v=0}^{k} (P_k - P_{v-1}) a_v\lambda_v\beta_v. \end{aligned} \tag{5.1}$$

Using Abel's transformation we have, with

$\Delta T_n := T_n - T_{n-1}$(normally Δ denotes the forward difference, not the backward difference),

$$\begin{aligned}
\Delta T_n &= \frac{1}{(n+1)}\sum_{k=0}^{n}\frac{1}{P_k}\sum_{v=1}^{k}(P_k - P_{v-1})a_v\lambda_v\beta_v - \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{P_k}\sum_{v=1}^{k}(P_k - P_{v-1})a_v\lambda_v\beta_v \\
&= \frac{1}{(n+1)}\frac{1}{P_n}\sum_{v=1}^{n}(P_n - P_{v-1})a_v\lambda_v\beta_v + \frac{1}{(n+1)}\sum_{k=0}^{n-1}\frac{1}{P_k}\sum_{v=1}^{k}(P_k - P_{v-1})a_v\lambda_v\beta_v \\
&\quad -\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{P_k}\sum_{v=1}^{k}(P_k - P_{v-1})a_v\lambda_v\beta_v \\
&= \frac{1}{(n+1)}\frac{1}{P_n}\sum_{v=1}^{n}(P_n - P_{v-1})a_v\lambda_v\beta_v - \frac{1}{n(n+1)}\sum_{k=0}^{n-1}\frac{1}{P_k}\sum_{v=1}^{k}(P_k - P_{v-1})a_v\lambda_v\beta_v \\
&= I_{n,1} + I_{n,2}. \qquad (5.2)
\end{aligned}$$

In order to prove Theorem 3.1, it is sufficient to show that

$$\sum_{n=1}^{m}\varphi_n^{k-1}|\Delta T_n|^k < \infty, \qquad (5.3)$$

Using Minkowski's inequality, $|\Delta T_n|^k \leq 2^k\Big(|I_{n,1}|^k + |I_{n,2}|^k\Big)$. Since k is finite, it is sufficient to show

$$\sum_{n=1}^{m}\varphi_n^{k-1}|I_{n,r}|^k < \infty, \ for\ r = 1, 2. \qquad (5.4)$$

$$\begin{aligned}
\sum_{n=1}^{m}\varphi_n^{k-1}|I_{n,1}|^k &= 2^k\left[\sum_{n=1}^{m}\varphi_n^{k-1}\left|\frac{1}{(n+1)}\sum_{v=1}^{n}a_v\lambda_v\beta_v\right|^k + \sum_{n=1}^{m}\varphi_n^{k-1}\left|\frac{1}{(n+1)P_n}\sum_{v=1}^{n}P_{v-1}a_v\lambda_v\beta_v\right|^k\right] \\
&= 2^k(L_1 + L_2). \qquad (5.5)
\end{aligned}$$

Since k is finite, it is sufficient to show that L_1 and L_2 are bounded. By using Abel's transformation and Minkowski's inequality, we have

$$\begin{aligned}
L_1 &= \sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{(n+1)^k}\left|\sum_{v=1}^{n}a_v\lambda_v\beta_v\right|^k \\
&= \sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{(n+1)^k}\left|\sum_{v=1}^{n-1}s_v\Delta(\lambda_v\beta_v)+\lambda_n\beta_n s_n\right|^k \\
&= \sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{(n+1)^k}\left|\sum_{v=1}^{n-1}s_v\lambda_v\Delta\beta_v+\sum_{v=1}^{n-1}s_v\Delta\lambda_v\beta_{v+1}+\lambda_n\beta_n s_n\right|^k \\
&\leq 3^k\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{(n+1)^k}\left|\sum_{v=1}^{n-1}s_v\lambda_v\Delta\beta_v\right|^k+3^k\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{(n+1)^k}\left|\sum_{v=1}^{n-1}s_v\Delta\lambda_v\beta_{v+1}\right|^k \\
&\quad +3^k\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k}|\lambda_n\beta_n s_n|^k \\
&= O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{(n+1)^k}\left|\sum_{v=1}^{n-1}s_v\lambda_v\Delta\beta_v\right|^k+O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{(n+1)^k}\left|\sum_{v=1}^{n-1}s_v\Delta\lambda_v\beta_{v+1}\right|^k \\
&\quad +O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k}|\lambda_n\beta_n s_n|^k \\
&= I_1+I_2+I_3. \qquad (5.6)
\end{aligned}$$

Using Hölder's inequality, conditions (3.3), (3.10), (3.1), (4.3) of Lemma 4.1 and (3.5),

$$\begin{aligned}
I_1 &= O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k}\left|\sum_{v=1}^{n-1}s_v\lambda_v\Delta\beta_v\right|^k \leq O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k}\left(\sum_{v=1}^{n-1}v^{-1}|s_v||\lambda_v||\beta_v|\right)^k \\
&= O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n}\sum_{v=1}^{n-1}v^{-k}|s_v|^k|\lambda_v|^k|\beta_v|^k\left(\frac{1}{n}\sum_{v=1}^{n-1}1\right)^{k-1}=O(1)\sum_{v=1}^{m}\frac{|\lambda_v|^k}{v^k}|s_v|^k|\beta_v|^k\sum_{n=v}^{m}\frac{\varphi_n^{k-1}}{n} \\
&= O(1)\sum_{v=1}^{m}\frac{\varphi_v^{k-1}}{v^k}|\lambda_v|^{k-1}|s_v|^k|\beta_v|^k|\lambda_v|=O(1)\sum_{v=1}^{m}\frac{\varphi_v^{k-1}}{v^k\chi_v^{k-1}}|s_v|^k|\beta_v|^k\sum_{n=v}^{\infty}\Delta|\lambda_n| \\
&= O(1)\sum_{v=1}^{m}\frac{\varphi_v^{k-1}}{v^k\chi_v^{k-1}}|s_v|^k|\beta_v|^k\sum_{n=v}^{\infty}|\Delta\lambda_n|=O(1)\sum_{n=1}^{\infty}|\Delta\lambda_n|\sum_{v=1}^{n}\frac{\varphi_v^{k-1}}{v^k\chi_v^{k-1}}|s_v|^k|\beta_v|^k \\
&= O(1)\sum_{n=1}^{\infty}\chi_n|\Delta\lambda_n|=O(1). \qquad (5.7)
\end{aligned}$$

Using Hölder's inequality, Abel's transformation, conditions (4.1), (3.9), (3.5) (3.4) and (4.2) of Lemma 4.1,

$$\begin{aligned}
I_2 &= O(1)\sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^k}\left|\sum_{v=1}^{n-1}s_v\beta_{v+1}\Delta\lambda_v\right|^k \\
&= O(1)\sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^k}\sum_{v=1}^{n-1}\frac{|\beta_v|^k}{\chi_v^{k-1}}|s_v|^k|\Delta\lambda_v|\left(\sum_{v=1}^{n-1}\chi_v|\Delta\lambda_v|\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m-1}\frac{|\beta_v|^k}{\chi_v^{k-1}}|s_v|^k|\Delta\lambda_v|\sum_{n=v+1}^{m}\frac{\varphi_n^{k-1}}{n^k} = O(1)\sum_{v=1}^{m}v|\Delta\lambda_v|\frac{\varphi_v^{k-1}|\beta_v|^k|s_v|^k}{v^k\chi_v^{k-1}} \\
&= O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\lambda_v|)\sum_{r=1}^{v}\frac{\varphi_r^{k-1}|\beta_r|^k|s_r|^k}{r^k\chi_r^{k-1}} + O(1)m|\Delta\lambda_m|\sum_{v=1}^{m}\frac{\varphi_v^{k-1}|\beta_v|^k|s_v|^k}{v^k\chi_v^{k-1}} \\
&= O(1)\sum_{v=1}^{m-1}v|\Delta^2\lambda_v|\chi_v + O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|\chi_v + O(1)m|\Delta\lambda_m|\chi_m = O(1). \qquad (5.8)
\end{aligned}$$

Using conditions (3.1), (3.5) and (4.1) of Lemma 4.1 (See I_1),

$$\begin{aligned}
I_3 &= O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k}|s_n|^k|\beta_n|^k|\lambda_n|^k \\
&= O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k}|s_n|^k|\beta_n|^k|\lambda_n|^{k-1}|\lambda_n| = O(1). \qquad (5.9)
\end{aligned}$$

Applying Abel's transformation and Minkowski inequality, we have

$$\begin{aligned}
L_2 &= \sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^kP_n^k}\left|\sum_{v=1}^{n}a_vP_{v-1}\lambda_v\beta_v\right|^k \\
&= \sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^kP_n^k}\left|P_{n-1}\lambda_n\beta_ns_n + \sum_{v=1}^{n-1}s_v\Delta(P_{v-1}\lambda_v\beta_v)\right|^k \\
&= O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^kP_n^k}\left|P_{n-1}\lambda_n\beta_ns_n\right|^k \\
&\quad + O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^kP_n^k}\left|\sum_{v=1}^{n-1}\big(-p_vs_v\lambda_v\beta_v + P_vs_v\lambda_v\Delta\beta_v + P_v\Delta\lambda_v\beta_{v+1}s_v\big)\right|^k \\
&= M_1 + M_2. \qquad (5.10)
\end{aligned}$$

Using conditions (3.1), (3.5) and (4.1) of Lemma 4.1 (See I_1),

$$\begin{aligned} M_1 &= \sum_{n=1}^{m} \frac{\varphi_n^{k-1}}{n^k P_n^k} \left| P_{n-1}\lambda_n\beta_n s_n \right|^k \\ &= O(1)\sum_{n=1}^{m} \frac{\varphi_n^{k-1}}{n^k P_n^k} |P_{n-1}|^k |s_n|^k |\beta_n|^k |\lambda_n|^{k-1} |\lambda_n| \\ &= O(1)\sum_{n=1}^{m} \frac{\varphi_n^{k-1}}{n^k P_n^k \chi_n^{k-1}} |P_{n-1}|^k |s_n|^k |\beta_n|^k \sum_{v=n}^{\infty} |\Delta\lambda_v| \\ &= O(1)\sum_{n=1}^{m} \frac{\varphi_n^{k-1}|s_n|^k|\beta_n|^k}{n^k \chi_n^{k-1}} \sum_{v=n}^{\infty} |\Delta\lambda_v| = O(1). \end{aligned}$$

$$\begin{aligned} M_2 &= O(1)\sum_{n=1}^{m} \frac{\varphi_n^{k-1}}{n^k P_n^k} \left| \sum_{v=1}^{n-1} \left(-p_v s_v \lambda_v \beta_v + P_v s_v \lambda_v \Delta\beta_v + P_v \Delta\lambda_v \beta_{v+1} s_v \right) \right|^k \\ &= M_{11} + M_{22} + M_{33}. \end{aligned} \tag{5.12}$$

Applying Hölder's inequality, conditions (3.8), (3.6), (4.3), (3.1), (3.5) and (4.1) (See I_1),

$$\begin{aligned} M_{11} &= O(1)\sum_{n=1}^{m} \frac{\varphi_n^{k-1}}{n^k P_n^k} \left| \sum_{v=1}^{n-1} p_v \lambda_v \beta_v s_v \right|^k \\ &= O(1)\sum_{n=1}^{m} \frac{\varphi_n^{k-1}}{n^k P_n} \left(\sum_{v=1}^{n-1} |p_v||s_v|^k |\beta_v|^k |\lambda_v|^k \right) \left(\sum_{v=1}^{n-1} \frac{p_v}{P_n} \right)^{k-1} \\ &\leq O(1)\sum_{v=1}^{m} |p_v||s_v|^k |\beta_v|^k |\lambda_v|^k \sum_{n=v}^{m} \frac{\varphi_n^{k-1}}{n^k P_n} \\ &= O(1)\sum_{v=1}^{m} |p_v||s_v|^k |\beta_v|^k |\lambda_v|^{k-1} |\lambda_v| \frac{\varphi_v^{k-1}}{v^{k-1} P_v} \\ &= O(1)\sum_{v=1}^{m} \frac{\varphi_v^{k-1}}{v^k} |s_v|^k |\beta_v|^k |\lambda_v|^{k-1} |\lambda_v| = O(1). \end{aligned} \tag{5.13}$$

Using Hölder's inequality, conditions (3.3), (3.6), (3.8), (3.1), (4.1), (3.5) and (4.3) (See I_1),

$$\begin{aligned}
M_{22} &= O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^kP_n^k}\left|\sum_{v=1}^{n-1}P_vs_v\lambda_v\Delta\beta_v\right|^k = O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^kP_n^k}\left(\sum_{v=1}^{n-1}v^{-1}P_v|s_v||\lambda_v||\beta_v|\right)^k\\
&= O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^kP_n}\sum_{v=1}^{n-1}v^{-k}p_v|s_v|^k|\lambda_v|^k|\beta_v|^k\left(\frac{P_v}{p_v}\right)^k\left(\frac{1}{P_n}\sum_{v=1}^{n-1}p_v\right)^{k-1}\\
&\le O(1)\sum_{v=1}^{m}p_v|\lambda_v|^k|s_v|^k|\beta_v|^k\sum_{n=v}^{m}\frac{\varphi_n^{k-1}}{n^kP_n}\\
&= O(1)\sum_{v=1}^{m}\frac{\varphi_v^{k-1}}{v^{k-1}}\frac{p_v}{P_v}|\lambda_v|^{k-1}|s_v|^k|\beta_v|^k|\lambda_v|\\
&= O(1)\sum_{v=1}^{m}\frac{\varphi_v^{k-1}}{v^k\chi_v^{k-1}}|s_v|^k|\beta_v|^k\sum_{n=v}^{\infty}\Delta|\lambda_n| = O(1). \qquad (5.14)
\end{aligned}$$

Applying Hölder's inequality, Abel's transformation, conditions (3.2), (3.8), (3.5), (3.4) and (4.1) (4.2) of Lemma 4.1, we have

$$\begin{aligned}
M_{33} &= O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^kP_n^k}\left|\sum_{v=1}^{n-1}P_vs_v\Delta\lambda_v\beta_{v+1}\right|^k\\
&= O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^kP_n^k}\left(\sum_{v=1}^{n-1}P_v|s_v||\Delta\lambda_v||\beta_v|\right)^k\\
&= O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^kP_n^k}\sum_{v=1}^{n-1}\frac{P_v^k}{\chi_v^{k-1}}|s_v|^k|\Delta\lambda_v||\beta_v|^k\left(\sum_{v=1}^{n-1}\chi_v|\Delta\lambda_v|\right)^{k-1}\\
&\le O(1)\sum_{v=1}^{m}\frac{P_v}{\chi_v^{k-1}}|\Delta\lambda_v||s_v|^k|\beta_v|^k\sum_{n=v+1}^{m}\frac{\varphi_n^{k-1}}{n^kP_n}\\
&= O(1)\sum_{v=1}^{m}\frac{\varphi_v^{k-1}}{v^{k-1}\chi_v^{k-1}}|\Delta\lambda_v||s_v|^k|\beta_v|^k = O(1)\sum_{v=1}^{m}v|\Delta\lambda_v|\frac{\varphi_v^{k-1}|\beta_v|^k|s_v|^k}{v^k\chi_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\lambda_v|)\sum_{r=1}^{v}\frac{\varphi_r^{k-1}|\beta_r|^k|s_r|^k}{r^k\chi_r^{k-1}} + O(1)m|\Delta\lambda_m|\sum_{v=1}^{m}\frac{\varphi_v^{k-1}|\beta_v|^k|s_v|^k}{v^k\chi_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}v|\Delta^2\lambda_v|\chi_v + O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|\chi_v + O(1)m|\Delta\lambda_m|\chi_m = O(1). \qquad (5.15)
\end{aligned}$$

Collecting (5.5)-(5.15), we have

$$\begin{aligned}
\sum_{n=2}^{m}\varphi_n^{k-1}|I_{n,1}|^k &= L_1 + L_2\\
&= (I_1+I_2+I_3)+(M_1+M_2)\\
&= (I_1+I_2+I_3)+(M_1+M_{11}+M_{22}+M_{33}) = O(1). \qquad (5.16)
\end{aligned}$$

Using condition (3.7), we have

$$\begin{aligned}\sum_{n=1}^{m}\varphi_n^{k-1}|I_{n,2}|^k &= \sum_{n=1}^{m}\varphi_n^{k-1}\left|\frac{1}{n(n+1)}\sum_{k=0}^{n-1}\frac{1}{P_k}\sum_{v=1}^{k}(P_k-P_{v-1})a_v\lambda_v\beta_v\right|^k\\ &= \sum_{n=1}^{m}\varphi_n^{k-1}\left|\frac{1}{n(n+1)}\sum_{k=0}^{n-1}\frac{1}{P_k}\sum_{v=0}^{k}p_v s_v\right|^k = \sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k(n+1)^k}\left|\sum_{k=0}^{n-1}t_k^{\bar{N}}\right|^k\\ &= O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k}\left|\frac{1}{n}\sum_{k=0}^{n-1}t_k^{\bar{N}}\right|^k = O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k}|t_n^{C\bar{N}}|^k = O(1).\end{aligned}$$

Now, from (5.16) and (5.17),

$$\sum_{n=1}^{m}\varphi_n^{k-1}|\Delta T_n|^k \leq \sum_{n=1}^{m}\varphi_n^{k-1}|I_{n,1}|^k+\sum_{n=1}^{m}\varphi_n^{k-1}|I_{n,2}|^k = O(1) \tag{5.18}$$

Therefore (5.4) is satisfied, and the series $\sum a_n\lambda_n\beta_n$ is $\varphi-|C_1\bar{N}_p|_k$ summable, which proves the main result.

6 Corollaries

Corollary 6.1. [20] Let $\{\varphi_n\}$, $\{\chi_n\}$ be sequences of positive real numbers, with $\{\chi_n\}$ a non-decreasing sequence, and conditions (3.1)-(3.6) and (3.8) are satisfied. Then the series $\sum a_n\lambda_n\beta_n$ is $\varphi-|\bar{N},p_n|_k$ summable for $k\geq 1$.

Remark 1. In Theorem 3.1 use $\varphi-|\bar{N},p_n|_k$ summability in place of $\varphi-|C_1\bar{N}_p|_k$.

Corollary 6.2. Let $\{\varphi_n\},\{\lambda_n\}$ be sequences of positive real numbers satisfying the following conditions:

$$\lambda_m = O(1),\ m\to\infty, \tag{6.1}$$

$$\sum_{n=1}^{m} n\ \log n\,|\Delta^2\lambda_n| = O(1),\ m\to\infty, \tag{6.2}$$

$$\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k}|s_n|^k = O(\log m)^k,\ \ as\ m\to\infty, \tag{6.3}$$

$$\sum_{v=1}^{m}\frac{\varphi_n^{k-1}}{n^{k+1}} = O(\frac{\varphi_v^{k-1}}{v^k}). \tag{6.4}$$

Then the series $\sum a_n\lambda_n$ is $\varphi-|C,1|_k$ summable for $k\geq 1$.

Remark 2. This result is an improved form of Theorem 2.2 by using $p_n=1$, $\beta_n=1$ and $\chi_n=\log n$ in Corollary 6.1.

7 Conclusion

This paper has focused on the study of generalized absolute Cesáro Riesz $\varphi - |C_1\bar{N}_p|_k$ summability factors obtained by the generalization of superimposition of summability $(C,1)$ on $(\bar{N},p_n)$ summability. The generalized product summability $\varphi - |C_1\bar{N}_p|_k$ plays an important role in signal processing as a double digital filter in finite impulse response.

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