Finding hybrid families of special matrix polynomials associated with Sheffer sequences

Ghazala Yasmin[1], Shahid Ahmad Wani[2] and Hibah Islahi[3]

[1,3]Department of Applied Mathematics, Aligarh Muslim University, Aligarh, India
[2]Department of Mathematics, Aligarh Muslim University, Aligarh, India
E-mail: ghazala30@gmail.com[2], shahidwani177@gmail.com[2], islahi.hibah@gmail.com[3]

Abstract

In this paper, three index three variable Hermite matrix based Sheffer polynomials (3I3VHMSP) are introduced by algebraic decomposition of exponential operators. The operational methods combined with the monomiality principle can be used to introduce 3I3VHMSP and also to establish rules of operational nature, framing the special polynomials within the context of particular solutions of generalized forms of partial differential equations of evolution type. The Appell and Sheffer sequences along with the operational formalism offer a powerful tool for investigation of the properties of a wide class of polynomials. Further, operational representation providing connections between 3I3VHMSP families and the known special polynomials are established, which are used to derive new identities and the results for the members of these new families. The approach presented is general.

2010 Mathematics Subject Classification. **33C45**. 33C99, 33E20
Keywords. 3 index 3 varible Hermite matrix based Sheffer polynomials, monomiality principle, operational techniques.

1 Introduction and preliminaries

An important generalization of special functions is special matrix functions. The study of special matrix polynomials is important due to their applications in certain areas of statistics, physics and engineering. The Hermite matrix polynomials and its extensions and generalizations have been introduced and studied in [2,5,6,8–10,15,18] for matrix in $\mathbb{C}^{N\times N}$, whose eigen values are all situated in the right open-half plane.

Throughout the paper unless otherwise stated, we assume that A is a positive stable matrix in $\mathbb{C}^{N\times N}$, that is, A satisfies the following condition:

$$Re(\mu) > 0, \text{ for all } \mu \in \sigma(A), \tag{1.1}$$

where $\sigma(A)$ denotes the set of all the eigenvalues of A.

If D_0 is the complex plane cut along the negative real axis and $\log(z)$ denotes the principal logarithm of z, then $z^{1/2}$ represents $\exp(\frac{1}{2}\log(z))$. If matrix $\mathbb{C}^{N\times N}$ with $\sigma(A) \subset D_0$, then $A^{1/2} = \sqrt{A}$ denotes the image by $z^{1/2}$ of the matrix functional calculus acting on the matrix A.

Tbilisi Mathematical Journal 12(4) (2019), pp. 43–59.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 15 April 2019.
Accepted for publication: 23 September 2019.

Recall that the 3-index 3-variable Hermite Matrix polynomials (3I3VHMP) $H_n^{(m,r)}(x,y,z;A)$ are defined by the series [16]:

$$H_n^{(m,r)}(x,y,z;A) = n!\sum_{k=0}^{[\frac{n}{r}]}\sum_{l=0}^{[\frac{n-rk}{m}]}\frac{(-1)^l z^k y^l (x\sqrt{mA})^{n-rk-ml}}{k!l!(n-rk-ml)!} \qquad (n\geq 0) \tag{1.2}$$

and specified by the generating function

$$\exp(xt\sqrt{mA} - yt^m I + zt^r I) = \sum_{n=0}^{\infty} H_n^{(m,r)}(x,y,z;A)\frac{t^n}{n!}, \tag{1.3}$$

where m and r are both positive integers.

It has been shown in [16], that 3I3VHMP are quasi-monomial under the action of the following multiplicative and derivative operators:

$$\hat{M}_H := x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} + rz(\sqrt{mA})^{-(r-1)}\frac{\partial^{r-1}}{\partial x^{r-1}}, \tag{1.4}$$

$$\hat{P}_H := \frac{1}{\sqrt{mA}}\frac{\partial}{\partial x}. \tag{1.5}$$

Taking suitable values of the indices and variables in equations (1.2) and (1.3), we get certain new and known special polynomials as special cases of the 3I3VHMP $H_n^{(m,r)}(x,y,z;A)$ and are mentioned in the following Table:

Table 1. Some special cases of the 3I3VHMP $H_n^{(m,r)}(x,y,z;A)$

S. No.	Values of the indices and variables	Relation between the 3I3VHMP $H_n^{(m,s)}(x,y,z;A)$ and its special cases	Name of the new and known polynomials
I.	$A=1\in\mathbb{C}^{1\times 1}$; $x\to\frac{x}{\sqrt{m}}$, $y\to -y$	$H_n^{(m,r)}\left(\frac{x}{\sqrt{m}},-y,z;1\right) = H_n^{(r,m)}(x,y,z)$	3-variable generalized Hermite polynomials (3VGHP) [3]
II.	$z=0$	$H_n^{(m,r)}(x,y,0;A) = H_{n,m}(x,y,A)$	2-index 2-variable Hermite matrix polynomials (2I2VHMP) [10]
III.	$y=0$	$H_n^{(m,r)}(x,0,z;A) = H_{n,m}^{(s)}(x,z;A)$	3-index 2-variable Hermite matrix polynomials (3I2VHMP) [16]
IV.	$r=m$; $y=0$, $z\to -z$	$H_n^{(m,m)}(x,0,-z;A) = H_{n,m}^{(m)}(x,-z;A) = H_{n,m}(x,z;A)$	2-index 2-variable Hermite matrix polynomials (2I2VHMP) [10]
V.	$A=1\in\mathbb{C}^{1\times 1}$; $x\to\frac{x}{\sqrt{m}}$, $y\to -y$, $z=0$	$H_n^{(m,r)}\left(\frac{x}{\sqrt{m}},-y,0;1\right) = H_n^{(m)}(x,y)$	Gould-Hopper polynomials (GHP) [7]
VI.	$m=2$; $z=0$	$H_n^{(2,r)}(x,y,0;A) = H_n(x,y,A)$	2-variable Hermite matrix polynomials (2VHMP) [2]
VII.	$m=2$; $x\to\frac{x}{2}$, $y\to -y$, $z=0$	$H_n^{(2,r)}\left(\frac{x}{2},-y,0;A\right) = \mathcal{H}_n(x,y;A)$	2-variable Hermite matrix polynomials of the second form (2VHMP) [18]
VIII.	$m=2$; $y=0$, $z=-1$	$H_n^{(2,r)}(x,0,-1;A) = H_{n,s}(x,A)$	Generalized Hermite matrix polynomials [15]
IX.	$m=2$; $y=1$, $z=0$	$H_n^{(2,r)}(x,1,0;A) = H_n(x,A)$	Hermite matrix polynomials (HMP) [5]

Most of the properties of families of polynomials, recognized as quasi-monomial, can be deduced by using operational rules associated with the relevant multiplicative and derivative operators. The notion of quasi-monomiality has been exploited within different contexts to deal with isospectral problems [19] and to study the properties of new families of special functions. Thus, we can define families of isospectral problems by using the correspondence

$$\begin{array}{lcl} \hat{M} & \Leftrightarrow & x, \\ \hat{P} & \Leftrightarrow & \dfrac{\partial}{\partial x}, \\ p_n(x) & \Leftrightarrow & x^n. \end{array} \tag{1.6}$$

There is a continuous demand of operational techniques in research fields like classical and quantum optics and in these fields the operational techniques provide powerful and efficient means of investigation.

According to the monomiality principle and in view of equations (1.4) and (1.5), we have

$$\hat{M}_H \ \{H_n^{(m,r)}(x,y,z;A)\} = H_{n+1}^{(m,r)}(x,y,z;A) \tag{1.7}$$

and

$$\hat{P}_H \ \{H_n^{(m,r)}(x,y,z;A)\} = n \ H_{n-1}^{(m,r)}(x,y,z;A), \tag{1.8}$$

respectively.

Now, since the 3I3VHMP $H_n^{(m,r)}(x,y,z;A)$ are quasi-monomial, the properties of these polynomials can be derived from those of the multiplicative and derivative operators $\hat{M}_H$ and $\hat{P}_H$ respectively. In fact, we have

$$\hat{M}_H \ \hat{P}_H\{H_n^{(m,r)}(x,y,z;A)\} = n \ H_n^{(m,r)}(x,y,z;A), \tag{1.9}$$

which yields the following differential equation satisfied by $H_n^{(m,r)}(x,y,z;A)$:

$$\left(my(\sqrt{mA})^{-m}\frac{\partial^m}{\partial x^m} - rz(\sqrt{mA})^{-r}\frac{\partial^r}{\partial x^r} - x\frac{\partial}{\partial x} + n \right) H_n^{(m,r)}(x,y,z;A) = 0. \tag{1.10}$$

Assuming here and in the sequel $H_0^{(m,r)}(x,y,z;A) = 1$, then $H_n^{(m,r)}(x,y,z;A)$ can be explicitly constructed as:

$$H_n^{(m,r)}(x,y,z;A) = \hat{M}_H^n\{1\}, \tag{1.11}$$

which yields the series definition for $H_n^{(m,r)}(x,y,z;A)$.

Equation (1.11), implies that the exponential generating function $H_n^{(m,r)}(x,y,z;A)$ can be given the form:

$$\exp(t\hat{M}_H)\{1\} = \sum_{n=0}^{\infty} H_n^{(m,r)}(x,y,z;A)\frac{t^n}{n!}, \qquad |t| < \infty, \tag{1.12}$$

which yields generating function (1.3).

The operators $\hat{M}_H$ and $\hat{P}_H$ also satisfy the commutation relation:

$$[\hat{P}_H, \hat{M}_H] = \hat{P}_H\hat{M}_H - \hat{M}_H\hat{P}_H = \hat{1} \tag{1.13}$$

and thus display the Weyl group structure.

Sequences of polynomials perform an important role in several branches of sciences. One of the important classes of polynomial sequences is the class of Sheffer sequences [14]. The Sheffer polynomial sequence arise in numerous problems of applied mathematics, theoretical physics, approximation theory and several other mathematical branches. According to Roman [12], the Sheffer sequence $s_n(x)$ is uniquely determined by two (formal) power series:

$$f(t) = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!}, \ f_0 = 0, \ f_1 \neq 0 \tag{1.14}$$

and

$$\mathfrak{g}(t) = \sum_{n=0}^{\infty} \mathfrak{g}_n \frac{t^n}{n!}, \ \mathfrak{g}_0 \neq 0. \tag{1.15}$$

Then there exists a unique sequence $s_n(x)$ of polynomials satisfying the orthogonality condition:

$$\left\langle g(t) \ f(t)^k | \ s_n(x) \right\rangle = n! \ \delta_{n,k} \ , \ \forall \ n, k \geq 0. \tag{1.16}$$

The exponential generating function of $s_n(x)$ is given by:

$$\frac{e^{xf^{-1}(t)}}{\mathfrak{g}(f^{-1}(t))} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \tag{1.17}$$

for all x in $\mathbb{C}$, where $f^{-1}(t)$ is the compositional inverse of $f(t)$.

It should be noted that for $\mathfrak{g}(t) = 1$, the Sheffer sequence $s_n(x)$ becomes the associated Sheffer sequence $\mathfrak{s}_n(x)$ and for $f(t) = t$, it becomes the Appell sequence $A_n(x)$ [1, 13].

The associate Sheffer and the Appell sequences are defined by the following generating functions:

$$e^{xf^{-1}(t)} = \sum_{n=0}^{\infty} \mathfrak{s}_n(x) \frac{t^n}{n!} \tag{1.18}$$

and

$$\frac{e^{xt}}{\mathfrak{g}(t)} = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!}. \tag{1.19}$$

Properties of Appell and Sheffer sequences are naturally handled within the framework of modern classical umbral calculus by Roman [12].

In this paper, the families of 3I3VHMSP are introduced by using the concepts and the methods associated with monomiality principle. In section 3, we construct certain examples of these polynomials . In Section 4, we consider certain special cases of 3I3VHMSP and obtain new and known special polynomials and frame these polynomials within the context of monomiality principle formalism and the operational representations are established for the 3I3VHMSP.

2 3-index 3-variable Hermite matrix based Sheffer polynomials

To introduceÂ the 3-index 3-variable Hermite matrix based Sheffer polynomials (3I3VHMSP) denoted by ${}_{H^{(m,r)}}s_n(x,y,z;A)$, we prove the following result:

Theorem 2.1. The 3I3VHMSP ${}_{H^{(m,r)}}s_n(x,y,z;A)$ are defined by the generating function:

$$\frac{1}{g(f^{-1}(t))}\exp\left(xf^{-1}(t)\sqrt{mA}-y(f^{-1}(t))^m I+z(f^{-1}(t))^r I\right)=\sum_{n=0}^{\infty}{}_{H^{(m,r)}}s_n(x,y,z;A)\frac{t^n}{n!}, \quad (2.1)$$

where m and r are both positive integers.

Proof. Replacing x in the l.h.s. and r.h.s. of equation (1.17) by the multiplicative operator $\hat{M}_H$ of the 3I3VHMP, we have

$$\frac{1}{g(f^{-1}(t))}\exp(\hat{M}_H f^{-1}(t))=\sum_{n=0}^{\infty}s_n(\hat{M}_H)\frac{t^n}{n!}. \quad (2.2)$$

Using the expression of $\hat{M}_H$ given in equation (1.4) and denoting the resultant 3I3VHMSP in the r.h.s. by ${}_{H^{(m,r)}}s_n(x,y,z;A)$ that is

$$s_n(\hat{M}_H)=s_n\left(x\sqrt{mA}-my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}+rz(\sqrt{mA})^{-(r-1)}\frac{\partial^{r-1}}{\partial x^{r-1}}\right)={}_{H^{(m,r)}}s_n(x,y,z;A), \quad (2.3)$$

we obtain

$$\frac{1}{g(f^{-1}(t))}\exp\left(\left(x\sqrt{mA}-my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}+rz(\sqrt{mA})^{-(r-1)}\frac{\partial^{r-1}}{\partial x^{r-1}}\right)f^{-1}(t)\right)$$

$$=\sum_{n=0}^{\infty}{}_{H^{(m,r)}}s_n(x,y,z;A)\frac{t^n}{n!}, \quad (2.4)$$

or

$$\frac{1}{g(f^{-1}(t))}\exp\left(\left(x\sqrt{mA}-my\frac{\partial^{m-1}}{\partial(x\sqrt{mA})^{m-1}}+rz\frac{\partial^{r-1}}{\partial(x\sqrt{mA})^{r-1}}\right)f^{-1}(t)\right)$$

$$=\sum_{n=0}^{\infty}{}_{H^{(m,r)}}s_n(x,y,z;A)\frac{t^n}{n!}, \quad (2.5)$$

Now, decoupling the exponential in l.h.s. of equation (2.5) by using the Crofton-type identity [4, p.12]:

$$f\left(y+m\lambda\frac{d^{m-1}}{dy^{m-1}}\right)\{1\}=\exp\left(\lambda\frac{d^m}{dy^m}\right)\{f(y)\}, \quad (2.6)$$

and then expanding the exponentials, we obtain the assertion (2.1).

Remark 2.1. Since for $g(t)=1$, the Sheffer polynomials $s_n(x)$ reduce to the associated Sheffer polynomials $\mathtt{s}_n(x)$. Therefore, on substitution of $g(t)=1$ in the l.h.s. of equation (2.1) and denoting the resultant 3-index 3-variable Hermite matrix based associated Sheffer polynomials

(3I3VHMASP) in the r.h.s. of equation (2.1) by ${}_{H^{(m,r)}}\mathtt{s}_n(x,y,z;A)$, we deduce the following consequence of theorem 2.1:

Corollary 2.1. The 3I3VHMASP ${}_{H^{(m,r)}}\mathtt{s}_n(x,y,z;A)$ are defined by the generating function:

$$\exp\left(xf^{-1}(t)\sqrt{mA}-y(f^{-1}(t))^m I+z(f^{-1}(t))^r I\right)=\sum_{n=0}^{\infty}{}_{H^{(m,r)}}\mathtt{s}_n(x,y,z;A)\frac{t^n}{n!}. \tag{2.7}$$

Remark 2.2. Since, for $f(t)=t$, the Sheffer polynomials $s_n(x)$ reduce to the Appell polynomials $A_n(x)$. Therefore, on substitution of $f(t)=t$ in the l.h.s. of equation (2.1) and denoting the resultant 3-index 3-variable Hermite matrix based Appell polynomials (3I3VHMAP) in the r.h.s. of equation (2.1) , we deduce the following consequence of Theorem 2.1:

Corollary 2.2. The 3I3VHMAP ${}_{H^{(m,r)}}A_n(x,y,z;A)$ are defined by the generating function:

$$\frac{1}{g(t)}\exp\left(xt\sqrt{mA}-yt^m I+zt^r I\right)=\sum_{n=0}^{\infty}{}_{H^{(m,r)}}A_n(x,y,z;A). \tag{2.8}$$

In order to frame the 3I3VHMSP ${}_{H^{(m,r)}}s_n(x,y,z;A)$ within the context of monomiality principle formalism, first we prove the following result:

Theorem 2.2. The 3I3VHMSP ${}_{H^{(m,r)}}s_n(x,y,z;A)$ are quasi monomial under the action of the following multiplicative and derivative operators:

$$\hat{M}_{Hs}:=\left(x\sqrt{mA}-my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}+rz(\sqrt{mA})^{-(r-1)}\frac{\partial^{r-1}}{\partial x^{r-1}}-\frac{g'(D_X)}{g(D_X)}\right)\frac{1}{f'(D_X)} \tag{2.9}$$

and

$$\hat{P}_{Hs}:=f(D_X) \tag{2.10}$$

respectively, where

$$\frac{D_x}{\sqrt{mA}}=D_X.$$

Proof. Consider the following identity:

$$D_x\left\{\frac{1}{g(f^{-1}(t))}\exp\left(xf^{-1}(t)\sqrt{mA}-y(f^{-1}(t))^m I+z(f^{-1}(t))^r I\right)\right\}$$

$$=\sqrt{mA}f^{-1}(t)\left(\frac{1}{g(f^{-1}(t))}\exp\left(xf^{-1}(t)\sqrt{mA}-y(f^{-1}(t))^m I+z(f^{-1}(t))^r I\right)\right). \tag{2.11}$$

Since $f^{-1}(t)$ denotes the compositional inverse of the function f and $f(t)$ has an expansion (1.14) in powers of t, therefore, we have

$$f\left(\frac{D_x}{\sqrt{mA}}\right)\left\{\frac{1}{g(f^{-1}(t))}\exp\left(xf^{-1}(t)\sqrt{mA}-y(f^{-1}(t))^mI+z(f^{-1}(t))^rI\right)\right\}$$
$$=t\left(\frac{1}{g(f^{-1}(t))}\exp\left(xf^{-1}(t)\sqrt{mA}-y(f^{-1}(t))^mI+z(f^{-1}(t))^rI\right)\right). \tag{2.12}$$

Differentiating equation (2.2) partially with respect to t and using equation (2.3) in the r.h.s. of the resultant equation, we find

$$\left(\left(\hat{M}_H-\frac{g'(f^{-1}(t))}{g(f^{-1}(t))}\right)\frac{1}{f'(f^{-1}(t))}\right)\frac{1}{g(f^{-1}(t))}\exp(\hat{M}_Hf^{-1}(t))=\sum_{n=0}^{\infty}{}_{H^{(m,r)}}s_{n+1}(x,y,z;A)\frac{t^n}{n!}, \tag{2.13}$$

which on using monomiality principle equation (1.12) with $t=f^{-1}(t)$ and then equation (1.3) gives

$$\left(\left(\hat{M}_H-\frac{g'(f^{-1}(t))}{g(f^{-1}(t))}\right)\frac{1}{f'(f^{-1}(t))}\right)\frac{1}{g(f^{-1}(t))}\exp\left(xf^{-1}(t)\sqrt{mA}-y(f^{-1}(t))^mI+z(f^{-1}(t))^rI\right)$$
$$=\sum_{n=0}^{\infty}{}_{H^{(m,r)}}s_{n+1}(x,y,z;A)\frac{t^n}{n!}. \tag{2.14}$$

Since $g(t)$ is an invertible series and $f(t)$ is a delta series of t therefore $\frac{g'(f^{-1}(t))}{g(f^{-1}(t))}$ and $\frac{1}{f'(f^{-1}(t))}$ possess power series expansions of $f^{-1}(t)$. Thus, in view of relation (2.11) and using $D_X=\frac{D_x}{\sqrt{mA}}$, the above equation becomes

$$\left(\left(\hat{M}_H-\frac{g'(D_X)}{g(D_X)}\right)\frac{1}{f'(D_X)}\right)\left\{\frac{1}{g(f^{-1}(t))}\exp(xf^{-1}(t)\sqrt{mA}-y(f^{-1}(t))^mI+z(f^{-1}(t))^rI)\right\}$$
$$=\sum_{n=0}^{\infty}{}_{H^{(m,r)}}s_{n+1}(x,y,z;A)\frac{t^n}{n!}. \tag{2.15}$$

which on using generating function (2.1) becomes

$$\left(\left(\hat{M}_H-\frac{g'(D_X)}{g(D_X)}\right)\frac{1}{f'(D_X)}\right)\left\{\sum_{n=0}^{\infty}{}_{H^{(m,r)}}s_n(x,y,z;A)\frac{t^n}{n!}\right\}=\sum_{n=0}^{\infty}{}_{H^{(m,r)}}s_{n+1}(x,y,z;A)\frac{t^n}{n!}. \tag{2.16}$$

Rearranging and then equating the coefficients of like powers of t on both sides of the above equation, we find

$$\left(\left(\hat{M}_H-\frac{g'(D_X)}{g(D_X)}\right)\frac{1}{f'(D_X)}\right){}_{H^{(m,r)}}s_n(x,y,z;A)={}_{H^{(m,r)}}s_{n+1}(x,y,z;A), \tag{2.17}$$

which, in view of equation (1.7) shows that multiplicative operator for ${}_{H^{(m,r)}}s_n(x,y,z;A)$ is given as:

$$\hat{M}_{Hs}=\left(\hat{M}_H-\frac{g'(D_X)}{g(D_X)}\right)\frac{1}{f'(D_X)}. \tag{2.18}$$

Finally, using equation (1.4) in the r.h.s of above equation, we get assertion (2.9).

Making use of generating function (2.1) on both sides of identity (2.12), we have

$$f\left(\frac{D_x}{\sqrt{mA}}\right)\left\{\sum_{n=0}^{\infty} {}_{H^{(m,r)}}s_n(x,y,z;A)\right\} = \sum_{n=1}^{\infty} {}_{H^{(m,r)}}s_{n-1}(x,y,z;A)\frac{t^n}{(n-1)!}, \tag{2.19}$$

Rearranging the summation in the r.h.s. of equation (2.18) and then equating the coefficients of like powers of t on both sides of the resultant equation, we find

$$f\left(\frac{D_x}{\sqrt{mA}}\right)\{{}_{H^{(m,r)}}s_n(x,y,z;A)\} = n\,{}_{H^{(m,r)}}s_{n-1}(x,y,z;A) \quad (n \geq 1), \tag{2.20}$$

which in view of equation (1.8) gives assertion (2.10).

Remark 2.3. Taking $g(t) = 1$ in equations (2.9) and (2.10), we deduce the following consequence of Theorem 2.2:

Corollary 2.3. The 3I3VHMASP ${}_{H^{(m,r)}}\mathtt{s}_n(x,y,z;A)$ are quasi monomial under the action of following multiplicative and derivative operators:

$$\hat{M}_{H\mathtt{s}} := \left(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} + rz(\sqrt{mA})^{-(r-1)}\frac{\partial^{r-1}}{\partial x^{r-1}}\right)\frac{1}{f'(D_X)} \tag{2.21}$$

and

$$\hat{P}_{H\mathtt{s}} := f(D_X), \tag{2.22}$$

respectively.

Remark 2.4. Taking $f(t) = t$ in equations (2.9) and (2.10), we deduce the following consequence of Theorem 2.2:

Corollary 2.4. The 3I3VHMAP ${}_{H^{(m,r)}}A_n(x,y,z;A)$ are quasi monomial under the action of following multiplicative and derivative operators:

$$\hat{M}_{HA} := \left(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} + rz(\sqrt{mA})^{-(r-1)}\frac{\partial^{r-1}}{\partial x^{r-1}} - \frac{g'(D_X)}{g(D_X)}\right) \tag{2.23}$$

and

$$\hat{P}_{HA} := D_X, \tag{2.24}$$

respectively.

Theorem 2.3. The 3I3VHMSP ${}_{H^{(m,r)}}s_n(x,y,z;A)$ satisfy the following differential equation:

$$\left(\left(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} + rz(\sqrt{mA})^{-(r-1)}\frac{\partial^{r-1}}{\partial x^{r-1}} - \frac{g'(D_X)}{g(D_X)}\right)\frac{f(D_X)}{f'(D_X)} - n\right) \times {}_{H^{(m,r)}}s_n(x,y,z;A) = 0. \tag{2.25}$$

Proof. Using equations (2.9) and (2.10) in equation (1.9), we get assertion (2.25).

Remark 2.5. Taking $g(t) = 1$ in equation (2.25), we deduce the following consequence of Theorem 2.3:

Corollary 2.5. The 3I3VHMASP ${}_{H^{(m,r)}}\mathtt{s}_n(x, y, z; A)$ satisfy the following differential equation:

$$\left(\left(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)} \frac{\partial^{m-1}}{\partial x^{m-1}} + rz(\sqrt{mA})^{-(r-1)} \frac{\partial^{r-1}}{\partial x^{r-1}} \right) \frac{f(D_X)}{f'(D_X)} - n \right)$$

$$\times \, {}_{H^{(m,r)}}\mathtt{s}_n(x, y, z; A) = 0. \tag{2.26}$$

Remark 2.6. Taking $f(t) = t$ in equations (2.25), we deduce the following consequence of Theorem 2.3:

Corollary 2.6. The 3I3VHMAP ${}_{H^{(m,r)}}A_n(x, y, z; A)$ satisfy the following differential equation:

$$\left(\left(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)} \frac{\partial^{m-1}}{\partial x^{m-1}} + rz(\sqrt{mA})^{-(r-1)} \frac{\partial^{r-1}}{\partial x^{r-1}} - \frac{g'(D_X)}{g(D_X)} \right) D_X - n \right)$$

$$\times {}_{H^{(m,r)}}A_n(x, y, z; A) = 0. \tag{2.27}$$

3 Examples

In this section, we consider some members of the Sheffer, associated Sheffer and Appell polynomials and obtain the corresponding members of the 3I3VHMSP ${}_{H^{(m,r)}}s_n(x, y, z; A)$.

We recall that the Hermite and Laguerre polynomials, $H_n(x)$ and $L_n(x)$ respectively are the Sheffer polynomials, defined by the generating functions [11]:

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \tag{3.1}$$

and

$$\frac{1}{1-t} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n(x) \frac{t^n}{n!}, \tag{3.2}$$

respectively.

Example 3.1 Since, for $g(t) = e^{t^2}$ and $f(t) = \frac{t}{2}$, the Sheffer polynomials $s_n(x)$ reduces to the Hermite polynomials $H_n(x)$. Therefore, for the same choice of $g(t)$ and $f(t)$ the 3I3VHMSP ${}_{H^{(m,r)}}s_n(x, y, z; A)$ reduced to the 3-index 3-variable Hermite matrix based Hermite polynomials (3I3VHMHP) ${}_{H^{(m,r)}}H_n(x, y, z; A)$.

Thus, by using these substitutions in equations (2.1), (2.9), (2.10) and (2.25), we get the following generating function, multiplicative and derivative operators and differential equation for the

3I3VHMHP ${}_{H^{(m,r)}}H_n(x,y,z;A)$:

$$e^{-t^2}\exp((2xt\sqrt{mA})-y(2t)^m I+z(2t)^r I)=\sum_{n=0}^{\infty}{}_{H^{(m,r)}}H_n(x,y,z;A), \tag{3.3}$$

$$\hat{M}:=\left(x\sqrt{mA}-my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}+rz(\sqrt{mA})^{-(r-1)}\frac{\partial^{r-1}}{\partial x^{r-1}}-2D_X\right)2, \tag{3.4}$$

$$\hat{P}:=\frac{D_X}{2}, \tag{3.5}$$

and

$$\left(\left(x\sqrt{mA}-my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}+rz(\sqrt{mA})^{-(r-1)}\frac{\partial^{r-1}}{\partial x^{r-1}}-2D_X\right)D_X-n\right)\times{}_{H^{(m,r)}}H_n(x,y,z;A)=0, \tag{3.6}$$

respectively.

Example 3.2 Since, for $g(t)=1-t$ and $f(t)=\frac{t}{t-1}$, the Sheffer polynomials $s_n(x)$ reduces to the Laguerre polynomials $L_n(x)$. Therefore, for the same choice of $g(t)$ and $f(t)$ the 3I3VHMSP ${}_{H^{(m,r)}}s_n(x,y,z;A)$ reduced to the 3-index 3-variable Hermite matrix based Laguerre polynomials (3I3VHMLP) ${}_{H^{(m,r)}}L_n(x,y,z;A)$.

Thus, by using these substitutions in equations (2.1), (2.9), (2.10) and (2.25) we get the following generating function, multiplicative and derivative operators and differential equation for the 3I3VHMLP ${}_{H^{(m,r)}}L_n(x,y,z;A)$:

$$\frac{1}{1-t}\exp\left(\left(\frac{xt}{t-1}\sqrt{mA}\right)-y\left(\frac{-t}{1-t}\right)^m I+z\left(\frac{-t}{1-t}\right)^r I\right)=\sum_{n=0}^{\infty}{}_{H^{(m,r)}}L_n(x,y,z;A), \tag{3.7}$$

$$\hat{M}:=\left(x\sqrt{mA}-my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}+rz(\sqrt{mA})^{-(r-1)}\frac{\partial^{r-1}}{\partial x^{r-1}}-\frac{1}{1-D_X}\right)\frac{-1}{(1-D_X)^2}, \tag{3.8}$$

$$\hat{P}:=\frac{D_X}{D_X-1}, \tag{3.9}$$

and

$$\left(\left(x\sqrt{mA}-my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}+rz(\sqrt{mA})^{-(r-1)}\frac{\partial^{r-1}}{\partial x^{r-1}}-\frac{1}{1-D_X}\right)\frac{-D_X}{(D_X-1)^3}-n\right)\times{}_{H^{(m,r)}}L_n(x,y,z;A)=0, \tag{3.10}$$

respectively.

Next, we recall that the Mittag-Leffler polynomials $M_n(x)$ are the associated Sheffer polynomials, defined by the generating functions [12]:

$$\left(\frac{1+t}{1-t}\right)^x=\sum_{n=0}^{\infty}m_n(x)\frac{t^n}{n!}. \tag{3.11}$$

Example 3.3 Since, for $g(t) = 1$ and $f(t) = \frac{e^t-1}{e^t+1}$, the associated Sheffer polynomials $\mathtt{s}_n(x)$ reduce to the Mittag-Leffer polynomials $m_n(x)$. Therefore, for the same choice of $g(t)$ and $f(t)$ the 3I3VHMASP ${}_{H^{(m,r)}}\mathtt{s}_n(x,y,z;A)$ reduced to the 3-index 3-variable Hermite matrix based-Mittag-Leffer polynomials (3I3VHMmLP)${}_{H^{(m,r)}}m_n(x,y,z;A)$. Thus, by using these substitutions in equations (2.7), (2.21), (2.22) and (2.26), we get the following generating function, multiplicative and derivative operators and differential equation for the 3I3VHMmLP ${}_{H^{(m,r)}}m_n(x,y,z;A)$:

$$\exp\left(x\ln\left(\frac{1+t}{1-t}\right)\sqrt{mA} - y\left(\ln\left(\frac{1+t}{1-t}\right)\right)^m I + z\left(\ln\left(\frac{1+t}{1-t}\right)\right)^r I\right) = \sum_{n=0}^{\infty} {}_{H^{(m,r)}}m_n(x,y,z;A), \tag{3.12}$$

$$\hat{M} := \left(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} + rz(\sqrt{mA})^{-(r-1)}\frac{\partial^{r-1}}{\partial x^{r-1}}\right)\frac{(1+e^{D_X})^2}{2e^{D_X}}, \tag{3.13}$$

$$\hat{P} := \frac{e^{D_X}-1}{e^{D_X}+1}, \tag{3.14}$$

and

$$\left(\left(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} + rz(\sqrt{mA})^{-(r-1)}\frac{\partial^{r-1}}{\partial x^{r-1}}\right)\frac{e^{2D_X}-1}{2e^{D_X}} - n\right) \times {}_{H^{(m,r)}}m_n(x,y,z;A) = 0, \tag{3.15}$$

respectively.

Now, recall that the Bernoulli and Euler polynomials, $B_n(x)$ and $E_n(x)$ respectively are the Appell polynomials, defined by the generating functions [11]:

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{3.16}$$

and

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \tag{3.17}$$

respectively.

Example 3.4 Since, for $g(t) = \frac{e^t-1}{t}$, the Appell polynomials $A_n(x)$ reduce to the Bernoulli polynomials $B_n(x)$. Therefore, for the same choice of $g(t)$, the 3I3VHMAP ${}_{H^{(m,r)}}A_n(x,y,z;A)$ reduced to the 3-index 3-variable Hermite matrix based Bernoulli polynomials (3I3VHMBP) ${}_{H^{(m,r)}}B_n(x,y,z;A)$. Thus, by using these substitutions in equations (2.8), (2.23), (2.24) and (2.27), we get the following generating function, multiplicative and derivative operators and differential equation for the 3I3VHMBP ${}_{H^{(m,r)}}B_n(x,y,z;A)$:

$$\frac{t}{e^t-1}\exp(xt\sqrt{mA} - yt^m I + zt^r I) = \sum_{n=0}^{\infty} {}_{H^{(m,r)}}B_n(x,y,z;A), \tag{3.18}$$

$$\hat{M} := \left(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} + rz(\sqrt{mA})^{-(r-1)}\frac{\partial^{r-1}}{\partial x^{r-1}} - \frac{1}{e^{D_X}-1}\right), \tag{3.19}$$

$$\hat{P} := D_X, \tag{3.20}$$

and

$$\left(\left(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} + rz(\sqrt{mA})^{-(r-1)}\frac{\partial^{r-1}}{\partial x^{r-1}} - \frac{1}{e^{D_X}-1}\right)D_X - n\right)$$
$$\times_{H^{(m,r)}} B_n(x,y,z;A) = 0, \tag{3.21}$$

respectively.

Example 3.5 Since, for $g(t) = \frac{e^t+1}{2}$, the Appell polynomials $A_n(x)$ reduce to the Euler polynomials $E_n(x)$. Therefore, for the same choice of $g(t)$, the 3I3VHMAP ${}_{H^{(m,r)}}A_n(x,y,z;A)$ reduced to the 3-index 3-variable Hermite matrix based Euler polynomials (3I3VHMEP) ${}_{H^{(m,r)}}E_n(x,y,z;A)$. Thus, by using these substitutions in equations (2.8), (2.23), (2.24) and (2.27), we get the following generating function, multiplicative and derivative operators for the 3I3VHMEP ${}_{H^{(m,r)}}E_n(x,y,z;A)$:

$$\frac{2}{e^t+1}\exp(xt\sqrt{mA} - yt^m I + zt^r I) = \sum_{n=0}^{\infty} {}_{H^{(m,r)}}E_n(x,y,z;A), \tag{3.22}$$

$$\hat{M} := \left(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} + rz(\sqrt{mA})^{-(r-1)}\frac{\partial^{r-1}}{\partial x^{r-1}} - \frac{e^{D_X}}{e^{D_X}+1}\right), \tag{3.23}$$

$$\hat{P} := D_X, \tag{3.24}$$

and

$$\left(\left(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} + rz(\sqrt{mA})^{-(r-1)}\frac{\partial^{r-1}}{\partial x^{r-1}} - \frac{e^{D_X}}{e^{D_X}+1}\right)D_X - n\right)$$
$$\times_{H^{(m,r)}} E_n(x,y,z;A) = 0, \tag{3.25}$$

respectively.

4 Appendix

By considering the special cases of the 3I3VHMP $H_n^{(m,r)}(x,y,z;A)$, we obtain the following results for the corresponding new or known special polynomials related to the 3I3VHMSP:

1. For the same choice of the values of the indices and variables taken in Table 1(I), for which the 3I3VHMSP ${}_{H^{(m,r)}}s_n(x,y,z;A)$ reduce to the 3-variable generalized Hermite based Sheffer polynomials (3VgHSP) ${}_{H^{(r,m)}}s_n(x,y,z)$. Thus, by these substitutions in Theorems 2.1, 2.2 and 2.3, we get the following results for 3VgHSP:

Table 2. Results for the 3VgHSP ${}_{H^{(r,m)}}s_n(x,y,z)$

S.No.	Results	Mathematical Expressions
1.	Generating function	$\frac{1}{g(f^{-1}(t))}\exp\left(xf^{-1}(t)+y(f^{-1}(t))^{m}I+z(f^{-1}(t))^{r}I\right)=\sum_{n=0}^{\infty}{}_{H^{(r,m)}}s_n(x,y,z)\frac{t^n}{n!}$
2.	Multiplicative and derivative operators	$\hat{M}:=\left(x+my\frac{\partial^{m-1}}{\partial x^{m-1}}+rz\frac{\partial^{r-1}}{\partial x^{r-1}}-\frac{g'(D_X)}{g(D_X)}\right)\frac{1}{f'(D_X)}$ $\hat{P}:=f(D_x)$
3.	Differential equation	$\left(\left(x+my\frac{\partial^{m-1}}{\partial x^{m-1}}+rz\frac{\partial^{r-1}}{\partial x^{r-1}}-\frac{g'(D_X)}{g(D_X)}\right)\frac{f(D_X)}{f'(D_X)}-n\right){}_{H^{(r,m)}}s_n(x,y,z)=0$

2. For the same choice of the values of the indices and variables taken in Table 1(II), the 3I3VHMSP ${}_{H^{(m,r)}}s_n(x,y,z;A)$ reduce to the 2-variable 2-index Hermite Matrix based Sheffer polynomials (2I2VHMSP) ${}_{H}s_{n,m}(x,y,A)$. Thus, by these substitutions in Theorems 2.1, 2.2 and 2.3, we get the following results for 2I2VHMSP:

Table 3. Results for the 2I2VHMSP ${}_{H}s_{n,m}(x,y,A)$

S.No.	Results	Mathematical Expressions
1.	Generating function	$\frac{1}{g(f^{-1}(t))}\exp\left(xf^{-1}(t)\sqrt{mA}-y(f^{-1}(t))^{m}I\right)=\sum_{n=0}^{\infty}{}_{H}s_{n,m}(x,y,A)\frac{t^n}{n!}$
2.	Multiplicative and derivative operators	$\hat{M}:=\left(x\sqrt{mA}-my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}-\frac{g'(D_X)}{g(D_X)}\right)\frac{1}{f'(D_X)}$ $\hat{P}:=f(D_x)$
3.	Differential equation	$\left(\left(x\sqrt{mA}-my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}-\frac{g'(D_X)}{g(D_X)}\right)\frac{f(D_X)}{f'(D_X)}-n\right){}_{H}s_{n,m}(x,y,A)=0$

3. For the same choice of the values of the indices and variables taken in Table 1(III), the 3I3VHMSP ${}_{H^{(m,r)}}s_n(x,y,z;A)$ reduce to the 3-index 2-variable Hermite Matrix based Sheffer polynomials (3I2VHMSP) ${}_{H^{(r)}}s_{n,m}(x,z;A)$. Thus, by these substitutions in Theorems 2.1, 2.2 and 2.3, we get the following results for 3I2VHMSP:

Table 4. Results for the 3I2VHMSP ${}_{H^{(r)}}s_{n,m}(x,z;A)$

S.No.	Results	Mathematical Expressions
1.	Generating function	$\frac{1}{g(f^{-1}(t))}\exp\left(xf^{-1}(t)\sqrt{mA}+z(f^{-1}(t))^{r}I\right)=\sum_{n=0}^{\infty}{}_{H^{(r)}}s_{n,m}(x,z;A)\frac{t^n}{n!}$
2.	Multiplicative and derivative operators	$\hat{M}:=\left(x\sqrt{mA}+rz(\sqrt{mA})^{-(r-1)}\frac{\partial^{r-1}}{\partial x^{r-1}}-\frac{g'(D_X)}{g(D_X)}\right)\frac{1}{f'(D_X)}$ $\hat{P}:=f(D_x)$
3.	Differential equation	$\left(\left(x\sqrt{mA}+rz(\sqrt{mA})^{-(r-1)}\frac{\partial^{r-1}}{\partial x^{r-1}}-\frac{g'(D_X)}{g(D_X)}\right)\frac{f(D_X)}{f'(D_X)}-n\right){}_{H^{(r)}}s_{n,m}(x,z;A)=0$

4. For the same choice of the values of the indices and variables taken in Table 1(V), the 3I3VHMSP ${}_{H^{(m,r)}}s_n(x,y,z;A)$ reduce to the Gould-Hopper based Sheffer polynomials (GHSP) ${}_{H^{(m)}}s_n(x,y)$ [17]. Thus, by these substitutions in Theorems 2.1, 2.2 and 2.3, we get the corresponding results for GHSP [17].

5. For the same choice of the values of the indices and variables taken in Table 1(VI), the 3I3VHMSP ${}_{H^{(m,r)}}s_n(x,y,z;A)$ reduce to the 2-variable Hermite matrix based Sheffer polynomials (2VHMSP) ${}_{H}s_n(x,y,A)$. Thus, by these substitutions in Theorems 2.1, 2.2 and 2.3, we get the following results for 2VHMSP:

Table 5. Results for the 2VHMSP ${}_H s_n(x, y, A)$

S.No.	Results	Mathematical Expressions
1.	Generating function	$\frac{1}{g(f^{-1}(t))} \exp\left(x f^{-1}(t)\sqrt{2A} - y(f^{-1}(t))^2 I\right) = \sum_{n=0}^{\infty} {}_H s_n(x, y, A) \frac{t^n}{n!}$
2.	Multiplicative and derivative operators	$\hat{M} := \left(x\sqrt{2A} - 2y(\sqrt{mA})^{-1} \frac{\partial}{\partial x} - \frac{g'(D_X)}{g(D_X)}\right) \frac{1}{f'(D_X)}$ $\hat{P} := f(D_X)$
3.	Differential equation	$\left(\left(x\sqrt{2A} - 2y(\sqrt{2A})^{-1} \frac{\partial}{\partial x} - \frac{g'(D_X)}{g(D_X)}\right) \frac{f(D_X)}{f'(D_X)} - n\right) {}_H s_n(x, y, A) = 0$

6. For the same choice of values of the indices and variables taken in Table 1(VIII), the 3I3VHMSP ${}_{H^{(m,r)}} s_n(x, y, z; A)$ reduce to the generalized Hermite matrix based Sheffer polynomials (GHMSP) ${}_H s_{n,r}(x, A)$. Thus, by these substitutions in Theorems 2.1, 2.2 and 2.3, we get the following results for GHMSP:

Table 6. Results for the GHMSP ${}_H s_{n,r}(x, A)$

S.No.	Results	Mathematical Expressions
1.	Generating function	$\frac{1}{g(f^{-1}(t))} \exp\left(x f^{-1}(t)\sqrt{2A} - (f^{-1}(t))^r I\right) = \sum_{n=0}^{\infty} {}_H s_{n,r}(x, A) \frac{t^n}{n!}$
2.	Multiplicative and derivative operators	$\hat{M} := \left(x\sqrt{2A} - r(\sqrt{mA})^{-(r-1)} \frac{\partial^{r-1}}{\partial x^{r-1}} - \frac{g'(D_X)}{g(D_X)}\right) \frac{1}{f'(D_X)}$ $\hat{P}_{Hs} := f(D_X)$
3.	Differential equation	$\left(\left(x\sqrt{2A} - r(\sqrt{2A})^{-(r-1)} \frac{\partial^{r-1}}{\partial x^{r-1}} - \frac{g'(D_X)}{g(D_X)}\right) \frac{f(D_X)}{f'(D_X)} - n\right) {}_H s_{n,r}(x, A) = 0$

7. For the same choice of values of the indices and variables taken in Table 1(IX), the 3I3VHMSP ${}_{H^{(m,r)}} s_n(x, y, z; A)$ reduce to the Hermite matrix based Sheffer polynomials (HMSP) ${}_H s_n(x, A)$. Thus, by these substitutions in Theorems 2.1, 2.2 and 2.3, we get the following results for HMSP:

Table 7. Results for the HMSP ${}_H s_n(x, A)$

S.No.	Results	Mathematical Expressions
1.	Generating function	$\frac{1}{g(f^{-1}(t))} \exp\left(x f^{-1}(t)\sqrt{2A} - y(f^{-1}(t))^2 I\right) = \sum_{n=0}^{\infty} {}_H s_n(x, A) \frac{t^n}{n!}$
2.	Multiplicative and derivative operators	$\hat{M} := \left(x\sqrt{2A} - 2y(\sqrt{mA})^{-1} \frac{\partial}{\partial x} - \frac{g'(D_X)}{g(D_X)}\right) \frac{1}{f'(D_X)}$ $\hat{P} := f(D_X)$
3.	Differential equation	$\left(\left(x\sqrt{2A} - 2y(\sqrt{2A})^{-1} \frac{\partial}{\partial x} - \frac{g'(D_X)}{g(D_X)}\right) \frac{f(D_X)}{f'(D_X)} - n\right) {}_H s_n(x, A) = 0$

Operational Representations

Next, certain operational representations of 3I3VHMSP ${}_{H^{(m,r)}} s_n(x, y, z; A)$ are obtained:

Theorem 4.1. The following operational representation connecting the 3I3VHMSP ${}_{H^{(m,r)}} s_n(x, y, z; A)$ with the Sheffer polynomials $s_n(x)$ holds true:

$$ {}_{H^{(m,r)}} s_n(x, y, z; A) = \exp\left(z\frac{\partial^r}{\partial(x\sqrt{mA})^r} - y\frac{\partial^m}{\partial(x\sqrt{mA})^m}\right) s_n(x\sqrt{mA}). \tag{4.1}$$

Proof. In view of equation (2.3), we have

$$ {}_{H^{(m,r)}}s_n(x,y,z;A) = s_n\left(x\sqrt{mA} - my\frac{\partial^{m-1}}{\partial(x\sqrt{mA})^{m-1}} + rz\frac{\partial^{r-1}}{\partial(x\sqrt{mA})^{r-1}}\right), \tag{4.2} $$

which on using equation (2.6) in the r.h.s. of equation (4.2), assertion (4.1) follows.

Theorem 4.2. The following operational representation connecting the 3I3VHMSP ${}_{H^{(m,r)}}s_n(x,y,z;A)$ with the 2I2VHMSP ${}_{H}s_{n,m}(x,y;A)$ holds true:

$$ {}_{H^{(m,r)}}s_n(x,y,z;A) = \exp\left((\sqrt{mA})^{-r}z\frac{\partial^r}{\partial x^r}\right)\{{}_{H}s_{n,m}(x,y;A)\}. \tag{4.3} $$

Proof. From equation (2.1), we have

$$ \frac{\partial^r}{\partial x^r}\,{}_{H^{(m,r)}}s_n(x,y,z;A) = (\sqrt{mA})^r\frac{\partial}{\partial z}\,{}_{H^{(m,r)}}s_n(x,y,z;A). \tag{4.4} $$

The solution of equation (4.4) is

$$ {}_{H^{(m,r)}}s_n(x,y,z;A) = \exp\left((\sqrt{mA})^{-r}z\frac{\partial^r}{\partial x^r}\right){}_{H^{(m,r)}}s_n(x,y,0;A) \tag{4.5} $$

Also in view of the fact, (Appendix 2.), we have

$$ {}_{H^{(m,r)}}s_n(x,y,0;A) = {}_{H}s_{n,m}(x,y;A). \tag{4.6} $$

Thus from equations (4.5) and (4.6), we get the assertion (4.3).

Theorem 4.3. The following operational representation connecting the 3I3VHMSP ${}_{H^{(m,r)}}s_n(x,y,z;A)$ with the 3I2VHMSP ${}_{H^{(r)}}s_{n,m}(x,z;A)$ holds true:

$$ {}_{H^{(m,r)}}s_n(x,y,z;A) = \exp\left(-(\sqrt{mA})^{-m}y\frac{\partial^m}{\partial x^m}\right)\{{}_{H^{(r)}}s_{n,m}(x,z;A)\}. \tag{4.7} $$

Proof. From equation (2.1), we have

$$ \frac{\partial^r}{\partial x^r}\,{}_{H^{(m,r)}}s_n(x,y,z;A) = -(\sqrt{mA})^m\frac{\partial}{\partial y}\,{}_{H^{(m,r)}}s_n(x,y,z;A). \tag{4.8} $$

The solution of equation (4.8) is

$$ {}_{H^{(m,r)}}s_n(x,y,z;A) = \exp\left(-(\sqrt{mA})^{-m}y\frac{\partial^m}{\partial x^m}\right){}_{H^{(m,r)}}s_n(x,0,z;A) \tag{4.9} $$

From the fact, (Appendix 3.), we have

$$ {}_{H^{(m,r)}}s_n(x,0,z;A) = {}_{H^{(r)}}s_{n,m}(x,z;A). \tag{4.10} $$

Thus using equations (4.9) and (4.10), we get the assertion (4.7).

In this article, special matrix polynomials are introduced by making use of operational identities for decoupling of exponential operators and the concepts associated with monomiality principle are used to establish their properties. The approach presented is general and can be extended to introduce other new families of special polynomials.

References

[1] P. Appell, *Sur une classe de polynômes,* Ann. Sci. *É*cole. Norm. Sup. **9** (1880) 119–144.

[2] R.S. Batahan, *A new extension of Hermite matrix polynomials and its applications*, *Linear Algebra Appl.* **419** (2006), no. 1, 82-92.

[3] G. Dattoli, P.E. Ricci, I. Khomasuridze, *On the derivation of new families of generating functions involving ordinary Bessel functions and Bessel-Hermite functions*, *Math. Comput. Modelling* **46** (2007), no. 3-4, 410-414.

[4] G. Dattoli, S. Lorenzutta, A.M. Mancho and A. Torre, *Generalized polynomials and associated operational identities*, *J. Comput. Appl. Math.* **108** (1-2) (1999) 209-218.

[5] E. Defez, L. Jódar, *Some applications of the Hermite matrix polynomials series expansions*, *J. Comput. Appl. Math.* **99** (1998), no. 1-2, 105-117.

[6] E. Defez, M. Garcia-Honrubia, R.J. Villanueva, *A procedure for computing the exponential of a matrix using Hermite matrix polynomials*, *Far East J. Appl. Math.* **6** (2002), no. 3, 217-231.

[7] H.W. Gould, A.T. Hopper, *Operational formulas connected with two generalizations of Hermite polynomials*, *Duke. Math. J.* **29** (1962) 51-63.

[8] L. Jódar, R. Company, *Hermite matrix polynomials and second order matrix differential equations*, *Approx. Theory Appl. (N.S.)* **12** (1996), no.2, 20-30.

[9] L. Jódar, E. Defez, *On Hermite matrix polynomials and Hermite matrix functions*, *Approx. Theory Appl. (N.S.)* **14** (1998), no. 1, 36-48.

[10] M. S. Metwally, M. T. Mohamed, A. Shehata, *Generalizations of two-index two-variable Hermite matrix polynomials*, *Demonstratio Math.* **42** (2009), no. 4, 687-701.

[11] E.D.Rainville, *Special Functions,* Macmillan, New York, NY. USA, 1960, reprinted by Chelsea, Bronx, NY, USA, 1971.

[12] S. Roman, *The Umbral Calculus* Academic Press, New York, 1984.

[13] S. Roman, *The theory of umbral calculus*, *I.J.Math.Anal* **87**(1) (1982) 58-115.

[14] I. M. Sheffer, *Some properties of polynomial sets of type zero,* Duke Math. J. **5** (1939) 590–622.

[15] K.A.M. Sayyed, M.S. Metwally, R.S. Batahan, *On generalized Hermite matrix polynomials*, *Electron. J. Linear Algebra* **10** (2003), 272-279.

[16] Subuhi Khan and A. A. Al-Gonah, *Multi-variable Hermite polynomials: Properties and applications*, *J. Math. Anal. Appl.* **412** (2014) 222-235.

[17] Subuhi Khan and Mumtaz Riyasat, *Determinantal approach to certain mixed special polynomials related to Gould-Hopper polynomials*, *Appl. Math. Comput.* **251** (2015) 599-614.

[18] Subuhi Khan, N. Raza, *2-variable generalized Hermite matrix polynomials and Lie algebra representation*, *Rep. Math. Phys.* **66** (2010), no. 2, 159-174.

[19] Y. Smirnov, A. Turbiner, *Lie algebraic discretization of differential equations*, *Modern Phys. Lett. A* **10**(24) (1995) 1795-1802.